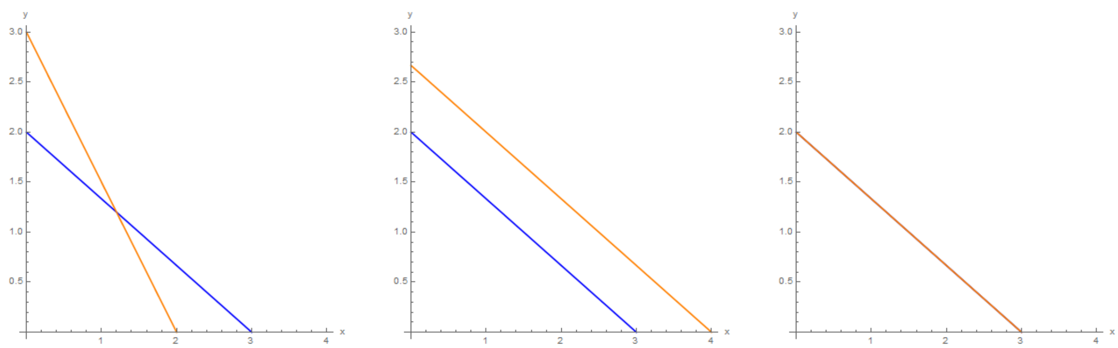
1 Systems of Linear Equations

1.1 Linear Equations and Linear Systems

By now you're quite familiar with linear equations. Indeed, you probably spent much of your time in high school learning to manipulate and plot solutions to equations of the form $Ax + By = C$ or $y = mx + b$. As you well know, the solution set to such an equation forms a line in the plane (which we'll denote by $\mathbf{R}^2$), provided the equation satisfies certain conditions. Now suppose we have two equations:

$$Ax + By = C \qquad \text{and} \qquad Dx + Ey = F, \tag{1}$$

and we want to identify the points $(x, y) \in \mathbf{R}^2$ that satisfy *both* equations. We call (1) a **system of linear equations** (**in two variables**). There are three possibilities for the lines determined by the two equations in this system.



(a) A unique solution. (b) No solution. (c) Infinitely many solutions.

Figure 1: Linear systems in two variables.

If the lines intersect, as depicted in Figure 1a, the point (x, y) where they intersect is a solution to both of our equations, and thus a **solution to the system of linear equations**. Moreover, the lines can't intersect at any other points, so this is the *unique* solution to our system. Another possibility is that the lines are parallel – as seen in the middle image – meaning that our system has no solutions. Finally, the last image shows the state of affairs when $Ax + By = C$ and $Dx + Ey = F$ determine the same line. In this case, every point (x, y) on one of the lines is also on the other line, and is thus a solution to the system. So the system of equations has infinitely many solutions. Here we considered a system of linear equations in two variables, but the possible outcomes are the same in any number of variables:

Solutions to a system of linear equations. A system of linear equations can have no solutions, exactly one solution, or infinitely many solutions. If the system has two or more distinct solutions, it must have infinitely many solutions.

Example 1. Consider the following systems of linear equations:

$$\text{(a)} \quad \begin{aligned} 2x + 3y + z &= 6 \\ x + y + z &= 17 \\ 4x + 6y + 2z &= 13 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x + y &= 12 \\ 3x + 3y &= 36 \end{aligned} \qquad \text{(c)} \quad \begin{aligned} 2x + 4y &= 8 \\ 4x + 2y &= 10. \end{aligned}$$

Determine whether each of these systems has a unique solution, infinitely many solutions, or no solution.

(*Solution*)

(a) Notice that the left side of the third equation in this system is twice the left side of the first equation:

$$2(2x + 3y + z) = 4x + 6y + 2z.$$

If (x, y, z) satisfies the first equation, then $2x + 3y + z = 6$, so $4x + 6y + 2z = 12$. In particular, (x, y, z) can't satisfy both the first and the third equation. Since the first and third equations don't share any solutions, this system has no solutions.

(b) We immediately notice that the second equation of this system is just the first equation multiplied by 3, and therefore doesn't give us any information about (x, y) that the first equation hasn't already given us. With just one condition on the two variables, we suspect that this system might have infinitely many solutions. We can exhibit this fact by pointing out that $(0, 12)$ and $(12, 0)$ are both solutions to this system. Since the system has at least two distinct solutions, it must in fact have infinitely many solutions.

