Goal: Describe a large class of embeddings for which the ECH capacities give a sharp obstruction.

Road map
1. Concave/convex toric domains
2. Reduction to a ball-packing problem
3. ECH capacities give sharp obstruction

Toric domains
The torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ has a natural action on $\mathbb{C}^2$ via $(\Theta_1, \Theta_2) \cdot (z_1, z_2) = (e^{2\pi i \Theta_1} z_1, e^{2\pi i \Theta_2} z_2)$. The moment map $\mu: \mathbb{C}^2 \rightarrow \mathbb{R}^2$ of this action is given by $\mu(z_1, z_2) = (\pi^1 |z_1|^2, \pi^1 |z_2|^2)$.

**Def.** A toric domain $X_\Omega \subset \mathbb{C}^2$ is the preimage of a region $\Omega \subset \mathbb{R}^2$ in the first quadrant under the map $\mu$.

**Ex.** The ellipsoid $E(a, b) = \{(z_1, z_2) \mid \frac{|z_1|^2}{a} + \frac{|z_2|^2}{b} \leq 1\}$ is the preimage under $\mu$ of the triangle with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and is thus a toric domain.
We call a toric domain concave or convex depending on the shape of \( \Omega \). Axes at an "upper bdy."

Example: Ellipsoids are both concave and convex.

Finally, we call a convex or concave toric domain rational if its upper boundary is piecewise linear with rational slopes. This will allow us to define a (finite) weight expansion, as we did for ellipsoids.

The main theorem

**Thm. (Cristofaro-Gardiner, 2014)** Let \( X_{\lambda_1} \) be a concave toric domain and let \( X_{\lambda_2} \) be a convex toric domain. Then there exists a symplectic embedding

\[
\text{int}(X_{\lambda_1}) \hookrightarrow \text{int}(X_{\lambda_2})
\]

iff

\[
\text{CECH}(\text{int}(X_{\lambda_1})) \leq \text{CECH}(\text{int}(X_{\lambda_2})).
\]

Notice that this generalizes the Hofer conjecture, previously established by McDuff.
Practicalities

For the main theorem to be useful, the ECH capacities of concave and convex toric domains must be computable. Indeed, we have a formula in each case, depending on the weight expansions.

We define the weight expansion of a (rational) concave toric domain recursively.

\[ W(T_0, a) := (a) \]
\[ W(\Omega) := W(\Omega_1) \cup W(\Omega_2) \cup W(\Omega_3) \]

The weight sequence is an unordered set, with repetition for ellipsoids, ordered.

\[ w(X_\omega) := w(\Omega) \]

The ECH capacities of a concave toric domain \( X_\omega \) were computed in terms of \( w(X_\omega) \) by Choi, Cristofaro-Gardiner, Frenkel, Hutchings, and Ramos:

**Thm (CCGFHR, 2013)** Let \( X_\omega \) be a rational concave toric domain, with \( w(X_\omega) = (a_1, \ldots, a_n) \). Then

\[ C_k(X_\omega) = C_k \left( \bigcup_{i=1}^n B(a_i) \right) \quad (\ast) \]
The RHS of (\(\ast\)) can be further reduced to a purely combinatorial expression in terms of \(a_1, \ldots, a_n\), using properties we've already seen.

The weight expansion of a (rational) convex toric domain is defined similarly, with a slightly different first step.

Choi \& Cristofaro-Gardiner computed the ECH capacities of convex toric domains.

**Thm. (CCG, 2014)** Let \(X_{\Omega}\) be a convex toric domain, let \(b\) be the head of the weight expansion for \(\Omega\), and let \(b_i\) be the \(i\)th term in the negative weight expansion for \(\Omega\). Then

\[
C_k = C_k(B(b)) - C_k\left(\bigcup_i B(b_i)\right),
\]

for all \(k\).
Reduction to ball packing

Recall that McDuff's proof of Hofer's conjecture reduces an ellipsoid embedding problem to a ball packing problem; we employ a similar strategy here.

Notation. For concave $\Omega$, let

$$B(\Omega) = \bigcup_i B(a_i),$$

where $W(\Omega) = (a_i)$. For convex $\Omega$, let

$$\hat{B}(\Omega) = \bigcup_i B(b_i),$$

where $W(\Omega) = (b_i, b_i)$

The key result in our proof of the main theorem is the following.

