1. Intro

Def. A symplectic structure on a sm. mfd $M$ is a closed, non-deg. 2-form $\omega$. A symplectomorphism is a diffeo $\Phi: (M, \omega) \rightarrow (M', \omega')$ s.t. $\Phi^* \omega' = \omega$.

Ex. The std sympl. str. on $\mathbb{R}^{2n}$ is

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Q. How do we characterize symplectos?

Thm. (Liouville, 1838) Symplectos preserve volume.
(Prf.) $\text{Vol} = \frac{1}{n!} \omega^n$. $\Phi^* \omega = \omega \rightarrow \Phi^* \text{Vol} = \text{Vol}$.

Liouville tells us that we must preserve a global inv — volume. Maybe we also preserve some local invols?

No.

Thm. (Darboux, 1882) Every sympl. mfd is locally symplecto to $(\mathbb{R}^{2n}, \omega_0)$. 
Maybe symplectic are just volume-preserving diffeos? This was disproven in 1985.

For any $r > 0$, let

$$\mathbb{B}^2(r) = \{ z \in \mathbb{R}^2 \mid |z| < r \}$$

and let

$$\mathbb{Z}^2(r) = \mathbb{B}^2(r) \times \mathbb{R}^2 = \{ z \in \mathbb{R}^2 \mid |z_1|^2 + |z_2|^2 < r^2 \}$$

Then (Gromov's nonsqueezing, 1985). If $\exists$

$$\mathbb{B}^2(r) \hookrightarrow \mathbb{Z}(r),$$

then $r \leq R$.

Gromov's NST distinguishes sympl. geom. from volume-preserving geometry via an embedding problem.

We want to study sympl. embedding problems.
\textbf{2. Symplectic capacities}

\textbf{Def.} Consider the class of all sympl. mflds (possibly w/ brdy) of a fixed dim. A symplectic capacity assigns \( c(M, \omega) \in [0, \infty] \) s.t.

1. (monotonicity) If \( (M, \omega) \hookrightarrow (M', \omega') \), then \( c(M, \omega) \leq c(M', \omega') \).
2. (conformality) For \( \alpha \in \mathbb{R} \setminus \{0\} \),
   \[ c(M, \alpha \omega) = |\alpha| c(M, \omega) \]
3. (non-triviality)
   \[ 0 < c(B(1), \omega_0) \leq c(S(1), \omega_0) < \infty. \]

\textbf{Remarks}

1. Not obvious that sympl. capacities exist.
2. These axioms do not determine a unique capacity — several exist.
3. A symplectic capacity is a sympl. invar.
4. Neither volume nor \( (\text{vol})^{\frac{1}{n}} \) is a capacity.
   for \( n > 1 \).

The existence of a capacity with
\[ c(B(1), \omega_0) = \pi = c(S(1), \omega_0) \]
is equivalent to Gromov's NST.
Suppose such a capacity exists, and that we have $B_2^n(r) \hookrightarrow \mathbb{Z}(R)$.

Then

$$r^2 \pi = c(B(1), r^2 \omega_0) = c(B(r), \omega_0)$$

$$\leq c(\mathbb{Z}(R), \omega_0) = c(\mathbb{Z}(1), R^2 \omega_0)$$

$$= R^2 \pi.$$

So $r \leq R$.

Assume NST. Define Gromov width

$$w_G(M, \omega) = \sup \left\{ r^2 \mid B_2^n(r) \hookrightarrow (M, \omega) \right\}.$$

NST: largest ball embedded in $\mathbb{Z}(1)$ is $B(1)$. So $w_G(\mathbb{Z}(1)) = \pi$.


Q. For real $a, b > 0$, let

$$E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{|z_1|^2}{a} + \frac{|z_2|^2}{b} < 1 \right\}.$$

When do we have

$$E(a, b) \hookrightarrow E(c, d)?$$

Thm (McDuff, 2010) $E(a, b) \hookrightarrow E(c, d)$ iff $N(a, b) \leq N(c, d).$
$N(a,b)$ is the sequence with entries of the form $(am + ln)_{m,n \geq 0}$, put in non-decreasing order with rep.

\[ N(1,2) = (0, 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, \ldots) \]

Sequences are ordered lexicographically.