(c) This system doesn't have any obvious problems like the first two, so perhaps we should try to solve it. First we can rearrange the first equation to write x in terms of y:

$$2x = 8 - 4y \quad \Rightarrow \quad x = 4 - 2y.$$

Next we can substitute this expression for x into our second equation:

$$4(4 - 2y) + 2y = 10 \quad \Rightarrow \quad 16 - 6y = 10.$$

This last equation is easily solved to conclude that $y = 1$. Our expression for x then tells us that $x = 4 - 2(1) = 2$. So $(2, 1)$ is a solution to this system, and in fact we've shown that if (x, y) satisfies both equations in the system, then $x = 2$ and $y = 1$. So $(2, 1)$ is the *u*nique solution to this system.

$\diamondsuit$

In finding a solution to the last system of equations, we used a somewhat *a*d hoc substitution method. This usually works well for systems with two variables, and it remains a viable (if tedious) approach to solving systems in three variables, but for systems with four or more variables it's simply untenable. Even with two- or three-variable systems, this substitution method is not quite as systematic as we might like. Much of the rest of this page is devoted to developing a systematic algorithm for solving systems of linear equations.

1.2 Elementary Row Operations

When searching for a solution to a system of linear equations, we often find ways to transform the system so that the solutions are more easily identified. For instance, the system of linear equations

$$\begin{aligned} 2x + 4y &= 8 \\ 4x + 2y &= 10 \end{aligned} \tag{2}$$

that we considered above is equivalent to the system of equations

$$\begin{aligned} x + 0y &= 2 \\ 0x + y &= 1 \end{aligned}, \tag{3}$$

but certainly we have an easier time identifying solutions to (3). We say that (3) is in **diagonal form**, and our goal with most systems will be to transform them until we have a diagonal system and then read off solutions. There are a few operations we can apply to a system of linear equations without altering their solutions. For starters, we can interchange any two equations in the system. Surely

$$\begin{aligned} 4x + 2y &= 10 \\ 2x + 4y &= 8 \end{aligned}$$

is equivalent to the system given in (2). Unsurpisingly, it's also safe to multiply (or divide) any equation in our system by any nonzero number. We restrict ourselves to nonzero numbers because multiplying by 0 would give us the unhelpful equation $0 = 0$ and dividing by 0 would lead to an apocalypse. Finally, we can change an equation by adding to it a multiple of another equation in our system. For instance, if both equations in (2) are true, then

$$6x + 6y = (4x + 2y) + (2x + 4y) = 10 + 8 = 18,$$

so we can replace the second equation with $6x + 6y = 18$:

$$\begin{aligned} 2x + 4y &= 8 \\ 6x + 6y &= 18 \end{aligned}.$$

Since the second equation of our original system was involved in the creation of our new equation, we don't lose any information by deleting the second equation and replacing it with the new one. Let's summarize the operations we have:

Elementary Row Operations. When manipulating a system of linear equations, we may apply any of the following operations:

1. Interchange any two equations.
2. Multiply (or divide) any equation by a nonzero number.
3. Change an equation by adding to it (or subtracting from it) a nonzero multiple of another equation.

Example 2. Solve the system of linear equations (2) by applying elementary row operations until the system is in diagonal form.

(*Solution*) First let's divide the top row by 2 to obtain

$$\begin{aligned} x + 2y &= 4 \\ 4x + 2y &= 10 \end{aligned}.$$

We can then subtract 4 times the first equation from the second equation:

$$\begin{aligned} x + 2y &= 4 \\ 0x - 6y &= -6 \end{aligned}$$

and then divide the second equation by -6:

$$\begin{aligned} x + 2y &= 4 \\ 0x + y &= 1 \end{aligned}.$$

Finally we subtract 2 times the second equation from the first equation:

$$\begin{aligned} x + 0y &= 2 \\ 0x + y &= 1 \end{aligned}.$$

From here we easily see that (2) has the same solution it did before: $(2, 1)$. $\Diamond$

The system of linear equations considered here is simple enough that we can apply row operations somewhat naïvely and still wind up with the diagonal form. We'll discuss shortly an algorithm for choosing row operations that will always lead us to the desired form (though this form won't always be diagonal).

1.3 Matrix Notation and the Reduced Row-Echelon Form

Before discussing this algorithm for choosing row operations, we'll introduce an extremely important bit of notation: the **matrix**. A system of m linear equations in n variables can always be written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_n + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}.$$

When working with such a system, it's often convenient to drop the variables $x_1, \ldots, x_n$ and consider the (**augmented**) **matrix of coefficients** of the system:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

The bar between the last two columns is to remind us of the location of the equals sign. Entries to the left of this bar are the coefficients of our system, and the column to the right is the collection of solutions by which we have augmented our system of equations. Each row of this matrix corresponds to one of the equations in our system, which helps to explain where elementary row operations get their name.

