ELIMINATING THURSTON OBSTRUCTIONS AND CONTROLLING DYNAMICS ON CURVES

MARIO BONK, MIKHAIL HLUSHCHANKA, AND ANNINA ISELI

Abstract. Every Thurston map \( f: S^2 \to S^2 \) on a 2-sphere \( S^2 \) induces a pull-back operation on Jordan curves \( \alpha \subset S^2 / \text{uni}_f \), where \( P_f \) is the postcritical set of \( f \). Here the isotopy class \( [f^{-1}(\alpha)] \) (relative to \( P_f \)) only depends on the class \([\alpha]\). We study this operation for Thurston maps with four postcritical points, where it is particularly simple. In this case a Thurston obstruction for the map \( f \) can be seen as a fixed point of the pull-back operation.

We show that if a Thurston map \( f \) with a hyperbolic orbifold and four postcritical points has a Thurston obstruction, then one can “blow up” suitable arcs in the underlying 2-sphere and construct a new Thurston map \( \tilde{f} \) for which this obstruction is eliminated. We prove that no other obstruction arises and so \( \tilde{f} \) is realized by a rational map. Our construction gives a large class of rational Thurston maps with four postcritical points.

We also study the dynamics of the pull-back operation under iteration. We exhibit a subclass of our rational Thurston maps with four postcritical points for which we can give positive answer to the global curve attractor problem.

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1. Introduction

In this paper, we consider Thurston maps and the dynamics of the induced pull-back operation on Jordan curves on the underlying 2-sphere. By definition, a Thurston map $f$ is a branched covering map $f: S^2 \to S^2$ on a topological 2-sphere $S^2$ such that $f$ is not a homeomorphism and every critical point of $f$ (points where $f$ is not a local homeomorphism) has a finite orbit under iteration of $f$. These maps are named after William Thurston who introduced them in his quest for a better understanding of the dynamics of postcritically-finite rational maps on the Riemann sphere. We refer to [BM17, Chapter 2] for general background on Thurston maps and related concepts.

For a branched covering map $f: S^2 \to S^2$ we denote by $C_f$ the set of critical points of $f$ and by $f^n$ the $n$-th iterate of $f$ for $n \in \mathbb{N}$. Then the postcritical set of $f$ is defined as

$$P_f = \bigcup_{n \in \mathbb{N}} \{ f^n(c) : c \in C_f \}.$$

For a Thurston map $f$ this set has finite cardinality $2 \leq \# P_f < \infty$ (for the first inequality see [BM17, Corollary 2.13]).

A Thurston map $f$ often admits a description in purely combinatorial-topological terms. In this context, it is an interesting question whether $f$ can be realized (in a suitable sense) by a rational map with the same combinatorics. Roughly speaking, this means that $f$ is conjugate to a rational map “up to isotopy” (see Section 3 for the precise definition).

It is not hard to see that each Thurston map with two or three postcritical points is realized. The situation is much more complicated for Thurston maps $f$ with $\# P_f \geq 4$. William Thurston found a necessary and sufficient condition when a Thurston map can be realized by a rational map [DH93]. Namely, if $f$ has an associated hyperbolic orbifold (this is always true apart from some well-understood exceptional maps), then $f$ is realized if and only if $f$ has no (Thurston) obstruction. Such an obstruction is given by a finite collection of disjoint Jordan curves in $S^2 \setminus P_f$ (up to isotopy) with certain invariance properties (see Section 3.2 for more discussion).

The “if part” of this statement gives a positive criterion for $f$ to be realized, but it is very hard to apply in practice, because, at least in principle, it involves the verification of infinitely many conditions for the map $f$. For this reason, in each individual case a successful
verification for a map, or a class of maps, is difficult and usually constitutes an interesting result in its own right.

We mention two results in this direction. The first one is the “arcs intersecting obstructions” theorem by Kevin Pilgrim and Tan Lei [PL98, Theorem 3.2] that gives control on the position of an obstruction and has many applications in holomorphic dynamics (see, for instance, [PL98, DMRS19]). The other one is the “mating criterion” by Tan Lei, Mary Rees, and Mitsuhiro Shishikura that addresses the question when two postcritically-finite quadratic polynomials can be topologically glued together to form a rational map (see [Lei92, Ree92, Shi00]).

The investigation of obstructions of a Thurston map \( f : S^2 \to S^2 \) is closely related to the study of the pull-back operation on Jordan curves. It is easy to show that if \( \alpha \subset S^2 / \text{unif}_P \) is a Jordan curve, then the isotopy class \([f^{-1}(\alpha)]\) (rel. \( P \)) only depends on the class \([\alpha]\) (see Lemma 3.3). In this paper, we focus on the simplest non-trivial case, namely Thurston maps \( f \) with \( \# P_f = 4 \). In this case the pull-back operation gives rise to a well-defined map, the slope map, on these isotopy classes \([\alpha]\) (we will discuss this in more detail below). The search for obstructions of \( f \) amounts to understanding the fixed points of the slope map.

There exist various natural constructions that allow one to combine or modify given (rational) Thurston maps to obtain a new dynamical system. The most studied constructions are mating (see [Lei92, SL00]), tuning (see [Ree92]), and capture (see [Hea88, Lei97]). In this paper, we study the operation of blowing up arcs, originally introduced by Kevin Pilgrim and Tan Lei in [PL98]. This operation can be applied to an arbitrary Thurston map \( f \) and results in a new Thurston map \( \widehat{f} \) that is of higher degree, but combinatorially closely related to the original map \( f \). In particular, \( f \) and \( \widehat{f} \) have the same set of postcritical points and the same dynamics on them. Nevertheless, the dynamical behavior of Jordan curves under the pull-back operation for the original map \( f \) and the new map \( \widehat{f} \) may differ drastically.

We show that, if a Thurston map \( f : S^2 \to S^2 \) with \( \# P_f = 4 \) has an obstruction \( \alpha \), then one can naturally modify \( f \) by blowing up certain arcs to produce a new Thurston map \( \widehat{f} \) for which this obstruction \( \alpha \) is eliminated. The main result of this paper is the fact that then no new obstructions arise for \( \widehat{f} \) and so it is realized by a rational map.

**Theorem 1.1.** Let \( f : S^2 \to S^2 \) be a Thurston map with \( \# P_f = 4 \) and a hyperbolic orbifold. Suppose that \( f \) has an obstruction represented by a Jordan curve \( \alpha \subset S^2 \setminus P_f \), and \( E \neq \emptyset \) is a finite set of arcs in \((S^2, f^{-1}(P_f))\) that satisfy the \( \alpha \)-restricted blow-up conditions.

Let \( \widehat{f} \) be a Thurston map obtained from \( f \) by blowing up arcs in \( E \) (with some multiplicities) so that \( \lambda_{\widehat{f}}(\alpha) < 1 \). Then \( \widehat{f} \) is realized by a rational map.

The technical verbiage and the notation in this formulation will be explained in subsequent sections (see in particular (3.3) for the definition of the “eigenvalue” \( \lambda_f(\alpha) \) and Definition 6.6 for \( \alpha \)-restricted blow-up conditions).

Recently, Dylan Thurston provided a positive characterization when a Thurston map is realized, at least in the case when each critical point eventually lands in a critical cycle under iteration. He proved that such a Thurston map \( f \) is realized by a rational map if and only if there is an “elastic spine” (that is, a planar embedded graph in \( S^2 \setminus P_f \) with a suitable metric on it) that gets “looser” under backwards iteration (see [Thu16, Thu20] for more details). In concrete cases, especially for Thurston maps that should be realized by rational maps with Julia sets homeomorphic to the Sierpiński carpet, the application of Dylan Thurston’s criterion is not so straightforward. Moreover, his criterion is only valid
for Thurston maps with periodic critical points. In contrast, for some maps for which Dylan Thurston’s criterion is not applicable or hard to apply, Theorem 1.1 can be used to verify that the maps are realized. In particular, many maps obtained by blowing up Lattès maps (see below) are of this type.

1.1. Blowing up Lattès maps. We will now discuss a special case of Theorem 1.1 in detail to give the reader some intuition for the geometric ideas behind this statement and its proof.

Let $\mathbb{P}$ be a pillow obtained from two copies of the unit square $[0, 1]^2 \subset \mathbb{R}^2 \cong \mathbb{C}$ glued together along their boundaries. We consider the two copies of $[0, 1]^2$ in $\mathbb{P}$ as the front and back side of $\mathbb{P}$ and call them the tiles of level 0 or simply 0-tiles. We denote by $A := (0, 0) \in \mathbb{P}$ the lower left corner of $\mathbb{P}$ (see the right part of Figure 1). The pillow $\mathbb{P}$ is a topological 2-sphere. Actually, if we consider $\mathbb{P}$ as an abstract polyhedral surface, then $\mathbb{P}$ carries a conformal structure making $\mathbb{P}$ conformally equivalent to the Riemann sphere $\hat{\mathbb{C}}$. See Section 2.4 for more discussion.

![Figure 1. The $(4 \times 4)$-Lattès map.](image)

We now fix $n \in \mathbb{N}$ with $n \geq 2$. We subdivide each of the two 0-tiles of $\mathbb{P}$ into $n^2$ small squares of sidelength $1/n$, called the 1-tiles. We color these 1-tiles in a checkerboard fashion black and white so that the 1-tile in the front 0-tile that contains the vertex $A$ on its boundary is colored white (see the left part of Figure 1). We map this 1-tile to the front 0-tile of the pillow by an orientation-preserving Euclidean similarity that fixes the vertex $A$. This similarity scales distances by the factor $n$. We can uniquely extend the similarity by a successive Schwarz reflection process to the whole pillow $\mathbb{P}$ to obtain a continuous map $L_n: \mathbb{P} \to \mathbb{P}$. Then on each 1-tile $S$ the map $L_n$ is a Euclidean similarity that sends $S$ to the front or back 0-tile of $\mathbb{P}$ depending on whether $S$ is white or black. We call $L_n$ the $(n \times n)$-Lattès map, because under a suitable conformal equivalence $\mathbb{P} \cong \hat{\mathbb{C}}$, the map $L_n$ is conjugate to a rational map obtained from $n$-multiplication of a Weierstrass $\wp$-function. See Figure 1 for an illustration of the map $L_4$. Here, the marked points on the left pillow $\mathbb{P}$ (the domain of the map) correspond to the preimage points $L_4^{-1}(A)$. Note that there is exactly one preimage of $A$ in the interior of the back side of the pillow.

It is easy to see that the $(n \times n)$-Lattès map $L_n: \mathbb{P} \to \mathbb{P}$ is a Thurston map with four postcritical points, namely, the four corners of the pillow $\mathbb{P}$. The map $L_n$ is realized by a rational map, because it is even conjugate to such a map.

We now modify the map $L_n$ by gluing in vertical or horizontal flaps to $\mathbb{P}$. This is a special case of the more general construction of blowing up arcs mentioned above. We will describe
this in detail in Section 4, but will illustrate the procedure in Figure 2, where we show how to glue in one horizontal flap.

