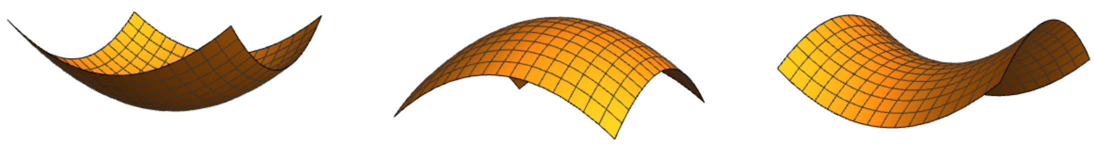
Math 33A Week 10
November 29 and December 1, 2016

Definiteness of Quadratic Forms

Given a quadratic form $q(\vec{x})$, we often care about the range of values the form might take. *A priori*, we know a few things about the values of q. We always have $q(\vec{0}) = 0$, and the range of q is unbounded, since

$$q(k\vec{x}) = k^2\vec{x}$$

for any scalar $k \in \mathbf{R}$. Questions whose answer we don't know, however, include whether or not q takes on any negative values, or whether q has a nontrivial kernel. Figure 1 shows plots of various forms $q\colon \mathbf{R}^2 \to \mathbf{R}$.



(a) A positive-definite form. (b) A negative-definite form. (c) An indefinite form.

(d) A positive semi-definite form. (e) A negative semi-definite form.

Figure 1: Plots of quadratic forms.

In these plots we see five distinct behaviors. The form in Figure 1a never takes on a negative value, and the only vector in its kernel is the zero vector. That is, $q(\vec{x}) > 0$ for all nonzero vectors $\vec{x}$. On the other hand, Figure 1b has the property that $q(\vec{x}) < 0$ for all nonzero vectors $\vec{x}$. We say that these forms are **positive-definite** and **negative-definite**, respectively. Figure 1c shows a form that takes on both positive and negative values, and we say that such a form is **indefinite**. The forms in Figures 1d and 1e are similar to those in Figures 1a and 1b, respectively, in that their range is of the form $[0, \infty)$ or $(-\infty, 0]$. The difference, though, is that the forms in Figures 1d and 1e both have nontrivial kernels. We say that the form in Figure 1d is **positive semi-definite**, meaning that $q(\vec{x}) \geq 0$ for all $\vec{x}$, but that there is some nonzero vector $\vec{x}$ so that $q(\vec{x}) = 0$. Similarly, the form in Figure 1e is called **negative semi-definite**. Our goal now is to classify quadratic forms according to these five categories. In particular, we want to take a quadratic form $q(\vec{x}) = ax_1^2 + bx_1x_2 + cx_2^2$ and determine which of these five terms is in effect[1].

[1] Of course, there's one form which doesn't fall into any of these five categories: the form $q(\vec{x}) = 0$.

A good first step towards determining the definiteness of $q(\vec{x}) = ax_1^2 + bx_1x_2 + cx_2^2$ is to recall that we can represent this quadratic form with a symmetric matrix A:

$$q(\vec{x}) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x}^T A \vec{x}.$$

Next, we recall the following (very important) result:

> **The Spectral Theorem.** A matrix is orthogonally diagonalizable if and only if it is symmetric.

Because the matrix A used to represent our quadratic form is symmetric, we may choose an orthonormal eigenbasis $\vec{u}_1, \vec{u}_2$, with associated eigenvalues λ_1 and λ_2. We then have

$$A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}^{-1} = SDS^T.$$

Since the columns of S are orthonormal, S is an orthogonal matrix, and this tells us that $S^{-1} = S^T$. So we may now write q as

$$q(\vec{x}) = \vec{x}^T A \vec{x} = \vec{x}^T SDS^T \vec{x} = \left(S^T \vec{x}\right)^T D(S^T \vec{x}).$$

At this point we introduce the following change of variables:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^T \vec{x}.$$

Notice that $S^T \vec{x} = S^{-1} \vec{x} = {}_{\mathcal{B}}[S]_{\mathcal{E}} \vec{x}$, so $\vec{x} = c_1 \vec{u}_1 + c_2 \vec{u}_2$. That is, our change of variables represents $\vec{x}$ in the eigenbasis $\vec{u}_1, \vec{u}_2$. We now substitute this into our expression for q to find that

$$q(\vec{x}) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda_1 c_1^2 + \lambda_2 c_2^2.$$

Now we must ask how this new expression for q aids our attempt to characterize the definiteness of the form. The benefit is that we no longer have a "mixed" term: a term of the form x_1x_2 or c_1c_2. Because we know that c_1^2 and c_2^2 will both be non-negative for all vectors $\vec{x}$, we should be able to characterize the definiteness of q from the signs of the eigenvalues λ_1 and λ_2.

