This week we’ll repeat some of what you heard in lecture about normal and geodesic curvatures, as well as some material about the second fundamental form and the Christoffel symbols.

Normal and geodesic curvatures

Earlier in the quarter we spent some time defining the curvature $\kappa$ of a curve in $\mathbb{R}^3$ and then the 
planar curvature $k$ of a curve in $\mathbb{R}^2$. The planar curvature allowed us to extract a little more 
information than the original curvature. For instance, by integrating the planar curvature over a 
closed curve in $\mathbb{R}^2$ we were able to obtain a topological invariant, the rotation index. The reason 
we were able to make sense of planar curvature is that in two dimensions we can choose a preferred 
normal vector for our curves. Specifically, if $t(s)$ was the unit tangent vector to $\alpha$ at $\alpha(s)$ then the 
unit normal vector was obtained by rotating $t(s)$ 90° counterclockwise.

Now suppose $x: U \to \mathbb{R}^3$ parametrizes a patch on a surface $S$. So $x$ produces coordinates on 
$S$, allowing us to (locally) treat $S$ as the two-dimensional object that it is. In particular, we can 
use $x$ to choose a preferred normal vector for curves in $S$. The parametrization $x$ gives us a unit 
normal vector $n(u, v)$ to our surface $S$. Let $\alpha(s)$ be a unit-speed curve in $S$, and let \{T, N, B\} be 
its Frenet-Serret frame. Now consider the vector 

$$S(s) := n(\alpha(s)) \times T(s).$$

This vector is perpendicular to $n$ — thus tangent to $S$ — and perpendicular to $T$. So $S(s)$ is a 
normal vector to $\alpha$ as it would be seen by an inhabitant of $S$. Just as in the case of $\mathbb{R}^2$, we have a 
preferred normal vector for our curves.

In order to define curvature as it would be experienced by an inhabitant of our surface we should 
recall the definition of planar curvature. Planar curvature measures the extent to which our unit 
tangent vector is turning towards our unit normal vector, and does so in a signed way. Concretely, 
if $\alpha(s)$ is a unit-speed curve in $\mathbb{R}^2$ we define 

$$k(s) := \langle \alpha''(s), n(s) \rangle,$$

where $n(s)$ is\footnote{This should be the only time in these notes that we use $n$ for a normal to a curve. Otherwise we’re using $n$ to denote a unit normal to $S$.} the preferred normal vector along $\alpha$. We play a similar game on the surface $S$. The geodesic curvature should tell us how much $\alpha'$ is turning towards $S$, which is the preferred normal vector along $\alpha$ from the point of view of $S$. So we define the geodesic curvature by 

$$\kappa_g(s) := \langle \alpha''(s), S(s) \rangle.$$ 

For emphasis we’ll repeat: the geodesic curvature represents the planar curvature, as it would be 
measured by an inhabitant of the surface.
Figure 1: A constant-height curve in a sphere. The red vectors represent $S$ the preferred normal to $\alpha$ in $S$. The green vectors are $\alpha''$. Notice that these are not perpendicular to $S$, so the geodesic curvature will be nonzero.

One example of geodesic curvature that might come to mind is that of a great circle in a sphere. Consider, for instance, Earth’s equator. Because our planet is round\textsuperscript{2} we know that the equator curves, but this is about as straight as we can hope for a curve on Earth to be. We’ll soon see that the geodesic curvature of the equator is indeed 0, meaning that it’s a straight line from our point of view — that is, as seen by Earthlings.

Before we compute any geodesic curvatures it’s worth noting that $S$ is a member of an obvious orthonormal basis at $\alpha(s)$, given by $(n, T, S)$. From its definition we see that the geodesic curvature is the $S$-component of $\alpha''(s)$, so we wonder about the other components. Because $\alpha$ is unit speed we know that $\alpha''$ is perpendicular to $T = \alpha'$, so the $T$-component is zero. We can then define the normal curvature of $\alpha$ to be

$$\kappa_n(s) := \langle \alpha''(s), n(s) \rangle,$$

so that

$$\alpha''(s) = \kappa_n(s)n(s) + \kappa_g(s)S(s).$$

The normal curvature can be used to tell us about the extent to which the surface itself is curving, and later in the quarter we’ll use it for this purpose.