We cut the pillow \( \mathbb{P} \) open along a horizontal side \( e \) of one of the 1-tiles. Note that in this process \( e \) is “doubled” into two arcs \( e' \) and \( e'' \) with common endpoints. We then take two disjoint copies of the Euclidean square \([0, 1/n]^2\) and identify them along three corresponding sides to obtain a flap \( F \). It has two “free” sides on its boundary. We glue each free side to one of the arcs \( e' \) and \( e'' \) of the cut in the obvious way.

In general, one can repeat this construction and glue several flaps at the location given by the arc \( e \). We assume that this has been done simultaneously for \( n_h \geq 0 \) flaps along horizontal edges and \( n_v \geq 0 \) flaps along vertical edges. By this procedure we obtain a “flapped” pillow \( \hat{\mathbb{P}} \), which is still a topological 2-sphere (see the left part of Figure 3). By construction it is tiled by \( 2n^2 + 2(n_h + n_v) \) squares of sidelength \( 1/n \), which we consider as the 1-tiles of \( \hat{\mathbb{P}} \). The checkerboard coloring of the base surface \( \mathbb{P} \) extends in a unique way to the new surface \( \hat{\mathbb{P}} \).

The original \((n \times n)\)-Lattès map \( \mathcal{L}_n: \mathbb{P} \to \mathbb{P} \) can naturally be “extended” to a continuous map \( \hat{\mathcal{L}}: \hat{\mathbb{P}} \to \mathbb{P} \) so that each 1-tile \( S \) of \( \hat{\mathbb{P}} \) is mapped to the front or back 0-tile of \( \mathbb{P} \) (depending on the color of \( S \)) by a Euclidean similarity scaling distances by the factor \( n \). See Figure 3 for an illustration of a map \( \hat{\mathcal{L}} \) obtained from the Lattès map \( \mathcal{L}_4 \) by gluing in flaps at a vertical and a horizontal edge. Similarly as in Figure 1, on the left we marked the preimages of \( A \) under \( \hat{\mathcal{L}} \).

In order to obtain a Thurston map \( f: \mathbb{P} \to \mathbb{P} \) from this construction, we need to choose a homeomorphism \( \phi: \hat{\mathbb{P}} \to \mathbb{P} \) that identifies \( \hat{\mathbb{P}} \) with \( \mathbb{P} \) and satisfies suitable conditions. The
precise choice of \( \phi \) is somewhat technical and so we refer to Section 4.2 for the details. Then \( f := \widehat{L} \circ \phi^{-1} \mathbb{P} \rightarrow \mathbb{P} \) is a Thurston map with a postcritical set \( P_f \) consisting of the four corners of the pillow \( \mathbb{P} \). The map \( f \) is uniquely determined up to Thurston equivalence (see Definition 3.2) independently of the choice of \( \phi \) under suitable restrictions. We refer to \( f \) as a Thurston map \textit{obtained from the \((n \times n)\)-Lattès map by gluing \( n_h \) horizontal and \( n_v \) vertical flaps to \( \mathbb{P} \).}

Now the following statement is true. As we will explain, it can be seen as a special case of our main result.

**Theorem 1.2.** Let \( n \in \mathbb{N} \) with \( n \geq 2 \) and \( f : \mathbb{P} \rightarrow \mathbb{P} \) be a Thurston map \textit{obtained from the \((n \times n)\)-Lattès map \( \mathcal{L}_n \) by gluing \( n_h \geq 0 \) horizontal and \( n_v \geq 0 \) vertical flaps to \( \mathbb{P} \), where \( n_h + n_v > 0 \). Then the map \( f \) has a hyperbolic orbifold. It has an obstruction if and only if \( n_h = 0 \) or \( n_v = 0 \). In particular, if \( n_h > 0 \) and \( n_v > 0 \), then \( f \) is realized by a rational map.

If \( n_h = n_v = 0 \), then no flaps were glued to \( \mathbb{P} \) and the map \( f \) coincides with the original \((n \times n)\)-Lattès map \( \mathcal{L}_n \) (strictly speaking, only if we choose the homeomorphism \( \phi \) used in the construction above to be the identity on \( \mathbb{P} \), as we may). Then \( f = \mathcal{L}_n \) has a parabolic orbifold. Therefore, Thurston’s criterion as formulated in Section 3.2 does not apply.

If \( n_h = 0 \) or \( n_v = 0 \), but \( n_h + n_v > 0 \) as in Theorem 1.2, then it is immediate to see that \( f \) has an obstruction (see Section 5.1) and therefore \( f \) cannot be realized by a rational map. So the interesting part of Theorem 1.2 is the claim that if \( n_h > 0 \) and \( n_v > 0 \), then \( f \) has no obstruction.

Even though Theorem 1.2 follows from our more general statement formulated in Theorem 1.1, we will give a complete proof. We will argue by contradiction and assume that a map \( f \) with \( n_h > 0 \) and \( n_v > 0 \) has an obstruction. In principle, there are infinitely many candidates represented by essential isotopy classes of Jordan curves \( \alpha \subset \mathbb{P} \setminus P_f \). These isotopy classes in turn are distinguished by different rational slopes in \( \mathcal{Q} = \mathbb{Q} \cup \{ \infty \} \) (as will be explained in Section 2.5). For such an isotopy class represented by \( \alpha \) to be an obstruction, it has to be \( f \)-invariant in the sense that \( f^{-1}(\alpha) \) should contain a component \( \overline{\alpha} \) isotopic to \( \alpha \) rel. \( P_f \). It seems to be a very intricate problem to find all slopes in \( \mathcal{Q} \) that give an invariant isotopy class for \( \alpha \). Since we have been able to decide this question only for very simple maps \( f \), we proceed in a more indirect manner.

We assume that the Jordan curve \( \alpha \subset \mathbb{P} \setminus P_f \) is \( f \)-invariant and gives an obstruction. We then investigate the mapping degrees of \( f \) on components of \( f^{-1}(\alpha) \) and consider intersection numbers of some relevant curves together with a careful counting argument. We heavily use the fact that the horizontal and vertical Jordan curves (see (2.5)) are \( f \)-invariant. Ultimately, we arrive at a contradiction. See Section 5 for the details of this argument.

Our idea to use intersection numbers (as in Lemma 5.5) to control possible locations of obstructions and dynamics on curves is not new (see, for example, [PL98, Theorem 3.2], [CPL16, Section 8], and [Par18]). However, the previously available results do not provide sharp enough estimates applicable in our situation.

One can think of Theorem 1.2 in the following way. Suppose that instead of directly passing from the Lattès map \( \mathcal{L}_n \) to a map, let us now call it \( \widehat{f} \), obtained by gluing \( n_h > 0 \) horizontal and \( n_v > 0 \) vertical flaps to \( \mathbb{P} \), we first create an intermediate map \( f \) obtained by gluing \( n_h > 0 \) horizontal, but no vertical flaps. Then \( f \) has a hyperbolic orbifold and an obstruction given by a “horizontal” Jordan curve \( \alpha \). In the passage from \( f \) to \( \widehat{f} \) we kill this obstruction, because we glue additional vertical flaps that serve as obstacles and increase the
mapping degree on some pullbacks of $\alpha$. Theorem 1.1 then says that no other obstructions arise for $\tilde{f}$. Therefore, Theorem 1.1 generalizes Theorem 1.2 if we interpret it in the way just described. The proof of Theorem 1.1 is based on a refinement of the ideas that we use to establish Theorem 1.2.

1.2. The global curve attractor problem. The mapping properties of Jordan curves play an important role in Thurston’s characterization of rational maps. The original proof of this statement associates with a given Thurston map $f: S^2 \to S^2$ with a hyperbolic orbifold a certain Teichmüller space $\mathcal{T}_f$ and a real-analytic map $\sigma_f: \mathcal{T}_f \to \mathcal{T}_f$, called Thurston’s pullback map. One can show that the map $f$ is realized by a rational map if and only if $\sigma_f$ has a fixed point [DH93]. This reduction to a fixed point problem in a Teichmüller space has also been successfully applied by Thurston in the other contexts such as uniformization problems and the theory of 3-manifolds (there is a rich literature on the subject; see, for example, [Thu88, Thu98, Ota01, FLP12, Hub16]).

In recent years, the pullback map $\sigma_f$ and its dynamical properties have been subject to deeper investigations (see, for example, [Pil12, Sel12, Lod13, KPS16]). In particular, Nikita Selinger showed in [Sel12] that $\sigma_f$ extends to the Weil-Petersson boundary of $\mathcal{T}_f$. The behavior of $\sigma_f$ on this boundary is closely related to the behavior of Jordan curves under pull-back by $f$. This in turn leads to the following difficult open question in holomorphic dynamics, called the global curve attractor problem (see [Lod13, Section 9]).

**Conjecture.** Let $f: S^2 \to S^2$ be a Thurston map with a hyperbolic orbifold that is realized by a rational map. Then there exists a finite set $\mathcal{A}(f)$ of Jordan curves in $S^2 \setminus P_f$ such that for every Jordan curve $\gamma \subset S^2 \setminus P_f$ and all sufficiently large $n \in \mathbb{N}$ all pullbacks $\tilde{\gamma}$ of $\gamma$ under $f^n$ are contained in $\mathcal{A}(f)$ up to isotopy rel. $P_f$.

A set of Jordan curves $\mathcal{A}(f)$ as in this conjecture is called a global curve attractor of $f$.

We will give a solution of this problem for maps as in Theorem 1.2 with $n = 2$ and $n_h, n_v \geq 1$. Unfortunately, our methods only apply for $n = 2$ and not for $n \geq 3$.

**Theorem 1.3.** Let $f: P \to P$ be a Thurston map obtained from the $(2 \times 2)$-Lattès map by gluing $n_h \geq 1$ horizontal and $n_v \geq 1$ vertical flaps to the pillow $P$. Then $f$ has a global curve attractor $\mathcal{A}(f)$.

One can show that the Julia set of a rational map $f$ as provided by Theorem 1.2 is either a Sierpiński carpet or the whole Riemann sphere depending on whether $f$ has periodic critical points or not (see Proposition 9.1). Accordingly, Theorem 1.3 provides the first examples of maps with Sierpiński carpet Julia set for which an answer to the global curve attractor problem is known. In fact, we obtain such maps with arbitrary large degrees.

Recently, Belk-Lanier-Margalit-Winarski proved the existence of a finite global curve attractor for all postcritically-finite polynomials [BLMW19]. The conjecture is also known to be true for all critically fixed rational maps (that is, rational maps for which each critical point is fixed) and some nearly Euclidean Thurston maps (that is, Thurston maps with exactly four postcritical points and only simple critical points); see Husn19 and Lod13. In KPS17 Gregory Kelsey and Russell Lodge verified the conjecture for all quadratic non-Lattès maps with four postcritical points. However, for general postcritically-finite rational maps the conjecture remains wide open.

Since the maps we consider have four postcritical points, it is convenient to reformulate the global curve attractor problem by introducing the slope map (it is closely related to the
Thurston pull-back map $\sigma_f$ on the Weil-Petersson boundary of $T_f$). To define it in the special case relevant for us, we consider the marked pillow $(P, V)$, where $V$ is the set consisting of the four corners of $P$, and assume that $f: P \rightarrow P$ is a Thurston map with $P_f = V$. Up to topological conjugacy, every Thurston map with four postcritical points can be assumed to have this form.