If $c = d$, we're looking for

\[ E(a,b) \leftrightarrow B(c) \]

Up to rescaling, \[ E(1,a) \leftrightarrow B(c), \ a \geq 1. \]

Define \( f : [1, \infty) \to [1, \infty) \)

\[ f(a) = \inf \left\{ c \mid E(1,a) \leftrightarrow B(c) \right\} \]

This function was computed explicitly by McDuff-Schlenk.

![Diagram showing the function $f(a)$ and the sequence $N(1,2)$](image)
Properties

- For \( 1 \leq a \leq T^4 \), \( f \) is p.w. linear, where \( T = \left( 1 + \sqrt{5} \right) / 2 \).
- The interval \([T^4, (\frac{17}{6})^2]\) is partitioned into finitely many intervals on which \( f \) is linear or \( f(a) = \sqrt{a} \).
- For \( a \geq \frac{17}{6} \), \( f(a) = \sqrt{a} \).
- The "landings" occur at heights \( \frac{2}{1}, \frac{\sqrt{5}}{2}, \frac{13}{5}, \ldots \), \( \frac{F_{2n+1}}{F_{2n-1}} \rightarrow 1+2 = T^2 \).

4. ECH capacities

Note that \( N(a, b) \) is not a capacity—it's a sequence of \#s.

Hutchings defined ECH capacities (in dim 4), a sequence
\[
0 = c_0(M, w) < c_1(M, w) \leq c_2(M, w) \leq \ldots \leq \infty
\]
\( C_*(M, w) \)

Properties of ECH capacities

- (monotonicity) Each \( c_k(M, w) \) is monotone.
- (conformality) Each \( c_k(M, w) \) is conformal.
- (ellipsoid) \( C_*(E(a, b)) = N(a, b) \)
(disjoint union)
\[ C_k \left( \bigcup_{i=1}^m (M_i, \omega_i) \right) = \max \left\{ \sum_{i=1}^m c_{k_i}(M_i, \omega_i) \mid \sum_{i=1}^m k_i = k \right\}. \]

5. Embeddings of toric domains

A natural class of embedding problems for which ECT capacities give sharp obstructions.

A subset \( X_\Omega \subset \mathbb{C}^d \) is a \underline{toric domain} if
\[ X_\Omega = \mu^{-1}(\Omega), \]
where \( \Omega \subset \mathbb{R}^2 \) is a region in \( \mathbb{Q}_1 \) and
\( \mu : \mathbb{C}^2 \to \mathbb{R}^2 \)
\[ (z_1, z_2) \mapsto (\prod |z_1|^2, \prod |z_2|^2). \]

\[ E(x, y) = X_\Omega \]
\[ \Omega = \begin{array}{c}
\text{(0,1)} \\
\text{-(q,0)}
\end{array} \]

We call \( X_\Omega \) \underline{convex} or \underline{concave} depending on \( \Omega \).

![Convex](convex.png)
![Concave](concave.png)
Thm (Cristofaro-Gardiner, 2014) Let $X_{a_1}$ be a concave toric domain, $X_{a_2}$ convex. Then
\[ \exists \text{ int}(X_{a_1}) \hookrightarrow \text{ int}(X_{a_2}) \]
iff
\[ C_k(\text{ int}(X_{a_1})) \leq C_k(\text{ int}(X_{a_2})) \]
for all $k$.

Higher dimensions
When does
\[ E(a_1, \ldots, a_n) \hookrightarrow E(a_1', \ldots, a_n') \]?
Volume: $a_1 \cdots a_n \leq a_1' \cdots a_n'$
Gromov: $a_i \leq a_i'$
Guth (2008): for some finite $R$, $\exists$
\[ E(1, s, \ldots, s) \hookrightarrow E(R, R, \infty, \ldots, \infty) \]
for arbitrarily large $s$.

Outline for quarter
1. Hofer-Zehnder capacity (2 talks)
2. ECH capacities (3 talks)
3. Ellipsoid embeddings (2 talks)
4. Embeddings of toric domains (1 talk)
5. Guth’s ellipsoid embeddings (1 talk)