Example 1. Let $S$ be the sphere of radius $R > 0$ in $\mathbb{R}^3$ centered at the origin. We have a familiar parametrization of this surface given by

$$x(u, v) = (R \sin v \cos u, R \sin v \sin u, R \cos v).$$

Now suppose $\alpha$ is cut out of $S$ by a plane of the form $z = c$, as seen in Figure 1. Along $\alpha$ the quantity $R \cos v$ is constant, and thus $v$ is constant. Let $\theta_0$ be this constant value. We’ll further

\textsuperscript{2}I hope I’ve not offended anyone.
assume that $-R < c < R$, so that $0 < \theta_0 < \pi$. Then we can produce a unit-speed parametrization of $\alpha$ as

$$\alpha(s) = \mathbf{x}\left(\frac{s}{R \sin \theta_0}, \theta_0\right) = \left(R \sin \theta_0 \cos\left(\frac{s}{R \sin \theta_0}\right), R \sin \theta_0 \sin\left(\frac{s}{R \sin \theta_0}\right), R \cos \theta_0\right).$$

Notice that

$$\alpha'(s) = \left(-\sin\left(\frac{s}{R \sin \theta_0}\right), \cos\left(\frac{s}{R \sin \theta_0}\right), 0\right)$$

and

$$\alpha''(s) = \frac{-1}{R \sin \theta_0} \left(\cos\left(\frac{s}{R \sin \theta_0}\right), \sin\left(\frac{s}{R \sin \theta_0}\right), 0\right),$$

so $\alpha(s)$ is indeed unit-speed. Now to compute the normal and geodesic curvatures of $\alpha$ we’ll need to get our hands on $\mathbf{S} = \mathbf{n} \times \mathbf{T}$. We already have $\mathbf{T}(s) = \alpha'(s)$, and because we’re on a sphere centered at the origin we have

$$\mathbf{n}(s) = \frac{1}{R} \alpha(s) = \left(\sin \theta_0 \cos\left(\frac{s}{R \sin \theta_0}\right), \sin \theta_0 \sin\left(\frac{s}{R \sin \theta_0}\right), \cos \theta_0\right).$$

Then

$$\mathbf{S} = \mathbf{n} \times \mathbf{T} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin \theta_0 \cos\left(\frac{s}{R \sin \theta_0}\right) & \sin \theta_0 \sin\left(\frac{s}{R \sin \theta_0}\right) & \cos \theta_0 \\ -\sin\left(\frac{s}{R \sin \theta_0}\right) & \cos\left(\frac{s}{R \sin \theta_0}\right) & 0 \end{vmatrix}$$

$$= \left(-\cos \theta_0 \cos\left(\frac{s}{R \sin \theta_0}\right), -\cos \theta_0 \sin\left(\frac{s}{R \sin \theta_0}\right), \sin \theta_0\right).$$

Finally we can compute the geodesic curvature:

$$\kappa_g(s) = \langle \alpha''(s), \mathbf{S}(s) \rangle$$

$$= \frac{-1}{R \sin \theta_0} \left[-\cos \theta_0 \cos^2\left(\frac{s}{R \sin \theta_0}\right) - \cos \theta_0 \sin^2\left(\frac{s}{R \sin \theta_0}\right)\right]$$

$$= \frac{1}{R \cot \theta_0}.$$

We can bring this computation back to reality. If $\alpha$ parametrizes the equator of $\mathbf{S}$ then $\theta_0 = \pi/2$, in which case the geodesic curvature is indeed zero. So the equator is seen as a straight line in $\mathbf{S}$. On the other hand, if $0 < \theta_0 < \pi/2$ then the geodesic curvature is positive, and $\alpha$ is turning towards $\mathbf{S}$. In the southern hemisphere we have $\pi/2 < \theta_0 < \pi$, so the geodesic curvature is negative. In this region $\alpha$ is turning away from $\mathbf{S}$, so this make sense.

At last we’ll compute the normal curvature of $\alpha$. We have

$$\kappa_n(s) = \langle \alpha''(s), \mathbf{n}(s) \rangle$$

$$= \frac{-1}{R \sin \theta_0} \left[\sin \theta_0 \cos^2\left(\frac{s}{R \sin \theta_0}\right) + \sin \theta_0 \sin^2\left(\frac{s}{R \sin \theta_0}\right)\right]$$

$$= \frac{-1}{R}.$$
Notice that this value doesn’t depend on $\theta_0$ at all. Indeed, we can check that every curve on the sphere of radius $R$ has geodesic curvature $\pm 1/R$. Indeed, let $\gamma$ be a unit-speed curve on this sphere, and continue letting $n$ be the outward-pointing normal. Then $n = x/R$, so we have

$$\kappa_n = \langle \gamma'', n \rangle = \left\langle \frac{\gamma''}{R}, \gamma \right\rangle.$$ 

Now because $\gamma$ takes its values on the sphere we know that $\langle \gamma, \gamma \rangle = R^2$ for all values of $s$. Differentiating this tells us that $\langle \gamma', \gamma \rangle = 0$ for all values of $s$. Differentiating again yields

$$\langle \gamma'', \gamma \rangle + \langle \gamma', \gamma' \rangle = 0.$$ 

But we’ve assumed that $\gamma$ is unit-speed, so we see that $\langle \gamma'', \gamma \rangle = -1$, and thus

$$\kappa_n = \frac{1}{R} \langle \gamma'', \gamma \rangle = -\frac{1}{R}.$$ 

Later in the quarter we’ll use this computation to describe the curvature of the sphere.