As we already mentioned, there is a bijective correspondence between isotopy classes $[\alpha]$ of essential Jordan curves $\alpha$ in $(P, V)$ and slopes $r/s \in \mathcal{Q}$ (see Lemma 2.2). We introduce the additional symbol $\odot$ to represent peripheral Jordan curves in $(P, V)$. We now define the slope map $\mu_f: \mathcal{Q} \cup \{\odot\} \rightarrow \mathcal{Q} \cup \{\odot\}$ associated with $f$ as follows. We set $\mu_f(\odot) := \odot$. This corresponds to the fact that each pullback of a peripheral Jordan curve $\alpha$ in $(P, V)$ under $f$ is peripheral (see Lemma 3.3). If $r/s \in \mathcal{Q}$ is an arbitrary slope, then we choose a Jordan curve $\alpha$ in $(P, V)$ whose isotopy class $[\alpha]$ is represented by $r/s$. If all pullbacks of $\alpha$ under $f$ are peripheral, we set $\mu_f(r/s) := \odot$. Otherwise, there exists an essential pullback $\tilde{\alpha}$ of $\alpha$ under $f$. Then the isotopy class $[\tilde{\alpha}]$ is independent of the choice of the essential pullback $\alpha$ (see Lemma 3.3) and so it is represented by a unique slope $r'/s' \in \mathcal{Q}$. In this case, we set $\mu_f(r/s) := r'/s'$. In this way, $\mu_f(x) \in \mathcal{Q} \cup \{\odot\}$ is defined for all $x \in \mathcal{Q} \cup \{\odot\}$. Since the map $\mu_f$ has the same source and target, we can iterate it. If $n \in \mathbb{N}_0$, then we denote by $\mu_f^n$ the $n$-th iterate of $\mu_f$. We will then prove the following statement.

**Theorem 1.4.** Let $f: P \rightarrow P$ be a Thurston map obtained from the $(2 \times 2)$-Lattès map by gluing $n_h \geq 1$ horizontal and $n_v \geq 1$ vertical flaps to the pillow $P$. Then there exists a finite set $S \subset \mathcal{Q} \cup \{\odot\}$ with the following property: for each $x \in \mathcal{Q} \cup \{\odot\}$ there exists $N \in \mathbb{N}_0$ such that $\mu_f^n(x) \in S$ for all $n \geq N$.

Note that $P_f = V$ in this case; so our previous considerations apply and the map $\mu_f$ is defined. It is clear that the previous theorem leads to the solution of the global curve attractor problem for the maps $f$ considered:

**Proof of Theorem 1.3 based on Theorem 1.4.** To obtain a finite attractor $\mathcal{A}(f)$, pick a Jordan curve in each isotopy class represented by a slope in $S$ and add five Jordan curves that represent the isotopy classes of peripheral Jordan curves in $(P, V)$ (one for null-homotopic curves and one for each corner of $P$).

For the proof of Theorem 1.4 we will establish a certain monotonicity property of the slope map $\mu_f$ for a map $f$ as in the statement (see Proposition 8.1). Roughly speaking, this monotonicity means that up to isotopy rel. $P_f = V$ complicated essential Jordan curves in $(P, V)$ get “simpler” and “less twisted” if we take successive preimages under $f$ and eventually end up in the global curve attractor.

Our methods again rely on the consideration of intersection numbers. The algebraic methods for solving the global curve attractor problem developed in [Pil12] (specifically, [Pil12, Theorem 1.4]) do not apply in general for the maps considered in Theorem 1.3 (see the discussion in Section 9.3).

Some of our ideas can also be used for the study of the global dynamics of the slope map for Thurston maps that are not covered by Theorem 1.4. In particular, we are able to describe the iterative behavior of $\mu_f$ for a specific obstructed Thurston map $f$ obtained by blowing up the $(2 \times 2)$-Lattès map (see Section 9.2 for the details). This provides an answer to a question by Kevin Pilgrim (see [Pil18, Question 4.4]).
While it is straightforward to compute $\mu_f(x)$ for individual values $x \in \hat{Q} \cup \{\infty\}$, we have been unable to give an explicit formula for $\mu_f$ for the maps $f$ we consider. In general, these slope maps show very complicated behavior. Currently, very few explicit computations of slope maps are known in the literature. Except for some very special situations (for example, when the slope map is constant, that is, when $\mu_f(x) = \infty$ for all $x \in \hat{Q} \cup \{\infty\}$), we are only aware of computations of slope maps for nearly Euclidian Thurston maps in [CFPP12, Section 5] and [Lod13, Section 6]. See also [FPP18] for some general properties of the slope map $\mu_f$.

An undergraduate student at UCLA, Darragh Glynn, performed some computer experiments to compute $\mu_f$ for maps $f$ as in Theorem 1.2 for $n \geq 3$ (and $n_h, n_v \geq 1$ corresponding to the rational case). His results show that in these cases the map $\mu_f$ does not have the monotonicity property as for $n = 2$, but indicate that these maps $f$ still have a global curve attractor (see Section 9.2 for more discussion).

1.3. Organization of this paper. Our paper is organized as follows. In the next two sections we will discuss some background. Namely, after fixing notation and stating some basic definitions, in Section 2 we discuss isotopy classes of Jordan curves on a sphere with four marked points, how isotopy classes of such curves correspond to slopes in $\hat{Q}$, and some relevant facts about intersection numbers. Even though all of this is fairly standard, we provided some of the proofs, because it is hard to track down this material in the literature with a detailed exposition.

In Section 3 we review some basics about Thurston maps and the relevant concepts for a precise formulation of Thurston’s characterization of rational maps for Thurston maps with four postcritical points—the only case relevant for us (see Section 3.2).

We explain the blow-up procedure for arcs in Section 4 and relate this to the procedure of gluing flaps to the pillow $P$ (see Section 4.2). The proof of Theorem 1.2 is then given in Section 5.

The proof of our main result, Theorem 1.1, requires more preparation. This is the purpose of Section 6. There we introduce the concept of essential circuit length that will allow us to formulate tight estimates for the number of essential pullbacks of a Jordan curve under a Thurston map with four postcritical points. This is formulated in the rather technical Lemma 6.2 which is of crucial importance though. The proof of Theorem 1.1 is then given in Section 7.

Section 8 is devoted to the proof of Theorem 1.4. In the final section, Section 9, we discuss some further directions related to this work.

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2. Preliminaries

In this section, we discuss background relevant for the rest of the paper.

2.1. Notation and basic concepts. We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers and by $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ the set of natural numbers including 0. The sets of integers, real numbers, and complex numbers are denoted by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. We write $i$ for the imaginary unit in $\mathbb{C}$.
As usual, $\mathbb{R}^2 := \{ (x, y) : x, y \in \mathbb{R} \}$ is the Euclidean plane and $\widehat{\mathbb{C}} := \mathbb{C} \cup \{ \infty \}$ is the Riemann sphere. Here and elsewhere, we write $A := B$ for emphasis when an object $A$ is defined to be another object $B$. When we consider two objects $A$ and $B$, and there is a natural identification between them that is clear from the context, we write $A \cong B$. For example, $\mathbb{R}^2 \cong \mathbb{C}$ if we identify a point $(x, y) \in \mathbb{R}^2$ with $x + iy \in \mathbb{C}$. We will freely switch back and forth between these different viewpoints of $\mathbb{R}^2 \cong \mathbb{C}$.

We use the notation $I := [0, 1] \subset \mathbb{R}$ for the closed unit interval, $D := \{ z \in \mathbb{C} : |z| < 1 \}$ for the open unit disk in $\mathbb{C}$, and $\mathbb{Z}^2 := \{ x + iy : x, y \in \mathbb{Z} \}$ for the square lattice in $\mathbb{C}$. If $z, w \in \mathbb{C}$, then we write $[z, w] := \{ z + t(w - z) : t \in I \}$ for the line segment in $\mathbb{C}$ joining $z$ and $w$.

The cardinality of a set $X$ is denoted by $|X| \in \mathbb{N}_0 \cup \{ \infty \}$ and the identity map on $X$ by $\text{id}_X$. If $X$ is a topological space and $M \subset X$, then $\text{cl}(M)$ denotes the closure, $\text{int}(M)$ the interior, and $\partial M$ the boundary of $M \subset X$.

Let $f : X \to Y$ be a map between sets $X$ and $Y$. If $M \subset X$, then $f|_M$ stands for the restriction of $f$ to $M$. If $N \subset Y$, then $f^{-1}(N) := \{ x \in X : f(x) \in N \}$ is the preimage of $N$ in $X$. Similarly, $f^{-1}(y) := \{ x \in X : f(x) = y \}$ is the preimage of a point $y \in Y$.

A Jordan curve $\alpha$ in a surface $S$ is the image $\alpha = \iota(\partial D)$ of a (topological) embedding $\iota : \partial D \to S$ of the unit circle $\partial D = \{ z \in \mathbb{C} : |z| = 1 \}$ into $S$. An arc $e$ in $S$ is the image $e = \iota(I)$ of an embedding $\iota : I \to S$. Then $\iota(0)$ and $\iota(1)$ are the endpoints of $e$, and we define $\partial e := \{ \iota(0), \iota(1) \}$. The set $\text{int}(e) := \epsilon \cap \partial e$ is called the interior of $e$. The notions of endpoints and interior of $e$ only depend on $e$ and not on the choice of the embedding $\iota$. Note that the notation $\partial e$ and $\text{int}(e)$ is ambiguous, because it should not be confused with the boundary and interior of $e$ as a subset of $S$. For arcs $e$ in a surface $S$, we will only use $\partial e$ and $\text{int}(e)$ with the meaning just defined.

A path $\gamma$ in $S$ is a continuous map $\gamma : [a, b] \to S$, where $[a, b] \subset \mathbb{R}$ is a compact (non-degenerate) interval. As is common, we will use the same notation $\gamma$ for the image $\gamma([a, b])$ of the path if no confusion can arise. A loop in $S$ based at $p \in S$ is a path $\gamma : [a, b] \to S$ such that $\gamma(a) = \gamma(b) = p$. The loop $\gamma$ is called simple if $\gamma$ is injective on $[a, b] := [a, b] \setminus \{ b \}$. So essentially a simple loop is a Jordan curve run through with some parametrization.

Let $Z \subset S$ be a finite set of points in a surface $S$. Then we refer to the pair $(S, Z)$ as a marked surface, and the points in $Z$ as the marked points in $S$. The most important case for us will be when $S = S^2$ is a 2-sphere and $Z \subset S^2$ consists of four points.

A Jordan curve $\alpha$ in a marked 2-sphere $(S^2, Z)$ is a Jordan curve $\alpha \subset S^2 \setminus Z$. An arc $e$ in $(S^2, Z)$ is an arc $e \subset S^2$ with $\partial e \subset Z$ and $\text{int}(e) \subset S^2 \setminus Z$. We say that a Jordan curve $\alpha$ in $(S^2, Z)$ is essential if each of the two connected components of $S^2 \setminus \alpha$ contains at least two points of $Z$; otherwise, we say that $\alpha$ is peripheral.
Let \((S^2, Z)\) be a marked sphere with \#Z = 4. A core arc of an essential Jordan curve \(\alpha\) in \((S^2, Z)\) is an arc in \((S^2, Z)\) that is contained in one of the two connected component of \(S^2 \setminus \alpha\) and joins the two points in \(Z\) that lie in this component.

