Notation: \((Y, \lambda)\) contact manifold of dimension 3
\[ S = \ker \lambda \]

Recall: Roman defined for us the embedded contact homology
\[ \text{ECH} (Y, \lambda, \Gamma) \]
- generators: \(\alpha = \{(x_i, m_i)\}\) sets of Reeb orbits with multiplicities \(\Sigma m_i [x_i] = \Gamma \in H_1(Y)\)
- differentials: Count of J-hol curves in \(\mathbb{R} \times Y\) with ECH index 1, limiting to generators at \(\pm \infty\).

- Given generator \(\alpha = \{(x_i, m_i)\}\), its symplectic action is \(A(\alpha) = \Sigma m_i \int_{x_i} \beta\).
- If \(\langle \omega, \alpha \rangle \neq 0\), then \(A(\alpha) > A(\beta)\).

- We defined the filtered ECH \(\text{ECH}^L (Y, \lambda, \Gamma)\) as the homology of the subcomplex spanned by generators \(\alpha\) with action \(< L\).
\textbf{Definition:} A weakly exact symplectic cobordism (\textit{WEsc}) from \((Y_+, \lambda_+)\) to \((Y_-, \lambda_-)\) is a compact symplectic \(\eta\)-manifold \((X, \omega)\) with \(d\omega = \lambda_+ - \lambda_-\) such that \(\omega\) is exact and \(\omega|_{Y_+} = d\lambda_+\).
Example: A 4-dim. Liouville domain is a W.E.S.C. from its boundary to the empty set.

Theorem 1: Let \((X,\omega)\) be a W.E.S.C. from \((Y_+,\lambda_+)\) to \((Y_-,\lambda_-)\), where \(Y_+, Y_-\) are closed and \(\lambda_+\) non-degenerate. Then there exist maps

\[ \phi^L(x,\omega) : ECH^L(Y_+,\lambda_+) \to ECH^L(Y_-,\lambda_-) \]

of \(\mathbb{Z}/2\mathbb{Z}\) modules such that for each \(L \leq L'\):

a) If \(L < L'\), the following commutes:

\[
\begin{array}{ccc}
ECH^L(Y_+,\lambda_+) & \xrightarrow{\phi^L(x,\omega)} & ECH^L(Y_-,\lambda_-) \\
\downarrow i_* & & \downarrow i_* \\
ECH^{L'}(Y_+,\lambda_+) & \xrightarrow{\phi^{L'}(x,\omega)} & ECH^{L'}(Y_-,\lambda_-)
\end{array}
\]

\[ \phi(x,\omega) : ECH(Y_+,\lambda_+) \to ECH(Y_-,\lambda_-) \]
\[ \phi(x, w) := \lim_{n \to \infty} \phi^n(x, w) \]

b) \( \phi(x, w)[y] = [y] \) isomorphic

c) If \( X = [0, 1] \times Y \), then \( \phi(x, w) \) is an isomorphism

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**Full ECH Spectrum**

We have a map \( i_x : ECH^L(y, z) \to ECH(y, z) \)

Let \( I^L(y, z) \) be the dimension of the image \( i_x(ECH^L(y, z)) \) inside \( ECH(y, z) \).

If \( L \leq L' \), then \( I^L(y, z) \leq I^{L'}(y, z) \).

**Definition:** For each positive integer \( k \),

\[ \hat{c}_k(y, z) := \inf \{ L \mid I^L(y, z) \geq k \} \]

We call \( \{ \hat{c}_k(y, z) \}_{k=1}^{\infty} \) the **Full ECH Spectrum**
Remark:

a) \( 0 \leq \tilde{\zeta}_1(y, \lambda) \leq \cdots \leq \tilde{\zeta}_k(y, \lambda) \leq \infty \)

b) \( \tilde{\zeta}_1(y, \lambda) > 0 \iff c(s) = 0 \in \text{ECH}(y, \lambda) \)

Proof: If \( \tilde{\zeta}_1(y, \lambda) > 0 \), \( \exists \lambda > 0 \) such that \( \tilde{\zeta}_1(Ech^+) = 0 \Rightarrow c(s) = 0 \).