It’s worth pointing out that if we didn’t care about signs, then we could have saved ourselves some trouble by first computing $\kappa$ and $\kappa_n$. Then we could have used the equation $\kappa^2 = \kappa_n^2 + \kappa_\gamma^2$ to solve for the geodesic curvature, up to sign.

The second fundamental form and the Christoffel symbols

We can think of the geodesic and normal curvatures as the tangential and normal components of the curvature, respectively. Indeed, you’ll show in homework that

$$\kappa^2 = \kappa_n^2 + \kappa_\gamma^2,$$

where $\kappa$ is the usual curvature of a curve in $\mathbb{R}^3$. Both of these curvatures are computed using the second derivative of our curve, so a natural next step is to wonder whether we can compute these more generally. That is, if we have a surface patch $x: U \to S$, can we compute the tangential and normal components of $x_{ij}$, for each $1 \leq i, j \leq 2$? First, let’s write

$$x_{ij} = c_1 x_1 + c_2 x_2 + c_3 n,$$ 

where $c_k: U \to \mathbb{R}$ is some function$^3$, for $k = 1, 2, 3$. How do we determine the components $c_1, c_2, c_3$? The normal component $c_3$ is easy enough. We simply notice that

$$\langle x_{ij}, n \rangle = c_3,$$

since $n$ is perpendicular to both $x_1$ and $x_2$, and $n$ has unit length. We thus define

$$L_{ij} := \langle x_{ij}, n \rangle,$$

and we call the quantities $L_{ij}$ the coefficients of the second fundamental form. These measure the normal components of our second derivatives, and are closely related to the curvature of our surface.

$^3$In particular, we should not expect $c_k$ to be constant.
Next we want to compute the tangential components $c_1$ and $c_2$. This is slightly trickier, because the basis $(x_1, x_2)$ for the tangent plane need not be orthonormal. However, we see that

$$\langle x_{ij}, x_1 \rangle = c_1 g_{11} + c_2 g_{12}$$

and

$$\langle x_{ij}, x_2 \rangle = c_1 g_{12} + c_2 g_{22}.$$

Now consider the matrix equation $(g_{ij})(g^{ij}) = I$. We may write this as

$$\left( \begin{array}{cc} g_{11} & g_{12} \\ g_{21} & g_{22} \end{array} \right) \left( \begin{array}{cc} g^{11} & g^{12} \\ g^{21} & g^{22} \end{array} \right) = \left( \begin{array}{cc} g_{11} g^{11} + g_{12} g^{12} & g_{11} g^{12} + g_{12} g^{22} \\ g_{21} g^{11} + g_{22} g^{12} & g_{21} g^{12} + g_{22} g^{22} \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),$$

where we’ve liberally used the facts that $g_{12} = g_{21}$ and $g^{12} = g^{21}$. From this matrix equation we see that

$$\langle x_{ij}, x_1 \rangle g^{11} + \langle x_{ij}, x_2 \rangle g^{12} = (c_1 g_{11} + c_2 g_{12}) g^{11} + (c_1 g_{12} + c_2 g_{22}) g^{12}$$

$$= c_1 (g_{11} g^{11} + g_{12} g^{12}) + c_2 (g_{12} g^{11} + g_{22} g^{12})$$

$$= c_1.$$

Similarly, we find that

$$c_2 = \langle x_{ij}, x_1 \rangle g^{12} + \langle x_{ij}, x_2 \rangle g^{22}.$$

This motivates the following definition: For each $1 \leq i, j, k$, we define the Christoffel symbols

$$\Gamma^k_{ij} := \sum_{\ell=1}^{2} \langle x_{ij}, x_\ell \rangle g^{\ell k}.$$

Then, according to equation (1) and the work we did above,

$$x_{ij} = \Gamma^1_{ij} x_1 + \Gamma^2_{ij} x_2 + L_{ij} n.$$

So we see that the Christoffel symbols measure the tangential components of the second derivatives of our coordinate patch.