2.2. Branched covering maps. Let \(X\) and \(Y\) be surfaces. Then a continuous map \(f: X \to Y\) is called a branched covering map if for each point \(q \in Y\) there exists an open set \(V \subset Y\) homeomorphic to \(\mathbb{D}\) with \(q \in V\) that is evenly covered in the following sense: for some index set \(J \neq \emptyset\) we can write \(f^{-1}(V)\) as a disjoint union

\[
(2.1) \quad f^{-1}(V) = \bigcup_{j \in J} U_j
\]

of open sets \(U_j \subset X\) such that \(U_j\) contains precisely one point \(p_j \in f^{-1}(q)\). Moreover, we require that for each \(j \in J\) there exists \(d_j \in \mathbb{N}\) and orientation-preserving homeomorphisms \(\varphi_j: U_j \to \mathbb{D}\) with \(\varphi_j(p_j) = 0\) and \(\psi_j: V \to \mathbb{D}\) with \(\psi_j(q) = 0\) such that

\[
(\psi_j \circ f \circ \varphi_j^{-1})(z) = z^{d_j}
\]

for all \(z \in \mathbb{D}\) (see Section [BM17 Section A.6] for more background on branched covering maps). For given \(f\), the number \(d_j\) is uniquely determined by \(p = p_j\), and called the local degree of \(f\) at \(p\) and denoted by \(\deg(f, p)\). A point \(p \in X\) with \(\deg(f, p) \geq 2\) is called a critical point of \(f\). The set of all critical points of \(f\) is a discrete set in \(X\) and denoted by \(C_f\). If \(f\) is a branched covering map, then it is a covering map (in the usual sense) from \(X \setminus f^{-1}(f(C_f))\) onto \(Y \setminus f(C_f)\).

In the following, suppose \(X\) and \(Y\) are compact surfaces, and \(f: X \to Y\) is a branched covering map. Then \(C_f \subset X\) is a finite set. Moreover, if \(\deg(f) \in \mathbb{N}\) denotes the topological degree of \(f\), then

\[
\sum_{p \in f^{-1}(q)} \deg(f, p) = \deg(f)
\]

for each \(q \in Y\).

If \(\gamma: [a, b] \to Y\) is a path, then we call a path \(\gamma: [a, b] \to X\) a lift of \(\gamma\) (under \(f\)) if \(f \circ \gamma = \gamma\). Every path \(\gamma\) in \(Y\) has a lift \(\gamma\) in \(X\) (see [BM17 Lemma A.18]), but in general \(\gamma\) is not unique. If \(\gamma([a, b]) \subset Y \setminus f(C_f)\) and \(x_0 \in f^{-1}(\gamma(a))\), then there exists a unique lift \(\tilde{\gamma}: [a, b] \to X\) of \(\gamma\) under \(f\) with \(\tilde{\gamma}(a) = x_0\). This easily follows from standard existence and uniqueness theorems for lifts under covering maps (see [BM17 Lemma A.6]).

If \(e \subset Y\) is an arc, then an arc \(\gamma \subset X\) is called a lift of \(e\) (under \(f\)) if \(f \circ \gamma\) is a homeomorphism of \(\gamma\) onto \(e\). It easily follows from the existence and uniqueness statements for lifts of paths just discussed that if \(e\) is an arc in \((Y, f(C_f)), y_0 \in \text{int}(e), \) and \(x_0 \in f^{-1}(y_0)\), then there exists a unique lift \(\tilde{\gamma} \subset X\) of \(e\) with \(x_0 \in \tilde{\gamma}\).

Let \(V \subset Y\) be an open and connected set and \(U \subset f^{-1}(V)\) be a (connected) component of \(f^{-1}(V)\). Then \(f|U: U \to V\) is also a branched covering map. Each point \(q \in V\) has the same number \(d \in \mathbb{N}\) of preimages counting local degrees. We set \(\deg(f|U) := d\). If the Euler characteristic \(\chi(V)\) is finite, then \(\chi(U)\) is also finite and we have the Riemann-Hurwitz formula

\[
(2.2) \quad \chi(U) + \sum_{p \in U \cap C_f} (\deg(f, p) - 1) = \deg(f|U) \cdot \chi(V).
\]
2.3. Planar embedded graphs. A planar embedded graph in a sphere $S^2$ is a pair $G = (V, E)$, where $V$ is a finite set of points in $S^2$ and $E$ is a finite set of arcs in $(S^2, V)$ with pairwise disjoint interiors. The sets $V$ and $E$ are called the vertex and edge sets of $G$, respectively. Note that our notion of a planar embedded graph does not allow loops, that is, edges that connect a vertex to itself, but it does allow multiple edges, that is, distinct edges that join the same pair of vertices. The degree of a vertex $v$ in $G$, denoted $\deg_G(v)$, is the number of edges of $G$ incident to $v$. Note that $2 \cdot \#E = \sum_{v \in V} \deg_G(v)$.

The realization of $G$ is the subset $\mathcal{G}$ of $S^2$ given by

$$\mathcal{G} := V \cup \bigcup_{e \in E} e.$$ 

A face of $G$ is a connected component of $S^2 \setminus \mathcal{G}$. Usually, we conflate a planar embedded graph $G$ with its realization $\mathcal{G}$. Then it is understood that $\mathcal{G}$ contains a finite set $V \subset \mathcal{G}$ of distinguished points that are the vertices of the graph. Its edges are the closures of the components of $\mathcal{G} \setminus V$.

A subgraph of a planar embedded graph $G = (V, E)$ is a planar embedded graph $G' = (V', E')$ with $V' \subset V$ and $E' \subset E$. A path of length $n$ between vertices $v$ and $v'$ in $G$ is a sequence $v_0, e_0, v_1, e_1, \ldots, e_{n-1}, v_n$, where $v_0 = v$, $v_n = v'$ and $e_k$ is an edge incident to the vertices $v_k$ and $v_{k+1}$ for $k = 0, \ldots, n-1$. A path that does not repeat vertices is called a simple path.

A path $v_0, e_0, v_1, e_1, \ldots, v_{n-1}, v_n$ with $v_0 = v_n$ and $n \geq 1$ is called a circuit of length $n$ in $G$ and is denoted by $(e_0, e_1, \ldots, e_{n-1})$. Such a circuit is called a simple cycle if all vertices $v_k$, $k = 0, \ldots, n-1$, are distinct.

A planar embedded graph $G$ is called connected if any two distinct vertices of $G$ can be joined by a path in $G$. Equivalently, $G$ is connected if its realization $\mathcal{G}$ is connected as a subset of $S^2$. Note that if $G$ is connected, then each face of $G$ is simply-connected.

As follows from [Die05, Lemma 4.2.2], the topological boundary $\partial U$ of each face $U$ of $G$ may be viewed as the realization of a subgraph of $G$. Moreover, a walk around a connected component of the boundary $\partial U$ traces a circuit $(e_0, e_1, \ldots, e_{n-1})$ in $G$ such that each edge of $G$ appears zero, one, or two times in the sequence $e_0, e_1, \ldots, e_{n-1}$. We will say that the circuit $(e_0, e_1, \ldots, e_{n-1})$ bounds the face $U$ or traces (a connected component of) the boundary $\partial U$. If $U$ is simply-connected, then $\partial U$ is connected, and the length of the (essentially unique) circuit that bounds $U$ is called the circuit length of $U$ in $G$.

A planar embedded graph $(V, E)$ is called bipartite if we can split $V$ into two disjoint subsets $V_1$ and $V_2$ such that each edge $e \in E$ has one endpoint in $V_1$ and one in $V_2$.

2.4. The Euclidean square pillow. As discussed in the introduction, we consider a square pillow $\mathbb{P}$ obtained from gluing two identical copies of the unit square $\mathbb{I}^2 \subset \mathbb{R}^2$ along their boundaries by the identity map. Then $\mathbb{P}$ is a topological 2-sphere. We equip $\mathbb{P}$ with the induced path metric that agrees with the Euclidean metric on each of the two copies of the unit square. We call this metric space $\mathbb{P}$ the Euclidean square pillow. The vertices and edges of the unit square $\mathbb{I}^2$ in $\mathbb{P}$ are called the vertices and edges of $\mathbb{P}$. One copy of $\mathbb{I}^2$ in $\mathbb{P}$ is called the front and the other copy the back side of $\mathbb{P}$. In a dynamical context, we also refer to these two copies of $\mathbb{I}^2$ as the 0-tiles of $\mathbb{P}$.

We label the vertices and edges of $\mathbb{P}$ in counterclockwise order by $A, B, C, D$ and $a, b, c, d$, respectively, so that $A \in \mathbb{P}$ correspond to the vertex $(0, 0) \in \mathbb{I}^2$ and the edge $a \in \mathbb{P}$ corresponds to $[0, 1] \times \{0\} \subset \mathbb{I}^2$. Then $a$ has the endpoints $A$ and $B$. We can view the boundary $\partial \mathbb{I}^2$ of $\mathbb{I}^2$ as a planar embedded graph in $\mathbb{P}$ with the vertex set $V := \{A, B, C, D\}$ and the edge set
$E := \{a, b, c, d\}$. We call $a$ and $c$ the horizontal edges, and $b$ and $d$ the vertical edges of $\mathbb{P}$; see Figure 4.

The pillow $\mathbb{P}$ is an example of a Euclidean polyhedral surface, that is, a surface obtained by gluing Euclidean polygons along boundary edges by using isometries. Note that the metric on $\mathbb{P}$ is locally flat except at its vertices, which are Euclidean conic singularities. So $\mathbb{P}$ is an orbifold (see, for example, [Mil06a, Appendix E] and [BM17, Appendix A.9]).