If \( c(s) = 0 \), \( \exists \lambda > 0 \) such that \( \tilde{\zeta}_1(Ech^+) = 0 \) \( \forall \lambda > 0 \)

\( \Rightarrow \tilde{\zeta}_1(y, \lambda) > 0 \).

c) If \( \lambda > 0 \),

\( \tilde{\zeta}_k(y, c\lambda) = c \cdot \tilde{\zeta}_k(y, \lambda) \)

Lemma 1: Let \((x, w)\) be a WESC from \((y_+, \lambda_+)\) to \((y_-, \lambda_-)\). Assume that \( \lambda_+ \) are nondegenerate and \( X \) diffeomorphic to \([0,1] \times Y\).

Then \( \tilde{\zeta}_k(y_-, \lambda_-) \leq \tilde{\zeta}_k(y_+, \lambda_+) \).
Proof: It suffices to show that for any \( \mathbf{C} \in \mathbb{R} \),

\[
\tau_\varepsilon^\mathbf{C}(Y_+) \leq \tau_\varepsilon^\mathbf{C}(Y_-)
\]

\[
\mathcal{E}(\mathbf{C})_+(Y_+) \xrightarrow{\varphi, \xi} \mathcal{E}(\mathbf{C})_+(Y_-)
\]

\[
\varphi: \mathcal{E}(\mathbf{C})_+(Y_-) \xrightarrow{\xi} \mathcal{E}(\mathbf{C})_+(Y_-)
\]

\[
\varphi \in (x_{\mathbf{C}}, (\mathcal{E}(\mathbf{C})_+(Y_-))) \mathcal{L}_x(\mathcal{E}(\mathbf{C})_+(Y_-))
\]

\( \varphi \) is an isomorphism.

So this proves the claim.

So far, we have only considered nondegenerate contact forms. We now extend the definition to any contact form.
Definition: Let $Y$ be any closed contact 3-manifold.

Define \( \hat{C}_c(Y, \alpha) := \sup \{ \hat{C}_c(Y, f \cdot \alpha) \} \),
\[ = \inf \{ \hat{C}_c(Y, f \cdot \alpha) \}_+ \]
where $f : Y \to (0, 1]$,
\[ f_+ : Y \to [1, \infty) \]

such that $f_+ \circ f$ is non-degenerate.

\[ \text{VESC} \quad (Y_+, \alpha_+) \to (Y_+, \alpha_+) \]
\[ (Y_-, \alpha_-) \to (Y_-, \alpha_-) \]

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**Full ECH capacities**

Definition: Let $(x, \omega)$ be a VESC from $Y$ to $\emptyset$.

Then \( \hat{C}_c(x, \omega) := \hat{C}_c(Y, \alpha) \)
where $\alpha$ is a contact form on $Y$ with $d\alpha = \omega|_Y$. 
Notice that this definition does not depend on the choice of contact form \( \alpha \).
Let's show that this is well-defined.

**Lemma 3:** Let \( \alpha, \alpha' \) be nondegenerate contact forms on \( Y \) with \( d\alpha = d\alpha' \).
Then we have isomorphisms

\[
ECH^*(Y, \alpha) \cong ECH^*(Y, \alpha').
\]

**Proof:** Let \( R, R' \) be the Reeb vector fields for \( \alpha, \alpha' \) respectively.

Then since \( d\alpha = d\alpha' \),
\[ R' = f \cdot R \]
for some positive \( f : Y \to \mathbb{R} \).

This gives a canonical bijection between the \( ECH \) generators of \( \alpha \) and those of \( \alpha' \).

Define \( \phi : \mathbb{R} \times Y \to \mathbb{R} \times Y \)
\[ (r, y) \mapsto (fr, y) \, , \]
$\phi$ is a diffeomorphism.