When solving the linear system (2) in Example 2, we applied row operations until the system was in diagonal form. The augmented matrix corresponding to the last linear system in that example is then

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right],$$

from which we read the solutions $x = 2$ and $y = 1$. But what about systems that don't have a unique solution? Such systems can't be transformed by elementary row operations into a system in diagonal form, so what form should we aim for with our row operations? We'll define now a form from which the solution(s) (or lack thereof) of a linear system can be easily identified.

Reduced Row-Echelon Form. A matrix is said to be in **reduced row-echelon form** (**rref**) if it satisfies the following conditions:

(i) If a row has any nonzero entries, then the first nonzero entry (from the left) is a 1, called the **leading 1** or the **pivot** in this row.

(ii) If a column contains a leading 1, then all the other entries in that column are 0.

(iii) If a row contains a leading 1, then each row above it contains a leading 1 that is further to the left.

Notice that if a matrix is in reduced row-echelon form and has zero rows, these rows are at the bottom of the matrix.

From the definition it isn't too difficult to determine whether or not a matrix is in reduced row-echelon form, but something that might be difficult is figuring out how this form is useful to us. We'll demonstrate with an example. Consider the following system of linear equations in four variables:

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 - x_4 &= 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 &= 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 &= -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 &= 11 \end{aligned}.$$

In this form, it's very difficult to quickly determine whether this system as a unique solution, infinitely many solutions, or no solutions at all. But we can apply elementary row operations to the augmented matrix corresponding to this system to obtain the following matrix, in reduced row-echelon form:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We can now reinterpret this matrix as a system of linear equations:

$$\begin{aligned} x_1 + x_4 &= 1 \\ x_2 - 3x_4 &= 2 \\ x_3 + 2x_4 &= -3 \end{aligned} \quad . \tag{4}$$

Notice that there is no condition on the variable x_4. This means that x_4 can take any value it likes, and as long as the remaining variables satisfy the equations in (4), we have a solution to the system. So our system must have infinitely many solutions. We see that the reduced row-echelon form allows us to determine the nature of the solution set to our system of equations, and what's more, we can give a simple characterization of this solution set:

$$x_4 = \text{any number}, \quad x_1 = 1 + x_4, \quad x_2 = 2 + 3x_4, \quad x_3 = -3 - 2x_4.$$

For instance, if $x_4 = 0$, then $x_1 = 1$, $x_2 = 2$, and $x_3 = -3$, so $(1, 2, -3, 0)$ is a solution of our system.

Example 3. Suppose the matrix corresponding to a system of linear equations has been reduced to the following:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k \end{array}\right],$$

where k is some real number. Describe the solution set of the original system.

(*Solution*) First, let's rewrite the matrix as a system of equations:

$$\begin{aligned} x_1 &= 3 \\ x_3 &= 4. \\ 0 &= k \end{aligned}$$

Certainly if $k \neq 0$, this system can't possibly be solved. So right away we see that if $k \neq 0$, the original system has no solutions. If we have $k = 0$, it can be tempting to say that our system has a unique solution in the form $x_1 = 3$ and $x_3 = 4$, but then what's the value of x_2? In fact, the system would have infinitely many solutions, with x_2 free to take any value, while $x_1 = 3$ and $x_3 = 4$. $\Diamond$

Note. If a system had a unique solution, then the reduced row-echelon form of its matrix is in diagonal form. That is, it has a 1 in each diagonal entry, and all other entries to the left of the bar are 0:

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & b_n \end{array}\right].$$

The unique solution to the system is then $(b_1, b_2, \ldots, b_n)$, and we call the matrix to the left of the bar the **$\mathbf{n \times n}$ identity matrix** (or just the **identity matrix** when the size is clear).

1.4 Gauss-Jordan Elimination

We're now ready to present our algorithm for transforming an augmented matrix into its reduced row-echelon form. The first step in developing this algorithm is to introduce the operation of **pivoting** about an entry of a matrix[1], which is a way to clear the associated variable out of the equations other than the one with which we're working.

> **Pivoting.** To pivot a matrix about a given nonzero entry, (1) transform the given entry into a 1 by multiplying its row by the appropriate number, and then (2) transform all other entries in the same column into zeros.