An alternative description for the pillow $\mathbb{P}$ can be given as follows. We consider the unit square $I^2 \subset \mathbb{R}^2 \cong \mathbb{C}$ and map it to the upper half-plane in $\hat{\mathbb{C}}$ by a conformal map, normalized so that the vertices $0, 1, 1+i, i$ are mapped to $0, 1, \infty, -1$, respectively. By Schwarz reflection, this map can be extended to a meromorphic function $\wp : \mathbb{C} \to \hat{\mathbb{C}}$. Then $\wp$ is a Weierstrass $\wp$-function (up to a postcomposition with a Möbius transformation) that is doubly-periodic with respect to the lattice $2\mathbb{Z}^2 := \{2k + 2ni : k, n \in \mathbb{Z}\} \subset \mathbb{C} \cong \mathbb{R}^2$. Actually, for $z, w \in \mathbb{C}$ we have

$$\wp(z) = \wp(w) \text{ if and only if } z - w \in 2\mathbb{Z}^2 \text{ or } z + w \in 2\mathbb{Z}^2. \tag{2.3}$$

We can push forward the Euclidean metric on $\mathbb{C}$ to the Riemann sphere $\hat{\mathbb{C}}$ by $\wp$. With respect to this metric, called the canonical orbifold metric for $\wp$, the sphere $\hat{\mathbb{C}}$ is isometric to the Euclidean square pillow $\mathbb{P}$. In the following, we identify the pillow $\mathbb{P}$ with $\hat{\mathbb{C}}$ by the orientation-preserving isometry that maps the vertices $A, B, C, D$ to $0, 1, \infty, -1$, respectively. Then we can consider $\wp : \mathbb{C} \to \mathbb{P} \cong \hat{\mathbb{C}}$ as a map onto the pillow $\mathbb{P}$. Actually, $\wp$ is the universal orbifold covering map for $\mathbb{P}$ (see [BM17, Section A.10] for more background). A very intuitive description of this map can be given if we color the squares $[k, k+1] \times [n, n+1], k, n \in \mathbb{Z}$, in checkerboard manner black and white. Restricted to such a square $S$, the map $\wp$ is an isometry that sends $S$ to the front or back side of $\mathbb{P}$ depending on the color of $S$; see Figure 5.

![Figure 4. The Euclidean square pillow $\mathbb{P}$.](image)

![Figure 5. The map $\wp : \mathbb{C} \to \mathbb{P}$.](image)
for an illustration. Here, the points in the complex plane $\mathbb{C}$ marked by a black dot (on the left) are mapped to $A$ by $\varphi$ and are elements of $\varphi^{-1}(A) = 2\mathbb{Z}^2$.

We denote by $\mathbb{T} := \mathbb{C}/2\mathbb{Z}^2$ the torus obtained as the quotient of $\mathbb{C}$ by the lattice $2\mathbb{Z}^2$ and by $\pi: \mathbb{C} \to \mathbb{T}$ be the corresponding quotient map. Note that for $z, w \in \mathbb{C}$ we have

$$\pi(z) = \pi(w) \text{ if and only if } z - w \in 2\mathbb{Z}^2.$$ 

We consider $\mathbb{T}$ as an Euclidean torus by equipping it with the unique locally Euclidean path metric such that $\pi: \mathbb{C} \to \mathbb{T}$ is a local isometry. One way to think about the torus $\mathbb{T}$ is to represent it as the square $Q := [0,2]^2 \subset \mathbb{R}^2 \cong \mathbb{C}$, where the sides of $Q$ are pairwise identified by the maps $z \in [0,2] \mapsto z + 2 \in [2,2+2i]$ and $z \in [0,2] \mapsto z + 2i \in [2i,2+2i]$.

From this geometric picture it is clear that $\varphi: \mathbb{C} \to \mathbb{P}$ descends to a branched covering map $\overline{\varphi}: \mathbb{T} \to \mathbb{P}$ such that $\varphi = \overline{\varphi} \circ \pi$. Essentially, this map is given by mapping the unit square $I^2 \subset Q = [0,2]^2$ to the front side of $S$ and then to the back side of $S$ by Schwarz reflection. From this representation of $\overline{\varphi}$ we see that the topological degree of $\overline{\varphi}$ is equal to 2; so each point $p \in \mathbb{P}$ has precisely two preimages in $\mathbb{T}$, counting local degrees. The map $\overline{\varphi}$ has four critical points, namely the preimages of the vertices of $\mathbb{P}$, and so $C_\overline{\varphi} = (\overline{\varphi}^{-1})'(V)$.

2.5. Isotopies and Jordan curves. Let $S$ and $S'$ be surfaces. Then a continuous map $H: S \times I \to S'$ is called an isotopy (from $S$ to $S'$) if $H_t := H(\cdot,t): S \to S'$ is a homeomorphism for each $t \in I$.

Suppose that $H: S \times I \to S'$ is an isotopy and $Z \subset S$. The isotopy $H$ is said to be an isotopy relative to $Z$ (abbreviated “$H$ is an isotopy rel. $Z$”) if $H_t(p) \in Z$ for all $p \in Z$ and $t \in I$. In other words, the image of each point in $Z$ remains fixed during the isotopy $H$.

Two homeomorphisms $h_0, h_1: S \to S'$ are called isotopic (rel. $Z \subset S$) if there exists an isotopy $H: S \times I \to S'$ (rel. $Z$) with $H_0 = h_0$ and $H_1 = h_1$.

Given $X, Y, Z \subset S$, we say that $X$ is isotopic to $Y$ rel. $Z$ (or $X$ can be isotoped into $Y$ rel. $Z$), denoted by $X \sim Y$ rel. $Z$, if there exists an isotopy $H: S \times I \to S$ rel. $Z$ with $H_0 = \text{id}_S$ and $H_1(X) = Y$. Recall that $\text{id}_S$ is the identity map on $S$. We are mostly interested in the case where the surface $S$ is a 2-sphere $S^2$ and $Z \subset S^2$ consists of precisely four points. Up to homeomorphism, we may then assume that $S^2$ is equal to the pillow $\mathbb{P}$, and $Z = V = \{A,B,C,D\}$ consists of the four vertices of $\mathbb{P}$. We will freely switch back and forth between a general marked sphere $(S^2, Z)$ with $\#Z = 4$ and $(\mathbb{P}, V)$.

The following statement gives a criterion when two Jordan curves in a marked sphere $(S^2, Z)$ are isotopic rel. $Z$.

**Lemma 2.1.** Let $\alpha$ and $\beta$ be disjoint Jordan curves in a marked sphere $(S^2, Z)$. Suppose there is an annulus $U \subset S^2 \setminus Z$ such that $\partial U = \alpha \cup \beta$. Then $\alpha$ and $\beta$ are isotopic rel. $Z$.

**Proof.** This is standard and we will only give a sketch of the proof. Since Jordan curves in surfaces are tame, one can slightly enlarge the annulus $U$ to an annulus $U' \subset S^2 \setminus Z$ that contains $\alpha$ and $\beta$. Then $\alpha$ can be isotoped into $\beta$ by an isotopy on $U'$ that is the identity near $\partial U'$. This isotopy on $U'$ can be extended to an isotopy on $S^2$ rel. $Z$ that isotopes $\alpha$ into $\beta$.

Let $\alpha$ be an essential Jordan curve in $(\mathbb{P}, V)$. Then its isotopy class $[\alpha]$ consists of all Jordan curves $\beta$ in $(\mathbb{P}, V)$ such that $\alpha \sim \beta$ rel. $V$. There is a natural way to define a bijection between the set of these isotopy classes $[\alpha]$ and the set of extended rational numbers $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$. Indeed, these isotopy classes $[\alpha]$ are naturally classified by slopes $r/s \in \hat{\mathbb{Q}}$. 

Throughout this paper, whenever we speak of a slope $r/s \in \widehat{\mathbb{Q}}$, we will assume that $r \in \mathbb{Z}$ and $s \in \mathbb{N}_0$ are two relatively prime integers. We allow $s = 0$ here in which case we assume $r = 1$. Then $r/s = 1/0 := \infty \in \widehat{\mathbb{Q}}$.

We say that a straight line $\ell \subset \mathbb{C}$ has slope $r/s \in \widehat{\mathbb{Q}}$ if it is given as

$$\ell = \{z_0 + (s + ir)t : t \in \mathbb{R}\} \subset \mathbb{C}$$

for some $z_0 \in \mathbb{C}$. We use the notation $\ell_{r/s}$ for a line with slope $r/s$.

If a line $\ell_{r/s}$ does not contain any lattice point in $\mathbb{Z}^2 = \mathbb{P}^{-1}(V)$ and so $\ell_{r/s} \subset \mathbb{C} \setminus \mathbb{Z}^2$, then $\tau_{r/s} := \varphi(\ell_{r/s})$ is an essential Jordan curve in $\mathbb{P} \setminus V$. In fact, $\tau_{r/s}$ is a simple closed geodesic in the Euclidean square pillow $\mathbb{P}$ (see Figure 6 for an illustration). Its isotopy class rel. $V$ depends only on the slope $r/s$ and not on the specific choice of $\ell_{r/s} \subset \mathbb{C} \setminus \mathbb{Z}^2$. Conversely, for each essential Jordan curve $\alpha \subset \mathbb{P} \setminus V$ there exists a unique rational slope $r/s \in \widehat{\mathbb{Q}}$ such that $\alpha \sim \tau_{r/s}$ rel. $V$. In fact, we have the following statement.

**Lemma 2.2.** Let $\alpha$ be an essential Jordan curve in $(\mathbb{P}, V)$. Then there exists a unique slope $r/s \in \widehat{\mathbb{Q}}$ with the following property. Let $\ell_{r/s}$ be any line in $\mathbb{C}$ with slope $r/s$ and $\ell_{r/s} \subset \mathbb{C} \setminus \mathbb{Z}^2$, and set $\tau_{r/s} := \varphi(\ell_{r/s})$. Then $\tau_{r/s}$ is an essential Jordan curve in $(\mathbb{P}, V)$ with $\alpha \sim \tau_{r/s}$ rel. $V$. Moreover, the map $[\alpha] \mapsto r/s$ gives a bijection between isotopy classes $[\alpha]$ of essential Jordan curves $\alpha$ in $(\mathbb{P}, V)$ and slopes $r/s \in \widehat{\mathbb{Q}}$.

**Proof.** This is well-known (see [FM12, Proposition 2.6]). We will give an outline of the proof, because we will use some of the ideas in the ensuing argument in later discussions.

A similar fact is much better known for isotopy classes $[\alpha]$ of essential (i.e., not null-homotopic) Jordan curves $\alpha$ on a torus $\mathbb{T}$. Here we may assume that the torus is equal to the quotient $\mathbb{T} = \mathbb{C}/2\mathbb{Z}^2$. We consider the quotient map $\pi : \mathbb{C} \to \mathbb{C}/2\mathbb{Z}^2 = \mathbb{T}$ and the branched cover $\overline{\varphi} : \mathbb{T} \to \mathbb{P}$ defined earlier.

It is easy to see that if $r/s \in \widehat{\mathbb{Q}}$ and $\ell_{r/s} \subset \mathbb{C} \setminus \mathbb{Z}^2$ is a line of slope $r/s$, then $\overline{\varphi}(r/s)$ is an essential Jordan curve in $\mathbb{T}$ whose isotopy class $[\tau_{r/s}]$ on $\mathbb{T}$ only depends on $r/s$ and not on the choice of the line $\ell_{r/s}$. It is somewhat harder to show that for each essential Jordan curve $\alpha$ in $\mathbb{T}$ there exists a unique slope $r/s \in \widehat{\mathbb{Q}}$ such that $\alpha$ is isotopic to the closed Jordan curve $\overline{\varphi}(r/s)$ on $\mathbb{T}$ and that the corresponding map $[\alpha] \mapsto r/s$ is a bijection between these isotopy classes on $\mathbb{T}$ and $\widehat{\mathbb{Q}}$ (see, for example, [FM12, Proposition 1.5]).