Given an admissible arc structure $J$ for $\gamma_1$, $J' = \phi_* J \circ \phi_*$ is admissible for $\gamma_2$.

Then the bijective on generators induces an isomorphism

$$\text{ECC}(\gamma_1, J) \cong \text{ECC}(\gamma_2, J').$$

Moreover, since $[\alpha J] = 0$ and $J - J'$ is a closed 1-form on $\gamma_1$, we have

$$\sum_m \int_{\alpha_i} \omega = \sum_m \int_{\alpha_i} \omega',$$

so this isomorphism respects the symplectic action filtrations.
Remark: \( \hat{c}_1(x, w) = 0 \).

Proof: By this, 1,

\[
\phi(x, w)[\phi_3], = [\phi], \phi \in \text{ECH}(\gamma)
\]

But \( \phi_3 \in \text{ECH}(\gamma) \) is non-zero.

Therefore \( \phi_3, \gamma \neq 0 \)

\[
\Rightarrow \hat{c}_1(\gamma, \chi) = 0.
\]

Proposition 2: Let \((x_0, w_0), (x_1, w_1)\)

be \(4\)-dimensional Liouville domains.

If \( \gamma: (x_0, w_0) \to (\text{int}(x_1, w_1)) \) is

a symplectic embedding such that

\[
x_1 \setminus \text{int}(\gamma(x_0)) = [0, 1] \times \gamma
\]

then \( \hat{c}_k(x_0, w_0) \leq \hat{c}_k(x_1, w_1) \) for \( k \in \mathbb{Z}^+ \).
Proof: Let $\Delta x_i = y_i, \Delta \gamma_i$ a contact from on $y_i$ s.t. $d \Delta x_i = \omega_1, \gamma_i$. Then $(x_i \setminus \text{int}((C(x_0)), \omega_1)$ is a WESE.

From $(y_i, x_i)$ to $(y_0, x_0)$.

Then $\hat{c}_k(y_0, x_0) \leq \hat{c}_k(y_i, x_i)$. \[ \leq \]

$\hat{c}_{k_1}(x_0, \omega_0) \leq \hat{c}_{k_1}(x_i, \omega_1)$. 

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**Full ECH capacities of an ellipsoid**

Recall: $E(a,b) := \{(z_1, z_2) \in \mathbb{C}^2 \mid \frac{\|z_1\|^2}{a} + \frac{\|z_2\|^2}{b} \leq 1\}$

- $(a,b)_k$ denotes the $k$th smallest entry of $(a\mathbf{r} + b\mathbf{v})_{\text{new}}$

\[ a \leq b \Rightarrow a + b \leq 2b \]
**Proposition:** The full ECH capacities of an ellipsoid \( E(a, b) \) are given by

\[
\widehat{c}_k (E(a, b)) = (a, b)_k
\]

**Proof:** \( R = 2\pi \left( a^{-1} \frac{\partial}{\partial \theta_1} + b^{-1} \frac{\partial}{\partial \theta_2} \right) \)

If \( \frac{a}{b} \) is irrational, then the only Kleeb orbits are \( \gamma_1 = (z_2 = 0) \)
\( \gamma_2 = (z_1 = 0) \)

These orbits are elliptic, nondegenerate and have symplectic actions a and b.

The differential vanishes.

The ECH generators have the form \( \gamma^m \gamma^n, \ m, n \in \mathbb{N} \).

\( A(\gamma^m \gamma^n) = am + bn \).