Note. We'll use different notation for elementary row operations than does the book, but hopefully this won't be cause for too much confusion.

Example 4. Pivot the following matrix about the circled entry:

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 3 & 5 & 14 \end{array}\right]$$

(*Solution*) First we transform the circled 2 into a 1, and then we "zero out" the 3 below:

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 3 & 5 & 14 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 5 & 14 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -1 & 2 \end{array}\right].$$

Notice that the first column of this matrix is what we would expect in a reduced row-echelon form, so this seems like a good first step in the process of finding such a form. $\Diamond$

We can now define the Gauss-Jordan elimination algorithm as a sequence of pivot operations, followed by the interchanging of some rows of our matrix.

[1] It should be noted that the textbook doesn't mention pivoting by name, though it does ask you to apply this operation.

The Gauss-Jordan Elimination Method. To transform a system of linear equations into its reduced row-echelon form,

(i) Write the augmented matrix corresponding to the linear system.

(ii) Working from top to bottom, pivot about the first nonzero entry of each row, provided this entry is to the left of the bar.

(iii) Interchange the rows of the matrix as needed, so that the leftmost pivot is in the top row, and so on.

Example 5. Find a solution to the system of linear equations given by $2x + 3y = 6$ and $3x + 2y = 6$ using the Gauss-Jordan elimination method.

(*Solution*) First we write down the matrix corresponding to this system:

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 3 & 2 & 6 \end{array}\right].$$

We then pivot about the first nonzero entry of the first row, which in this case is the first entry:

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 3 & 2 & 6 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 3/2 & 3 \\ 3 & 2 & 6 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{cc|c} 1 & 3/2 & 3 \\ 0 & -5/2 & -3 \end{array}\right].$$

Next, we pivot about the first nonzero entry of the second row:

$$\left[\begin{array}{cc|c} 1 & 3/2 & 3 \\ 0 & -5/2 & -3 \end{array}\right] \xrightarrow{-\frac{2}{5}R_2} \left[\begin{array}{cc|c} 1 & 3/2 & 3 \\ 0 & 1 & 6/5 \end{array}\right] \xrightarrow{R_1-\frac{3}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 6/5 \\ 0 & 1 & 6/5 \end{array}\right].$$

From here we see that $(6/5, 6/5)$ is the unique solution to our system. $\Diamond$

Example 6. Use the Gauss-Jordan elimination method to find the reduced row-echelon form for the system of linear equations given by $2x + 3y = 6$ and $2x + 3y = 8$.

(*Solution*) This system is clearly inconsistent, but we'll apply our algorithm anyways. First, we pivot about the first nonzero entry of the first row:

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 2 & 3 & 8 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 3/2 & 3 \\ 2 & 3 & 8 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c} 1 & 3/2 & 3 \\ 0 & 0 & 2 \end{array}\right].$$

We then move to the second row, only to find that there's no nonzero entry about which we could pivot. (Remember that we only pivot about entries to the left of the bar.) Since the bottom row produces the equation $0 = 2$, we see that our system of equations must have no solutions. The system we started with was pretty obviously inconsistent; often, we'll start with an inconsistent system whose inconsistency is not obvious. The Gauss-Jordan method will bring the inconsistency to light. $\Diamond$

Example 7. Find the solution set of the system of linear equations given by

$$\begin{aligned} x - 7y + 6w &= 5 \\ z - 2w &= -3 \\ -x + 7y - 4z + 2w &= 7 \end{aligned}$$

(*Solution*) First we write the matrix corresponding to this system:

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right].$$

We can now pivot about the first entry of the first column:

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right].$$

The first nonzero entry of the second row is the 1 in the third column, so we pivot about this entry:

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right] \xrightarrow{R_3+4R_2} \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This matrix is in reduced row-echelon form, and has associated linear system

$$\begin{aligned} x - 7y + 6w &= 5 \\ z - 2w &= -3. \end{aligned} \tag{5}$$

Notice that the first and third columns (associated to x and z, respectively) have pivots, and we can solve (5) for x and z in terms of y and w. So we can let y and w be **free variables**, allowed to take on any real value, and as long as x and z satisfy their respective equations, we have a solution. That is, we may write the general solution to our system as

$$\begin{aligned} y &= \text{any real number} \\ w &= \text{any real number} \\ x &= 7y - 6w + 5 \\ z &= 2w - 3. \end{aligned}$$

For example, $(5, 0, -3, 0)$ is a solution to our system. $\Diamond$