**Thm.** Let $X_{\Omega_1}$ be a rational concave toric domain, $X_{\Omega_2}$ a rational convex toric domain, and $b$ the head of the weight expansion for $\Omega_2$. Then $\exists$

$$\int (X_{\Omega_1}) \leftrightarrow \int (X_{\Omega_2})$$

iff $\exists$

$$\int (B(\Omega_1)) \cup \int (\hat{B}(\Omega_2)) \leftrightarrow \int (B(b)).$$

Necessity is easy; we'll put some work into sufficiency.
Only if. "Traynor trick"

Halfway there: embedding spheres into blowups
Towards reducing to the ball packing problem, we first reduce our embedding problem to a question of embedding a certain collection of spheres into a symplectic blowup of \( \mathbb{C}P^2 \).

Blowing up domains
Given a concave or convex toric domain \( X_\Omega \), we want to define a symplectic mfd \( (\mathbb{C}P^2 \# N \, \overline{\mathbb{C}P^2}, \omega_1) \) which we call the symplectic blowup of \( X_\Omega \).

Recall that symplectic blowup occurs along a ball (as opposed to a point). Given a symplectic embedding
\[
\bigcup_{i=1}^m B(a_i) \hookrightarrow (\mathbb{C}P^2, \omega_0)
\]
\( B(a_i) \rightharpoonup S^2 \)

(where \( \omega_0 \) is the Fubini–Study form), we obtain the blowup \( (\mathbb{C}P^2 \# m \, \overline{\mathbb{C}P^2}, \omega_1) \) by removing the interiors of the \( B(a_i) \) and collapsing their boundaries via the Hopf fibration. The form \( \omega_1 \) is canonical.
Each $\partial B(a_i)$ is collapsed to an exceptional divisor in the blowup, a sphere with homology class $E_i$. Then
\[ \text{PD}[\omega_1] = L - \sum_{i=1}^{m} a_i E_i, \]
where $L$ is the homology class of the line.

**Blowing up concave domains**

We define the blowup of $X_\Omega$, where $\Omega$ is concave, by mimicking the weight sequence. First, choose
\[ X_\Omega \hookrightarrow \text{int}(B(R)) \hookrightarrow (\mathbb{C}P^2, \omega), \]
for some $R > 0$ and some sympl. form $\omega$ on $\mathbb{C}P^2$.

We let
\[ a = \sup \{ r \mid T_{r, r} \subset \Omega^2 \}, \]
and choose $\delta > 0$ very small.

\[ \exists B(a + \delta) \hookrightarrow B(R), \]
so we blow up along $B(a + \delta)$.

The regions $\Gamma_1, \Gamma_2$ are, up to affine equivalence, concave. So we can iterate the procedure.

**Result:** $(\mathbb{C}P^2 \# m \overline{\mathbb{C}P^2}, \omega_1)$, with a configuration of $m$ sympl. spheres $C_{\Omega_1, \delta_1}$. 
Blowing up convex domains. We have a similar construction for rational convex domains. Here, we have (affine-) concave domains in the complement of $\Omega$ which sit inside $T_{b-S, b-S}$, where $w(\Omega) = (b; b)$. So we blow up $\mathbb{C}P^2 \setminus (b-S)w_0$.

No choice of $R$. \[
\bigcup_{i} \mathbb{B}(b_i) \hookrightarrow (\mathbb{C}P^2 \setminus (b-S)w_0)
\]

Result: $\left(\mathbb{C}P^2 \# n \mathbb{C}P^2, w_2\right)$, with a configuration of symplectic spheres $\hat{\mathcal{C}}_{\Omega, \delta_0}^i$.

Note that in the concave case we obtain an outer approximation $\Omega^\text{out}_S$, given by the solid red lines, with the property that the blowup removes $\text{int}(\Omega^\text{out}_S)$ and collapses $\partial \Omega^\text{out}_S$ to $\mathcal{C}_{\Omega, \delta_0}$. Similarly, the convex case has an inner approximation $\Omega^\text{in}_S$ s.t. blowup removes $\left(T_{b-S, b-S} - \Omega^\text{in}_S\right)$ and collapses $\partial \Omega^\text{in}_S$ to $\mathcal{C}_{\Omega, \delta_0}$.