\( \overline{d^L} = \dim \left( \mathcal{I}(ECH^L(\mathbb{R}^2, 1, 2)) \right) \) is
\[ \{ (m,n) \in \mathbb{N}^2 \mid ma + bn < L \} \]

\( \widehat{C}_m (\mathcal{E}(a,b)) \) is the minimal \( L \)

such that \( \{ (m,n) \in \mathbb{N}^2 \mid ma + bn \leq L \} = k. \)

\( \widehat{C}_m (\mathcal{E}(a,b)) = (a,b)_k. \)

\[ \begin{array}{c}
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 a+b \\
 0 \\
 b \\
 b
\end{array}
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\end{array} \]

If \( \frac{a}{b} \) rational, choose \( a_-, a_+, b_-, b_+ \)

\( a_- = \frac{a}{b}, \quad b_- = \frac{b}{b_+} \)

\[ \frac{a_-}{b_-} < \frac{a_+}{b_+} \text{ irrational.} \]

\( (a_-, b_-)_k \leq \widehat{C}_m (\mathcal{E}(a,b)) \leq (a_+, b_+)_k. \)

\[ \downarrow \quad \downarrow \]

\[ \text{WEFC} \quad \text{WEFC} \]

Take \( a_+ \to a, \quad b_+ \to b. \)
Corollary:

If \( E(a,b) \) symplectically embeds into the interior of \( E(c,d) \), then

\[
(C_{a,b})_k \leq (c,d)_k \quad \text{for all } k.
\]

Calculation:

Given \((m,n) \in \mathbb{N}^2\), let \( T_{a,b}(m,n) \)

denote the triangle in \( \mathbb{R}^2 \) where
edges are \( x\)-axis, \( y\)-axis, line through \( C_{a,b}(m,n) \)
with slope \( -\frac{a}{b} \).

Then \((a,b)_k = am + bn\), where

\[
k = |T_{a,b}(m,n) \cap \mathbb{N}^2|.
\]
Example: Let $a = 2$, $b = 1 - \varepsilon$

\begin{align*}
(a, b)_1 &= 0 & (u, n) &= (0, 0) \\
(a, b)_2 &= 1 - \varepsilon & (u, n) &= (0, 1) \\
(a, b)_3 &= 2 - 2\varepsilon & (u, n) &= (0, 2) \\
(a, b)_4 &= 2 & (u, n) &= (1, 0) \\
(a, b)_5 &= 3 - 3\varepsilon & (u, n) &= (0, 3) \\
(a, b)_6 &= 3 - \varepsilon & (u, n) &= (1, 1)
\end{align*}

In general,

\[ (a, b)_2 = \begin{cases} 
  a & \varepsilon \leq 1 \\
  2 & \varepsilon \geq 1
\end{cases} \]

This means that if $(a, b) \leq (c, d)$,

then $\min(a, b) \leq \min(c, d)$

Assuming $a \geq b$

\[ (a, b)_3 = \begin{cases} 
  2b & 2 \leq a\varepsilon \\
  a & 1 \leq a \frac{\varepsilon}{2} \leq 2
\end{cases} \]
\[(a, b)_6 = \begin{cases} 
5b, & 5 \leq a/b, \\
4b, & 4 \leq a/b \leq 5, \\
3b, & 3 \leq a/b \leq 4, \\
2b, & 2 \leq a/b \leq 3, \\
\frac{3}{2}b, & \frac{3}{2} \leq a/b \leq 2, \\
a + b, & 1 \leq a/b \leq \frac{3}{2}.
\end{cases}\]

**Final Remark:**

Let \( L = am+bn \), then the area of \( T_{a,b}(m,n) \) is \( \frac{L^2}{2ab} \).

When \( L \) is large,

\[
\left| T_{a,b}(m,n) \cap W \right| \sim \frac{L^2}{2ab}
\]

\[
\lim_{k \to \infty} \frac{\mathbb{E}_k (E(a,b))^2}{k} = \frac{L^2}{\left( \frac{L^2}{2ab} \right)} = 2ab = 4 \text{vol}(E(a,b))
\]

This means that \((a,b)_k \leq (c,d)_k\) for \( k \) large tells us that

\[\text{vol}(E(a,b)) \leq \text{vol}(E(c,d))\]