This week we’ll discuss the solution to a homework problem before reviewing fractional age assumptions from chapter 3. Hopefully we’ll have time for some group work afterwards.

**Fractional age assumptions**

Life tables were introduced in chapter 3 as a way of presenting a survival model. Because life tables will usually present data for integer ages, we must perform some sort of interpolation for non-integer ages. We call this interpolation a *fractional age assumption*. There is no unique way to interpolate from a life table, but our book discusses two common fractional age assumptions.

**Uniform distribution of deaths**

The first fractional age assumption presented by the book is the *uniform distribution of deaths*, or *UDD*, assumption. Stated simply, in this interpolation we assume that the number of lives remaining in a population is a piecewise linear function, decreasing from \( l_x \) to \( l_{x+1} \) at a constant rate. For instance, consider the life table extract in Table 1, taken from Table 3.1 of the textbook. After plotting the ten points \((x, l_x)\), the UDD assumption tells us that we recover the value \( l_x \) for non-integer ages \( x \) by simply “connecting the dots” via nine straight lines. See Figure 1.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( l_x )</th>
<th>( d_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>10000.00</td>
<td>34.78</td>
</tr>
<tr>
<td>31</td>
<td>9965.22</td>
<td>38.10</td>
</tr>
<tr>
<td>32</td>
<td>9927.12</td>
<td>41.76</td>
</tr>
<tr>
<td>33</td>
<td>9885.35</td>
<td>45.81</td>
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<tr>
<td>34</td>
<td>9839.55</td>
<td>50.26</td>
</tr>
<tr>
<td>35</td>
<td>9789.29</td>
<td>55.17</td>
</tr>
<tr>
<td>36</td>
<td>9734.12</td>
<td>60.56</td>
</tr>
<tr>
<td>37</td>
<td>9673.56</td>
<td>66.49</td>
</tr>
<tr>
<td>38</td>
<td>9607.07</td>
<td>72.99</td>
</tr>
<tr>
<td>39</td>
<td>9534.08</td>
<td>80.11</td>
</tr>
</tbody>
</table>

Table 1: A life table extract

In practice, the UDD assumption manifests itself as an equation for computing \( s q_x \) when \( x \) is an integer and \( s \in [0, 1) \). Remember that the numbers \( d_x \) appearing in Table 1 estimate the number of deaths that will occur in our population over the next year, among those lives aged \( x \). The UDD assumes that these deaths are distributed evenly throughout the year, so that 0 of the deaths have occurred the start of the year, \( 0.5d_x \) deaths have occurred six months later, and \( d_x \) deaths have occurred after one year. If we introduce the notation \( s d_x \) for the number of deaths we expect to occur within \( s \) years \((0 \leq s < 1)\) among lives aged \( x \), then we have

\[
s d_x = s d_x,
\]
Figure 1: The function $l_x$, under the uniform distribution of deaths assumption.

According to the uniform distribution of deaths assumption. From this it follows that

$$sq_x = s\frac{d_x}{l_x} = s\frac{d_x}{l_x} = s\left(\frac{d_x}{l_x}\right) = sq_x,$$

so we typically record the UDD assumption as

$$sq_x = sq_x. \quad (1)$$

We will revisit this equation in an example below.

**Constant force of mortality**

According to our textbook, the *constant force of mortality* fractional age assumption is less common than the UDD assumption. Here our simplifying assumption is that the force of mortality $\mu_x$ is a step function, changing only when $x$ is an integer. So for an integer age $x$ we have $\mu_x = \mu^*_x$ for some constant $\mu^*_x$, and then

$$\mu_{x+s} = \mu^*_x$$

for all $s \in [0, 1)$. This creates jump discontinuities in the force of mortality function, but the UDD assumption did this as well. The defining equation for this assumption involves $sp_x$ instead of $sq_x$. In particular, note that

$$p_x = \exp\left\{ -\int_0^1 \mu_{x+s} ds \right\} = \exp\left\{ -\int_0^1 \mu^*_x ds \right\} = e^{-\mu^*_x},$$

so

$$sp_x = \exp\left\{ -\int_0^s \mu_{x+u} du \right\} = \exp\left\{ -\int_0^s \mu^*_x du \right\} = e^{-sp^*_x} = (e^{-\mu^*_x})^s = (p_x)^s.$$

The defining equation of the constant force of mortality assumption is then

$$sp_x = (p_x)^s. \quad (2)$$

Again, we will revisit this equation in an example.