The statement of the lemma can be reduced to the torus case. Indeed, we know that the map $\overline{\varphi} : \mathbb{T} \to \mathbb{P}$ is a branched covering map of degree 2. One can show that for each essential Jordan curve $\alpha$ in $(\mathbb{P}, V)$ its preimage $\overline{\varphi}^{-1}(\alpha)$ under $\overline{\varphi}$ consists of two distinct...
Jordan curves $\alpha, \alpha' \subset \mathbb{T}$ (the alternative that $\varphi^{-1}(\alpha)$ consists of one Jordan curve can be ruled out by topological arguments). The map $\overline{\varphi}$ restricted to either of these Jordan curves is a homeomorphism onto $\alpha$. Moreover, $\alpha$ and $\alpha'$ are essential Jordan curves in $\mathbb{T}$ that are isotopic. By an isotopy lifting argument (see [BM17] Proposition 11.3 for a related statement) one can show by that $[\overline{\alpha}]$ only depends on $[\alpha]$; so we get a well-defined map by first sending $[\alpha]$ to the isotopy class $[\overline{\alpha}]$ on $\mathbb{T}$ and then to a unique slope $r/s \in \mathbb{Q}$. This actually gives the desired bijection.

One can describe this map explicitly as follows. Pick a basepoint $p_0 \in \alpha$ and consider $\alpha$ as a simple loop starting and ending in $p_0$ (running through $\alpha$ with some orientation). Pick a point $z_0 \in \mathbb{C}$ with $\varphi(z_0) = p_0$. Then we can find a unique lift $\beta$ of $\alpha$ under the map $\varphi$ such that $\beta$ has the initial point $z_0$. Note that $\varphi$ is a covering map over $\mathbb{P} \setminus V$, and that $\beta$ is either a simple loop or an arc in $\mathbb{C}$. We will see momentarily that $\beta$ is actually an arc.

Since $\pi(\beta) \subset \mathbb{T}$ is a connected set with $\overline{\varphi}(\pi(\beta)) = \varphi(\beta) = \alpha$, and $\overline{\varphi}$ restricted to either $\alpha$ and $\alpha'$ is a homeomorphism onto $\alpha$, it follows that $\pi(\beta)$ is one of the curves $\alpha$ or $\alpha'$, say $\pi(\beta) = \alpha$. If $\overline{\varphi}_0 = \pi(z_0) \in \alpha$, then we can consider $\alpha$ as a loop with base point $\overline{\phi}_0$ and $\beta$ as a lift of $\alpha$ under the covering map $\pi$. This implies that $\beta$ is actually an arc, because if $\beta$ were a loop, then $\alpha = \pi(\beta)$ would be null-homotopic and not essential in $\mathbb{T}$.

Let $w_0 = z_0$ be the endpoint of the arc $\beta$ distinct from $z_0$. Since $\beta$ is a lift of the simple loop $\alpha$ under $\pi$, the difference $w_0 - z_0 \neq 0$ must be a primitive element in $2\mathbb{Z}^2$, i.e., $w_0 - z_0 = 2(s + ir)$, where $s$ and $r$ are integers with greatest common divisor 1. By possibly interchanging the roles of $z_0$ and $w_0$ (corresponding to a change of the orientation how one runs though $\alpha$ starting at $p_0$), we may assume that $s \geq 0$, and that $r = 1$ in case $s = 0$. Under these restrictions on $(r, s)$, the pair $(r, s)$ gives a slope $r/s \in \mathbb{Q}$ which in turn uniquely determines the pair $(r, s)$. The slope $r/s$ obtained in this way is independent of the initial choice of $p_0$ and $z_0$, and only depends on the isotopy class of $\alpha$. \hfill \Box

In the following, we denote by $\alpha^h = \tau_0 = \varphi(\ell_0)$ a horizontal essential Jordan curve (corresponding to slope 0 and separating the edges $a$ and $c$ of $\mathbb{P}$) and by $\alpha^v = \tau_\infty = \varphi(\ell_\infty)$ a vertical essential Jordan curve (corresponding to slope $\infty$ and separating $b$ from $d$) in $\mathbb{(P, V)}$. To be specific, we set

\begin{equation}
\alpha^h := \varphi(\mathbb{R} \times \{1/2\}) \text{ and } \alpha^v := \varphi(\{1/2\} \times \mathbb{R}).
\end{equation}

A statement very similar to Lemma 2.2 is true for arcs $\xi$ in $(\mathbb{P}, V)$. If $\xi$ is such an arc, then there exists a unique rational slope $r/s \in \mathbb{Q}$ such that $\xi \sim \xi_{r/s}$ rel. $V$, where $\xi_{r/s} := \varphi(\ell_{r/s})$ is again the image of a straight line $\ell_{r/s} = \{z_0 + (s + ir)t : t \in \mathbb{R}\}$ in $\mathbb{C}$ with slope $r/s \in \mathbb{Q}$, but here $z_0 \in \mathbb{Z}^2$. More precisely, $z_0$ is a preimage point of an endpoint of $\xi$ under $\varphi$. Note that $\xi_{r/s}$ is a geodesic arc in the isotopy class of $\xi$ rel. $V$. Moreover, $\xi_{r/s}$ is a core curve of each closed geodesic $\tau_{r/s} = \varphi(\ell_{r/s})$ with $\ell_{r/s} \subset \mathbb{R}^2 \setminus \mathbb{Z}^2$.

2.6. Intersection numbers. Let $(S, Z)$ be a marked surface with a finite (possibly empty) set $Z \subset S$ of marked points. Again we are mostly interested in the case $S = S^2$ and a set $Z \subset S^2$ with $\#Z = 4$. As we already remarked, in this situation we may assume that $(S^2, Z) \cong (\mathbb{P}, V)$ up to homeomorphism, whenever this is convenient.

If $\alpha$ and $\beta$ are arcs or Jordan curves in $(S, Z)$, we define their (unsigned) intersection number as

$$i(\alpha, \beta) := \inf \{\#(\alpha' \cap \beta') : \alpha \sim \alpha' \text{ rel. } Z \text{ and } \beta \sim \beta' \text{ rel. } Z\}.$$
The relevant marked surface \((S, Z)\) here will be understood from the context, and we suppress it from our notation for intersection numbers. If we want to emphasize it, we say that we consider intersection numbers in \((S, Z)\). The intersection number is always finite. If \(\alpha\) and \(\beta\) satisfy \(i(\alpha \cap \beta) = \#(\alpha \cap \beta)\), then we say that \(\alpha\) and \(\beta\) are in minimal position (in their isotopy classes rel. \(Z\)).

Suppose \(\alpha\) and \(\beta\) are arcs or Jordan curves in \((S, Z)\). Then we say that the intersection of \(\alpha\) and \(\beta\) is transverse or that \(\alpha\) and \(\beta\) meet transversely if the following conditions are true. First, we require that the set \(\alpha \cap \beta\) consists of isolated points. Second, we require that at each point \(p \in \alpha \cap \beta \cap (S \setminus Z)\), the curve \(\alpha\) crosses \(\beta\) in the following sense: there exists a (small) subarc \(\sigma \subset \alpha\) containing \(p\) as an interior point such that \(\sigma \cap \beta = \{p\}\). Moreover, if \(\sigma^L\) and \(\sigma^R\) are the two subarcs of \(\sigma\) into which \(\sigma\) is split by \(p\), then with suitable orientation of \(\beta\) near \(p\) the arc \(\sigma^L\) lies to the left and \(\sigma^R\) to the right of \(\beta\).

**Lemma 2.3.** Let \((S, Z)\) be a marked surface with \(#Z < \infty\). Suppose \(\alpha\) and \(\beta\) are Jordan curves or arcs in \((S, Z)\). If \(#(\alpha \cap \beta) = i(\alpha \cap \beta)\), then \(\alpha\) and \(\beta\) meet transversely.

**Proof.** This is essentially a standard fact and we will only give an outline of the proof. Since \(#(\alpha \cap \beta) = i(\alpha \cap \beta)\), the set \(\alpha \cap \beta\) is finite and consists of finitely many isolated points. To reach a contradiction, we require that \(\alpha\) and \(\beta\) do not meet transversely at some point \(p \in \alpha \cap \beta \cap (S \setminus Z)\). Then there exists an arc \(\sigma \subset \alpha\) containing \(p\) as an interior point such that \(\sigma \cap \beta = \{p\}\) and with the following property: if \(\sigma_1\) and \(\sigma_2\) denote the two subarcs of \(\sigma\) into which \(\sigma\) is split by \(p\), then \(\sigma_1\) and \(\sigma_2\) lie on the same side of \(\beta\) (equipped with some orientation locally near \(p\)). In other words, \(\alpha\) touches \(\beta\) locally near \(p\) from one side and does not cross \(\beta\) at \(p\).

We can then modify the curve \(\alpha\) by an isotopy rel. \(Z\) near \(p\) that pulls the subarc \(\sigma\) away from \(\beta\) so that the new curve \(\hat{\alpha}\) does not have the intersection point \(p\) with \(\beta\) while no new intersection points of \(\alpha\) and \(\beta\) arise. This contradicts our assumption that for the original curve \(\alpha\) we have \(#(\alpha \cap \beta) = i(\alpha \cap \beta)\). \(\square\)

Before we turn to intersection numbers of curves in \((\mathbb{P}, V)\), we first record some related facts for the flat torus \(\mathbb{T} = \mathbb{C}/2\mathbb{Z}^2\). For each simple closed geodesic \(\tau\) on \(\mathbb{T}\) there exists a line \(\ell_{r/s} \subset \mathbb{C}\) with rational slope \(r/s \in \hat{\mathbb{Q}}\) such that \(\tau = \pi(\ell_{r/s})\). We say that \(r/s\) is the slope of the geodesic \(\tau\).

**Lemma 2.4.** Let \(\tau\) and \(\sigma\) be two closed geodesics on the torus \(\mathbb{T}\) with distinct slopes \(r/s\) and \(r'/s'\) in \(\hat{\mathbb{Q}}\), respectively. Then the intersection \(\tau \cap \sigma\) consists of \(N := |r s' - s r'|\) points and these points partition each curve \(\tau\) and \(\sigma\) into arcs of equal length.

**Proof.** By our standing assumption, \(r \in \mathbb{Z}\) and \(s \in \mathbb{N}_0\) are relatively prime here and so we can find integers \(p\) and \(q\) with \(pr + qs = 1\). Then \(\omega = 2(s + i r)\) and \(\bar{\omega} = 2(-p + i q)\) generate the lattice \(2\mathbb{Z}^2\). The torus \(\mathbb{T} = \mathbb{C}/2\mathbb{Z}^2\) can be obtained by identifying (via translation) the opposites sides of the parallelogram \(Q \subset \mathbb{C}\) spanned by \(\omega\) and \(\bar{\omega}\). It follows that \(\pi^{-1}(\tau)\) is a collection of equally spaced parallel lines in \(\mathbb{C}\) with slope \(r/s\) such that any two consecutive lines differ by a translation by \(\bar{\omega}\).