**Def.** A symplectic embedding of $\mathcal{C}_{\Omega, \delta_0} \cup \mathcal{C}_{\Omega, \delta_0}^i$ into $(X, \omega)$ is a map s.t. the image spheres intersect $\partial \mathcal{C}_{\Omega, \delta_0}$, restricts to each sphere as a symplectic embedding, and s.t. the intersection matrix of the spheres is preserved.
We can now state our criterion for an embedding \( X_{\Omega_1} \hookrightarrow \text{int}(X_{\Omega_2}) \) to exist.

**Prop. (CG, 2014)** Let \( \Omega_1, \Omega_2 \) be rational concave, convex domains, respectively, with \( w(\Omega_1) = (a_1, \ldots, a_m) \) and \( w(\Omega_2) = (b_1, \ldots, b_n) \). If there exists a symplectic embedding

\[
C_{\Omega_1, \delta_{\Omega_1}} \cup \hat{C}_{\Omega_2, \delta_{\Omega_2}} \hookrightarrow (\mathbb{C}P^1 \# (m+n)\mathbb{C}P^1, \omega),
\]

for some \( \omega \), then \( \exists X_{\Omega_1} \hookrightarrow \text{int}(X_{\Omega_2}) \).

We'll prove this using a result of Gromov–McDuff. "Uniqueness of symplectic forms on star-shaped subsets of \( \mathbb{R}^4 \) that are standard near the boundary."

**Thm. (Gromov–McDuff)** Let \( (M, \omega) \) be a minimal, connected symplectic 4-manifold, and suppose we have a symplectomorphism

\[
\Phi : M \setminus K \to \mathbb{R}^4 \setminus V,
\]

where \( K \) is cpt and \( V \) is star-shaped about the origin. Then for any open nbhd \( U \supset K \), there is a symplectomorphism

\[
\Phi_U : M \to \mathbb{R}^4
\]

which agrees with \( \Phi \) on \( M \setminus U \).
Our strategy is then to construct a sympl. mfld $(M, \omega)$ and a symplecto \( \Phi : M \setminus K \to \mathbb{R}^q \setminus V \) s.t.

- \( K \) is compact, and admits a nbhd \( \tilde{Z} \subset M \) into which \( X_{\tilde{\omega}_1} \) embeds;
- \( V = X_{\tilde{\omega}_2} \) for some \( r < 1 \) close to 1 (and thus \( \text{int} (X_{\tilde{\omega}_2}) \subset \mathbb{R}^q \) is a nbhd of \( V \)).

(Proof Sketch.)

- We start with a sympl. embedding
  \[ C_{\omega_1, \delta_0, m} \sqcup \bar{C}_{\omega_2, \delta_0} \to (\mathbb{C}P^2 \# (m+n) \bar{\mathbb{C}P}^2, \omega). \]
  Assume that all intersections are sympl. orthogonal.
- Use a sympl. nbhd thm to identify a nbhd of \( C_{\omega_1, \delta_0} \subset (\mathbb{C}P^2 \# (m+n) \bar{\mathbb{C}P}^2, \omega) \) with a nbhd of \( C_{\omega_2, \delta_0} \subset (\mathbb{C}P^2 \# m \bar{\mathbb{C}P}^2, \omega_1) \).
- Since \( (\mathbb{C}P^2 \# m \bar{\mathbb{C}P}^2, \omega_1) \) is the blowup of \( X_{\tilde{\omega}_1} \),
  we may identify this nbhd with \( X_{\tilde{\omega}_1} \). Construct a sympl. mfld \( \tilde{Z} \) by removing \( C_{\omega_1, \delta_0} \) from \( (\mathbb{C}P^2 \# (m+n) \bar{\mathbb{C}P}^2, \omega) \) and replacing with \( X_{\tilde{\omega}_1} \).
  (Think of this as a "blowdown" operation.)
Notice that \( X_\omega \) embeds into \( \tilde{\omega} \), avoiding a nbhd of \( \widehat{\omega}_\omega, \delta_\omega \). Call the complement of this nbhd \( \tilde{\omega} \). Similar to above, we can replace the nbhd of \( \widehat{\omega}_\omega, \delta_\omega \) with \( \mathbb{R}^n \setminus \text{int} (\tilde{\omega}^c) \) to produce \( (M, \omega) \), with \( \omega \in M \).