For instance, suppose that both λ_1 and λ_2 are positive. Then, since c_1^2 and c_2^2 are non-negative, so is $q(\vec{x})$. Moreover, if $\vec{x} \neq \vec{0}$, then at least one of c_1, c_2 is positive, and $q(\vec{x})$ must then be positive. We see that $q(\vec{x}) \geq 0$ for all vectors $\vec{x}$, and that $q(\vec{x}) > 0$ whenever $\vec{x} \neq \vec{0}$, so q is positive definite. If we instead have $\lambda_1 > 0$ and $\lambda_2 = 0$, then $q(\vec{u}_2) = 0$, since in this case $c_1 = 0$ and $c_2 = 1$. This would mean that while q does not take any negative values, it has a nontrivial kernel, and is thus positive semi-definite. We can apply a similar analysis to q for the remaining combinations of signs of λ_1 and λ_2. The results are tabulated below.

Definiteness of a quadratic form. Consider a quadratic form $q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a 2×2 symmetric matrix. Suppose A has eigenvalues λ_1 and λ_2, with $\lambda_1 \geq \lambda_2$. Then

- if $\lambda_1 = \lambda_2 = 0$, $q(\vec{x}) = 0$ for all vectors $\vec{x}$;
- if $\lambda_1, \lambda_2 > 0$, q is positive-definite;
- if $\lambda_1 > 0$ and $\lambda_2 = 0$, q is positive semi-definite;
- if $\lambda_1, \lambda_2 < 0$, q is negative-definite;
- if $\lambda_1 = 0$ and $\lambda_2 < 0$, q is negative semi-definite;
- if $\lambda_1 > 0$ and $\lambda_2 < 0$, q is indefinite.

Consequently, if $\det A = 0$, then q is neither positive-definite nor negative-definite. If $\det A < 0$, then q is indefinite, and if $\det A > 0$, then q is either positive-definite or negative-definite[2].

Example 1. Determine the definiteness of the quadratic form $q(\vec{x}) = x_1^2 + 2x_1x_2 + x_2^2$.

(*Solution*) This form can be written as

$$q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x},$$

so we'll compute the eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We have

$$\det(A - \lambda I) = \left| \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \right| = (1-\lambda)^2 - 1 = \lambda(\lambda - 2),$$

so the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 0$. Since both eigenvalues are non-negative, q takes on only non-negative values. Moreover, since $\lambda_2 = 0$, q has a nontrivial kernel, and is thus positive semi-definite. ♢

Example 2. For which real numbers k is the quadratic form

$$q(\vec{x}) = kx_1^2 - 6x_1x_2 + kx_2^2$$

positive-definite?

[2]Warning: This second sentence is only true because A is 2×2. For more general quadratic forms this determinant analysis is not as useful.

(*Solution*) To determine the definiteness of this form we'll need to consider the matrix

$$A = \begin{bmatrix} k & -3 \\ -3 & k \end{bmatrix},$$

whose characterstic polynomial is

$$\det(A - \lambda I) = \left| \begin{bmatrix} k - \lambda & -3 \\ -3 & k - \lambda \end{bmatrix} \right| = (k - \lambda)^2 - 9 = \lambda^2 - 2k\lambda + (k^2 - 9).$$

We can either factor this polynomial as

$$\det(A - \lambda I) = (\lambda - (k + 3))(\lambda - (k - 3))$$

or use the quadratic equation to find its roots:

$$\lambda = \frac{2k \pm \sqrt{4k^2 - 4(k^2 - 9)}}{2} = \frac{2k \pm \sqrt{36}}{2} = k \pm 3.$$

Whichever method we use, we find that $\lambda_1 = k + 3$ and $\lambda_2 = k - 3$. In order for q to be positive-definite, both of these eigenvalues must be positive, and in particular we must have $\lambda_2 > 0$. So $k > 3$ is a necessary and sufficient condition for q to be a positive-definite quadratic form. ◊