Since \(\sigma\) has slope \(r'/s' \in \hat{\mathbb{Q}}\) distinct from \(r/s\), there exists a point \(z_0 \in \pi^{-1}(\tau) \cap \pi^{-1}(\sigma)\). We can find a lift \(\beta\) of \(\sigma\) under \(\pi\) that starts at \(z_0\) and has the endpoint \(w_0 = z_0 + \omega'\), where \(\omega' = 2(s' + i r')\). Note that \(\beta\) is a segment contained in a line \(\ell_{r'/s'} \subset \pi^{-1}(\sigma)\). The endpoints \(z_0\) and \(w_0\) of \(\beta\) are mapped to the same point \(p_0 \in \tau \cap \sigma\) under \(\pi\), and so \(z_0, w_0 \in \ell_{r'/s'} \cap \pi^{-1}(\tau)\). It follows that the equally spaced parallel lines in \(\pi^{-1}(\tau)\) cut \(\beta\) into subsegments of equal
length. Applying the map $\pi$, we see that the points in $\tau \cap \sigma$ partition the curve $\sigma$ into arcs of equal length. Since the roles of $\tau$ and $\sigma$ are symmetric, we get a similar partition for $\tau$.

The number $N$ of points in $\tau \cap \sigma$ is equal to the number of arcs in the partition of $\sigma$. A simple geometric argument (based on the fact that $\pi^{-1}(\tau)$ consists of parallel lines translated by $\overline{\omega}$) shows that the number $N$ of these arcs can be obtain as follows: if we write $\omega' \in \mathbb{Z}^2$ uniquely in the form

$$\omega' = k\omega + n\overline{\omega}$$

with $k, n \in \mathbb{Z}$, then $N = |n|$. Now if we multiply the terms here with the complex conjugate $\overline{\omega}$ and take imaginary parts, we see that $N = |n| = |rs' - sr'|$ as claimed.

If $\overline{\alpha}$ and $\overline{\beta}$ are two non-isotopic Jordan curves in the torus $\mathbb{T}$, then, as follows from the discussion in the proof of Lemma 2.2, their isotopy classes can be represented by two closed geodesics $\tau, \sigma \in \mathbb{T}$ of distinct slopes $r/s, r'/s' \in \mathbb{Q}$, respectively. One can show that (see [FM12, p. 29])

$$i(\overline{\alpha}, \overline{\beta}) = |rs' - sr'| = |(\tau \cap \sigma)|. \tag{2.6}$$

Here the last equation follows from Lemma 2.4 and implies that the closed geodesics $\tau$ and $\sigma$ are in minimal position on $\mathbb{T}$.

The following lemma summarizes the intersection properties of Jordan curves and arcs in $(\mathbb{P}, V)$. Recall that $a, c$ denote the horizontal, and $b, d$ the vertical edges of $\mathbb{P}$. The curves $\alpha^h$ and $\alpha^v$ (see (2.5)) represent the “horizontal” and “vertical” isotopy classes of Jordan curves in $(\mathbb{P}, V)$ corresponding to slopes $0$ and $\infty$, respectively.

**Lemma 2.5.** Let $\alpha$ and $\beta$ be essential Jordan curves in $(\mathbb{P}, V)$ and $r/s, r'/s' \in \mathbb{Q}$ be the unique slopes such that $\alpha \sim \tau_{r/s}$ and $\beta \sim \tau_{r'/s'}$ rel. $V$, where $\tau_{r/s}$ and $\tau_{r'/s'}$ are simple closed geodesics in $(\mathbb{P}, V)$ corresponding to slopes $r/s$ and $r'/s'$, respectively. Let $\xi$ in $(\mathbb{P}, V)$ be a core arc of $\beta$. Then the following statements are true:

(i) If $r/s \neq r'/s'$, then $i(\alpha, \beta) = |(\tau_{r/s} \cap \tau_{r'/s'})| = 2|rs' - sr'| > 0$.

(ii) $i(\alpha, \xi) = \frac{1}{2}i(\alpha, \beta) = |rs' - sr'|$.

(iii) $i(\alpha, a) = |(\tau_{r/s} \cap a)| = |r|$, $i(\alpha, c) = |(\tau_{r/s} \cap c)| = |r|$.

(iv) $i(\alpha, b) = |(\tau_{r/s} \cap b)| = |r|$, $i(\alpha, d) = |(\tau_{r/s} \cap d)| = |r|$.

(v) $i(\alpha, \alpha^h) = 2|r|$ and $i(\alpha, \alpha^v) = 2s$.

Note that if $r/s = r'/s'$ in (i), then $i(\alpha, \beta) = 0$. Figure 7 illustrates the statements (iii), (v) of this lemma when $\alpha = \tau_2$.

**Proof.** (i) This is essentially well-known and we will only give an outline of the proof. The easiest way to see this is again to reduce to the torus case (2.6).

Let $\alpha$ and $\beta$ be essential Jordan curves in $(\mathbb{P}, V)$ as in the statement. We may assume that $\alpha$ and $\beta$ are in minimal position and so $|(\alpha \cap \beta)| = i(\alpha \cap \beta)$. We again consider the torus $\mathbb{T} = \mathbb{C}/\mathbb{Z}^2$ and the branched covering map $\overline{\phi}: \mathbb{T} \to \mathbb{P}$ of degree 2. Under $\overline{\phi}$ there exist precisely two lifts $\overline{\alpha}$ and $\overline{\alpha}'$ of $\alpha$, and two lifts $\overline{\beta}$ and $\overline{\beta}'$ of $\beta$. These lifts are Jordan curves in $\mathbb{T}$ corresponding to the slopes $r/s$ and $r'/s'$, respectively. Now if we consider intersection numbers of curves on $\mathbb{T}$, then by (2.6) we have

$$i(\overline{\alpha}, \overline{\beta}) = i(\overline{\alpha}, \overline{\beta}') = N := |rs' - sr'| > 0.$$
It follows that
\[ i(\alpha, \beta) = \#(\alpha \cap \beta) = \#(\overline{\alpha} \cap \overline{\beta}) = \#(\overline{\alpha} \cap \overline{\beta}) + \#(\alpha \cap \beta) \geq i(\overline{\alpha}, \overline{\beta}) + i(\alpha, \beta) = 2N. \]

To see that an inequality in the other direction is true, consider the simple closed geodesics \( \tau_{r/s} \) and \( \tau_{r'/s'} \) as in the statement. Then \( \tau_{r/s} = \varphi(\ell_{r/s}) \) and \( \tau_{r'/s'} = \varphi(\ell_{r'/s'}) \) for some lines \( \ell_{r/s}, \ell_{r'/s'} \subset \mathbb{C} \setminus \mathbb{Z}^2 \) with slopes \( r/s \) and \( r'/s' \), respectively. Since \( r/s \neq r'/s' \), these lines have exactly one point in common, say \( z_0 \in \ell_{r/s} \cap \ell_{r'/s'} \subset \mathbb{C} \setminus \mathbb{Z}^2 \). We also consider the line
\[ \ell'_{r'/s'} = -\ell_{r'/s'} = \{-z_0 - t(s' + ir') : t \in \mathbb{R}\}. \]

Let \( \tau = \pi(\ell_{r/s}), \sigma = \pi(\ell_{r'/s'}) \), \( \sigma' = \pi(\ell_{r'/s'}) \subset T \) be the corresponding closed geodesics on \( T \). Then \( \overline{\sigma} \cap \overline{\sigma'} = \emptyset \) as follows from the fact that \( \ell_{r'/s'} \subset \mathbb{C} \setminus \mathbb{Z}^2 \). By (2.6) we also know that
\[ \#(\tau \cap \sigma) = \#(\tau \cap \sigma') = N. \]

Using (2.3), one can see that \( \varphi \) is a homeomorphism of \( \tau \) onto \( \tau_{r/s} \) and that \( \varphi^{-1}(\tau_{r'/s'}) = \sigma \cup \sigma' \). Since \( \alpha \sim \tau_{r/s} \) and \( \beta \sim \tau_{r'/s'} \) rel. \( V \), it follows that
\[ 2N \leq i(\alpha, \beta) \leq \#(\tau_{r/s} \cap \tau_{r'/s'}) = \#(\tau \cap \sigma') = \#(\tau \cap \sigma') + \#(\tau \cap \sigma' \cap \sigma) = 2N. \]

We conclude that all inequalities must be equalities here and the statement follows.

(ii) This is a variant of the argument in [i]. Again we may assume that \( \alpha \) and \( \xi \) are in minimal position, and denote by \( \overline{\sigma} \) one of the two preimage components of \( \varphi^{-1}(\alpha) \). The preimage \( \overline{\xi} := \overline{\varphi^{-1}(\xi)} \) now consists of a single Jordan curve in \( T \). It corresponds to slope \( r'/s' \).

So we have
\[ i(\alpha, \xi) = \#(\alpha \cap \xi) = \#(\overline{\alpha} \cap \overline{\varphi^{-1}(\xi)}) = \#(\overline{\alpha} \cap \overline{\xi}) \geq i(\overline{\alpha}, \overline{\xi}) = N = |r's' - sr'|. \]

To obtain an inequality in the opposite direction, one again considers \( \tau_{r/s} = \varphi(\ell_{r/s}) \sim \alpha \) rel. \( V \) and \( \tau = \pi(\ell_{r/s}) \) with a line \( \ell_{r/s} \subset \mathbb{C} \setminus \mathbb{Z}^2 \). We pick a point \( u_0 \in \mathbb{Z}^2 \subset \mathbb{C} \) such that \( \varphi(u_0) \in V \) is the initial point of \( \xi \) and consider a line \( \ell_{r'/s'} \) passing through \( u_0 \). We define \( \xi_{r'/s'} = \varphi(\ell_{r'/s'}) \sim \xi \) rel. \( V \) and \( \sigma = \pi(\ell_{r'/s'}) \).

If \( r/s \neq r'/s' \), then again there exists a unique point \( z_0 \in \ell_{r/s} \cap \ell_{r'/s'} \), and by (2.6) we have
\[ \#(\tau \cap \sigma) = N = |r's' - sr'|. \]
lines $\ell_{r/s}$ and $\ell_{r/s'}$ are parallel, but distinct, because the first line does not contain points in $\mathbb{Z}^2$ while the second one does. This implies that $\tau \cap \sigma = \emptyset$.

Note that now $\overline{\varphi}^{-1}(\xi_{r/s'}) = \sigma$, and so

$$N \leq i(\alpha, \xi) \leq \#(\tau_{r/s} \cap \xi_{r/s'}) = \#(\tau \cap \overline{\varphi}^{-1}(\xi_{r/s'})) = \#(\tau \cap \sigma) = N.$$  

Hence $i(\alpha, x_i) = N$. We also see that

$$(2.7) \quad \#(\tau_{r/s} \cap \xi_{r/s'}) = N.$$ 

(iii)–(v) These are special cases of (i), (ii), and (2.7). For example, $a$ is a core curve of $\alpha^h$ corresponding to slope $r'/s' = 0/1 = 0$. If $\ell_0$ is the real axis, then $a = \xi_0 = \varphi(\ell_0)$. Hence by (ii) and (2.7) we have

$$i(\alpha, a) = |r \cdot 1 - s \cdot 0| = |r| = \#(\tau_{r/s} \cap \xi_0) = \#(\tau_{r/s} \cap a).$$

The other statements follow from similar considerations.