By construction, we get a symplecto
\[
\Phi : M / K \to \mathbb{R}^n \setminus X_{r, \omega_1},
\]
for some \( r < 1 \) close to 1 and \( K \subset \tilde{\omega} \) cpt.

Check : \( H_2(M) = 0 \), so \( M \) is minimal.

From ball packing to sphere embedding

Via the blowing up operation, we've reduced our embedding problem to the problem of embedding a configuration of spheres into \( (\mathbb{C}P^2 \# (m+n) \mathbb{C}P^2, \omega) \). Now we reduce this to a ball packing problem.

We accomplish this reduction using "inflation." For sufficiently small \( r > 0 \), we have a symplectic embedding
\[
C_r \cdot \omega_1, \delta_1 \cup \tilde{\omega}_\omega, \delta_\omega \hookrightarrow (\mathbb{C}P^2 \# (m+n) \mathbb{C}P^2, \omega)
\]
for some \( \omega \), via the blowing up procedure. (We simply need \( r \cdot \omega_1 \subset \tilde{\omega}_\omega \).) Inflation will amount to changing the symplectic form \( \omega \) so that the areas of the spheres in \( C_r \cdot \omega_1, \delta_1 \) are increased to the appropriate size.
Prop. (CG, 2014) Let $\Omega_1, \Omega_2$ be rational concave, convex domains, respectively, with $w(\Omega_1) = (a_1, \ldots, a_m)$ and $w(\Omega_2) = (b_1, \ldots, b_n)$. If $\exists$ a sympl. emb. $\text{int}(B(\Omega_1)) \cup \text{int}(B(\Omega_2)) \hookrightarrow \text{int}(B(\mathbb{W}))$, then $\exists$ a sympl. embedding

$$\hat{\Omega}_{\Omega_1, \delta_{\Omega_1}} \cup \hat{\Omega}_{\Omega_2, \delta_{\Omega_2}} \hookrightarrow (\mathbb{CP}^n \# (m+n) \overline{\mathbb{CP}}^1, w),$$

for some symplectic form $w$.

**Note:** The rationality assumption is not necessary.

**Recall:**

- If $(M, \omega)$ is a blowup of $(\mathbb{CP}^n, \omega_F)$ and $A \in H_2(M; \mathbb{Z})$, then we have Taubes' Gromov invariant, $Gr(A)$. (Of course this is defined more generally.) This is a count of certain $J$-hol. curves in the class $A$.
- A symplectic divisor in $(M, \omega)$ is a union of sympl. emb. surfaces which intersect transversely orthogonally.
- An exceptional class $E \in H_2(M)$ is one represented by a symplectically embedded $-1$ sphere.

We're now ready to state the key result that we need.
Prop. (McDuff–Opeshtein, 2015) Let \((X, \omega)\) be a sympl. mfd, \(A \in H_2(X; \mathbb{Z})\), and \(S = X\) a symplectic divisor. Assume:

(i) \(A \cdot A > 0\);
(ii) \(A \cdot E \equiv 0\) for all exceptional classes \(E\);
(iii) \(A \cdot S_i \geq 0\) for all components \(S_i\) of \(S\);
(iv) \(Gr(A) \neq 0\).

Then for any \(s \geq 0\), the class \([\omega] + sPD(A)\) has a sympl. rep. that is nondegenerate on \(S\).

"inflating the divisor"

Idea: \(Gr(A) \neq 0 \Rightarrow \exists\) symplectic \(T \subset X\) with \(A = [T]\).

Other conditions allow us to deform \(\omega\) locally around \(T\) w/out becoming degenerate along \(S\).

To apply inflation, we need elements of \(H_2(M; \mathbb{Z})\) with \(Gr(A) \neq 0\). Kronheimer–Mrowka computed SW for blowups of \(CP^2\); together with Taubes’ \(Gr = SW\), we get the following.