It’s worth pointing out that the two fractional age assumptions discussed today shouldn't typically differ by too much. In particular, the constant force of mortality assumption tells us that
$sp_x = e^{-s\mu^*_x}$, and thus $sq_x = 1 - e^{-s\mu^*_x}$. So, for instance, $q_x = 1 - e^{-\mu^*_x}$. Now a local linear approximation of 

$$f(t) = 1 - e^{-t}$$

near $t = 0$ tells us that $1 - e^{-t} \approx t$ for small values of $t$. So if $\mu^*_x$ is reasonably small, then the constant force of mortality assumption gives

$$sq_x = 1 - e^{-s\mu^*_x} \approx s\mu^*_x \approx s(1 - e^{-\mu^*_x}) = sq_x,$$

which is the defining equation of the UDD assumption.

**Example**

Suppose we know the values of $q_{63}$ and $q_{64}$ for some survival model. Among a population of lives aged 63.5 years, consider the problem of determining the proportion of this population that will die within 9 months.

(a) Estimate this proportion using the UDD assumption.

(b) Estimate this proportion using the constant force of mortality assumption.

(c) Compare these estimates, assuming that $q_{63} = 0.004$ and $q_{64} = 0.0042$.

**(Solution)**

(a) We need to compute the quantity $0.75q_{63.5}$, since 9 months is 0.75 years. Certainly we have

$$0.75q_{63.5} = 0.5q_{63.5} + 0.5p_{63.5.25}q_{64}$$

$$= (1 - 0.5p_{63.5}) + 0.5p_{63.5.25}q_{64}.$$  

At the same time, we can use Bayes’ theorem to compute

$$0.5p_{63.5} = \frac{p_{63}}{0.5p_{63}} = \frac{1 - q_{63}}{1 - 0.5q_{63}} = \frac{1 - q_{63}}{1 - 0.5 \cdot q_{63}},$$

where we’ve used the UDD assumption in the last equality. So we can write $0.75q_{63.5}$ in terms of $q_{63}$ and $q_{64}$:

$$0.75q_{63.5} = \left(1 - \frac{1 - q_{63}}{1 - 0.5 \cdot q_{63}}\right) + \left(1 - \frac{1 - q_{63}}{1 - 0.5 \cdot q_{63}}\right) 0.25 \cdot q_{64}. \quad (3)$$

(b) As before, we may write

$$0.75q_{63.5} = (1 - 0.5p_{63.5}) + 0.5p_{63.5.25}(1 - 0.25p_{64}).$$

But this time the constant force of mortality assumption (namely, equation (2)) tells us that

$$0.5p_{63.5} = \frac{p_{63}}{0.5p_{63}} = \frac{p_{63}}{(p_{63})^{0.5}} = (p_{63})^{0.5},$$

so

$$0.75q_{63.5} = (1 - (p_{63})^{0.5}) + (1 - (p_{63})^{0.5})(1 - (p_{64})^{0.25}), \quad (4)$$

and each of the values $p_{63}, p_{64}$ is immediately recovered from $q_{63}, q_{64}$.
(c) Substituting these values into (3) and 4) tell us that

\[ 0.75q_{63.5} \approx 0.00200611 \]

under the UDD assumption, while

\[ 0.75q_{63.5} \approx 0.00200411 \]

under the constant force of mortality assumption. So the two assumptions do not lead to substantially different results here.

\[ \diamond \]

**Group work**

1. Using the life table in Table 1, compute each of the following twice — once using UDD, and once using the constant force of mortality assumption.

   (a) \( 1.7q_{33} \)
   
   (b) \( 1.7q_{33.5} \)
   
   (c) \( 1.70.5q_{33} \)