Prop. Let \((M, \omega)\) be a sympl. blowup of \(CP^2\), and let \(A \in H_2(M, \omega)\). Let \(L\) be the homology class of the line in \(CP^2, E_1, \ldots, E_m\) the exceptional classes, and let

\[ PD(K) := -3L + \sum E_i \]

be the canonical class. If

\[ A^2 - K \cdot A > 0 \quad \text{and} \quad [\omega] \cdot (PD(K) - A) < 0, \]

then \(Gr(A) \neq 0\).
We're now ready to sketch the construction of \( C_{\omega_1, \delta, a} \cup \hat{C}_{\omega_2, \delta, \alpha} \rightarrow (\mathbb{CP}^1 \# (m+n) \mathbb{CP}^1, \omega) \) from the existence of
\[
\bigcup_{i=1}^{\infty} \text{int}(B(a_i)) \cup \bigcup_{j=1}^{\infty} \text{int}(B(b_j)) \leftrightarrow \text{int}(B(b)).
\]

- First, choose \( r \) suff. small s.t. \( r \Omega_1 \subset \text{int}(\Omega_2) \). Then we have a symplectic divisor \( S_0 = C_{r,\Omega_1, \delta, a} \cup \hat{C}_{\omega_2, \delta, \alpha} \rightarrow (\mathbb{CP}^2 \# (m+n) \mathbb{CP}^2, \omega) \).

We have homology classes \( L, E_1, \ldots, E_m, \hat{E}_1, \ldots, \hat{E}_n \) and the cohomology class of \( \omega_1 \) is given by
\[
[\omega_1] = b \cdot PD(L) - \sum_{i=1}^{m} a_i \cdot PD(E_i) - \sum_{j=1}^{n} b_j \cdot PD(\hat{E}_j) - \text{err}(S).
\]
\[
\text{err}(S) \rightarrow 0 \text{ as } S \rightarrow 0
\]

- We now want to inflate along the divisor \( S \). We have
\[
A := bL - \sum_{i=1}^{m} a_i E_i - \sum_{j=1}^{n} b_j \hat{E}_j \in H_2(M; \mathbb{Q}),
\]
so we choose \( k \) s.t. \( kA \) is integral.

**Exercise.** Check that by choosing \( k \) suff. large, \( kA \) will satisfy the conditions of the McDuff-Ohstein result.

- Applying the result gives a sympl. form \( \omega_{us} \) with
\[
[\omega_{us}] = [\omega_1] + sk \cdot PD(A),
\]
for any \( s \geq 0 \). Thus \( \frac{1}{1 + sk} \omega_{us} \) has cohom. class
\[
\frac{1}{1 + sk} [\omega, s] = \frac{1}{1 + sk} [\omega_1] + \frac{sk}{1 + sk} \mathbf{PD}(A)
\]

\[
= b \cdot \mathbf{PD}(L) - \sum_{i=1}^{m} a_i \left( \frac{r + sk}{1 + sk} \right) \mathbf{PD}(E_i) - \sum_{j=1}^{n} b_j \mathbf{PD}(\overline{E}_j) - \frac{1}{1 + sk} \mathbf{cn}(s),
\]

and thus

\[
\frac{1}{1 + sk} [\omega, s] \cdot [s] = \sum_{i=1}^{m} a_i^2 \left( \frac{r + sk}{1 + sk} \right) + \sum_{j=1}^{n} b_j^2.
\]

\[
\lambda^2
\]

\[
\cdot \text{By choosing } s \gg 0, \text{ we see that for all } \lambda < 1 \text{ we have a form } \omega \text{ s.t.}
\]

\[
\mathcal{C}_{\lambda, x_1, s_{x_2}, \mathcal{L}} \cup \mathcal{C}_{\lambda_2, s_{x_2}} \leftarrow (\mathcal{C}^+ \# (m+n)\mathcal{C}^+, \omega).
\]

At last, we proven the following:

**Thm.** Let \( X_{\Omega_1} \) be a rational concave toric domain, \( X_{\Omega_2} \) a rational convex toric domain, and \( b \) the head of the weight expansion for \( \Omega_2 \). Then \( \exists \)

\[
\text{int} (X_{\Omega_1}) \leftarrow \text{int} (X_{\Omega_2})
\]

iff \( \exists \)

\[
\text{int} (B(\Omega_1)) \cup \text{int} (\overline{B}(\Omega_2)) \leftarrow \text{int} (B(b)).
\]

The ECH capacities are known to give a sharp obstruction to the ball packing problem, and Zach showed last week how we can use the above theorem to get sharp obstructions for our original embedding problem.