Let $\gamma$ be a Jordan curve or an arc in $(\mathbb{P}, V)$, and $M_1, M_2 \subset \mathbb{P}$ be two disjoint sets with $0 < \#(\gamma \cap M_j) < \infty$ for $j = 1, 2$. We say that the points in $\gamma \cap M_1$ and $\gamma \cap M_2$ alternate on $\gamma$ if any two points in one of the sets are separated by the other, that is, any subarc $\beta \subset \gamma$ with both endpoints in either of the sets $\gamma \cap M_1$ or $\gamma \cap M_2$ must contain a point in the other set.

More intuitively, this situation when the points in $\gamma \cap M_1$ and $\gamma \cap M_2$ alternate on $\gamma$ can be described as follows. Suppose we run through $\gamma$. This means that if $\gamma$ is an arc, then we start from one of its endpoints, and traverse $\gamma$ until we end up at the other endpoint; or if $\gamma$ is a Jordan curve, we start from any point on $\gamma$ and run through $\gamma$ indefinitely with some orientation. Then we will first meet a point in either $M_1$ or $M_2$, say in $M_1$. Then if we continue along $\gamma$, we will meet a point in $M_2$, then a point in $M_1$, etc.

**Lemma 2.6.** Let $\gamma$ be an essential Jordan curve or an arc in $(\mathbb{P}, V)$ with $i(\alpha^h, \gamma) > 0$. Suppose $\gamma$ is in minimal position with the edge $a$ of $\mathbb{P}$ and also with the edge $c$ of $\mathbb{P}$. Then the sets $a \cap \gamma$ and $c \cap \gamma$ are non-empty and their points alternate on $\gamma$.

A similar statement is also true if we replace $a$, $c$, $\alpha^h$ with $b$, $d$, $\alpha^v$, respectively. Figure 7 again illustrates this lemma (here $\gamma = \tau_2$).

**Proof.** We first assume that $\gamma$ is a Jordan curve. In this case, $\gamma$ is isotopic rel. $V$ to a Jordan curve $\tau_{r/s} = \varphi(\ell_{r/s})$, where $\ell_{r/s}$ is a straight line in $\mathbb{C} \setminus \mathbb{Z}^2$ with slope $r/s \in \mathbb{Q}$. Then

$$i(a, \gamma) = i(c, \gamma) = |r| = \frac{1}{2}i(\alpha^h, \gamma) > 0.$$ 

Since $\gamma$ and $a$, as well as $\gamma$ and $c$, are in minimal position, we have $\#(a \cap \gamma) = \#(c \cap \gamma) = |r|$. We pick a basepoint $p_0 \in \gamma \setminus (a \cup c)$, and consider $\gamma$ as a loop starting at $p_0$ with some parametrization. We choose a preimage $z_0\in \varphi^{-1}(p_0) \subset \mathbb{C} \setminus \mathbb{Z}^2$. Note that then $z_0$ is not contained in any of the horizontal lines $\mathbb{R} \times \{k\} \subset \mathbb{R}^2 \cong \mathbb{C}$, $k \in \mathbb{Z}$, mapping to $a$ or $c$ under $\varphi$.

We can find a unique lift $\beta$ of $\gamma$ under $\varphi$ starting at $z_0$. Let $w_0$ be the endpoint of $\beta$ distinct from $z_0$. As was explained in the proof of Lemma 2.2, we have $w_0 - z_0 = \pm(2(s + ir))$. Now there are precisely $|r|$ distinct horizontal lines mapping to $a$ and $|r|$ distinct horizontal lines mapping to $c$ under $\varphi$ that separate $z_0$ and $w_0$. The lift $\beta$ must meet each of these lines at least once. Since $\varphi: \beta \setminus \{z_0, w_0\} \to \gamma \setminus \{p_0\}$ is a homeomorphism and $z_0 \notin a \cup c$, the intersection points of $\beta$ with these lines corresponds to the $|r|$ distinct point in $a \cap \gamma$ and the $|r|$ distinct points in $c \cap \gamma$. 

Since \( \#(a \cap \gamma) = |r| = \#(c \cap \gamma) \), the arc \( \beta \) actually meets each of these lines precisely once and there are no other intersections with preimages of \( a \) and \( c \) under \( \varphi \). Now the lines in \( \varphi^{-1}(a) \) and \( \varphi^{-1}(c) \) separating \( z_0 \) and \( w_0 \) alternate. This implies that the points in \( a \cap \gamma \) and \( c \cap \gamma \) alternate on \( \gamma = \varphi(\beta) \). The statement follows in this case.

If \( \gamma \) is an arc in \( (\mathbb{P}, V) \) the proof runs along similar lines. Again \( \gamma \) corresponds to a rational slope \( r/s \in \mathbb{Q} \). Then \( |r| = (a^b, \gamma) > 0 \). Let \( p_0 \) be one of the endpoints of \( \gamma \). Then \( p_0 \in V \subset a \cup c \).

We choose a preimage \( z_0 \in \varphi^{-1}(p_0) \subset \mathbb{C} \) and can find a lift \( \beta \) of \( \gamma \) under \( \varphi \) starting at \( z_0 \). The lift \( \beta \) is not unique (another lift can be obtained by reflecting \( \beta \) in \( z_0 \)), but again, if \( w_0 \) is the other endpoint of \( \beta \), then \( w_0 - z_0 = \pm 2(s + ir) \). The arc \( \beta \) must meet each horizontal line of the form \( \mathbb{R} \times \{k\}, k \in \mathbb{Z} \), that separates \( z_0 \) and \( w_0 \) at least once. It cannot meet such a line more than once, because otherwise \( a \) and \( \gamma \) or \( c \) and \( \gamma \) would not be in minimal position, because then the image \( \tau = \varphi([z_0, w_0]) \) of the line segment \([z_0, w_0]\) under \( \varphi \) would be an arc isotopic rel. \( V \) with fewer intersection with \( a \) or with \( c \) than \( \gamma \). Again we conclude that the points in \( \varphi^{-1}(a) \cap \beta \neq \emptyset \) and in \( \varphi^{-1}(c) \cap \beta \neq \emptyset \) alternate on \( \beta \), and hence the points in \( a \cap \gamma \) and \( c \cap \gamma \) alternate on \( \gamma \), because \( \varphi \) is a homeomorphism from \( \beta \) onto \( \gamma \).

We record a related fact formulated for an arbitrary sphere with four marked points.

**Lemma 2.7.** Let \((S^2, Z)\) be a marked sphere with \( \#Z = 4 \). Let \( \alpha \) and \( \gamma \) be essential Jordan curves in \((S^2, Z)\). Suppose that \( a_\alpha \) and \( c_\alpha \) are core curves of \( \alpha \) that lie in different components of \( S^2 \setminus \alpha \). Then the following statements are true:

(i) \( \text{i}(\alpha, \gamma) = 2\text{i}(a_\alpha, \gamma) = 2\text{i}(c_\alpha, \gamma) \).

(ii) There exists a Jordan curve \( \gamma' \) in \((S^2, Z)\) with \( \gamma' \sim \gamma \) rel. \( Z \) such that \( \gamma \) is in minimal position with \( \alpha, a_\alpha, \) and \( c_\alpha \).

(iii) If \( \text{i}(\alpha, \gamma) > 0 \) and \( \gamma' \) as in (ii), then the sets \( a_\alpha \cap \gamma \) and \( c_\alpha \cap \gamma \) are non-empty and alternate on \( \gamma' \).

**Proof.** We may identify the marked sphere \((S^2, Z)\) with the pillow \((\mathbb{P}, V)\) by a homeomorphism that sends \( \alpha, a_\alpha, c_\alpha \) to \( a^b, a, c \), respectively. Statements [i] and [ii] then follow from Lemma 2.5, because for \( \gamma' \) we can take a simple closed geodesic with the appropriate slope. Statement [iii] follows from Lemma 2.6. \( \square \)

3. *Thurston maps*

We provide a very brief overview here and refer the reader to [BM17, Chapter 2] for more details.

Let \( f: S^2 \to S^2 \) be a branched covering map of a topological 2-sphere \( S^2 \). Recall that \( C_f \) denotes the set of all critical points of \( f \). The union

\[
P_f = \bigcup_{n \in \mathbb{N}} f^n(C_f)
\]

of the orbits of critical points is called the *postcritical set* of \( f \). Note that

\[
f(P_f) \subset P_f \subset f^{-1}(P_f).
\]

The map \( f \) is said to be *postcritically-finite* if its postcritical set \( P_f \) is finite, in other words, if the orbit of every critical point of \( f \) is finite.

**Definition 3.1.** A *Thurston map* is a postcritically-finite branched covering map \( f: S^2 \to S^2 \) of topological degree \( \text{deg}(f) \geq 2 \).
Natural examples of Thurston maps are given by rational Thurston maps, that is, postcritically-finite rational maps on the Riemann sphere $\hat{\mathbb{C}}$.

The ramification function of a Thurston map $f: S^2 \to S^2$ is a function $\alpha_f: S^2 \to \mathbb{N} \cup \{\infty\}$ such that $\alpha_f(p)$ for $p \in S^2$ is the lowest common multiple of all local degrees $\text{deg}(f^n, q)$, where $q \in f^{-n}(p)$ and $n \in \mathbb{N}$ are arbitrary. In particular, $\alpha_f(p) = 1$ for $p \in S^2 \setminus P_f$ and $\alpha_f(p) \geq 2$ for $p \in P_f$.

The orbifold $\mathcal{O}_f$ associated with $f$ is the pair $(S^2, \alpha_f)$. The Euler characteristic of $\mathcal{O}_f$ is

\[
\chi(\mathcal{O}_f) = 2 - \sum_{p \in P_f} \left(1 - \frac{1}{\alpha_f(p)}\right).
\]

Here we set $1/\infty := 0$.

The Euler characteristic of the orbifold $\mathcal{O}_f$ satisfies $\chi(\mathcal{O}_f) \leq 0$. We call $\mathcal{O}_f$ hyperbolic if $\chi(\mathcal{O}_f) < 0$, and parabolic if $\chi(\mathcal{O}_f) = 0$.

A point $p \in S^2$ is called periodic (for $f$) if $f^n(p) = p$ for some $n \in \mathbb{N}$. A Lattés map is a rational Thurston map with parabolic orbifold that does not have periodic critical points.

**Definition 3.2.** Two Thurston maps $f: S^2 \to S^2$ and $g: S^2 \to \hat{S}^2$, where $\hat{S}^2$ is another topological 2-sphere, are called Thurston equivalent if there are homeomorphisms $h_0, h_1: S^2 \to \hat{S}^2$ that are isotopic rel. $P_f$ such that $h_0 \circ f = g \circ h_1$.

We say that a Thurston map is realized (by a rational map) if it is Thurston equivalent to a rational map. Otherwise, we say that it is obstructed.

3.1. The $(n \times n)$-Lattés map. Here we provide the analytic definition for the Lattés maps that we use in this paper and interpret this from a more geometric perspective (see [Mil06b] and [BML] Chapter 3 for a general discussion of Lattés maps).

Let $\mathbb{P} \cong \hat{\mathbb{C}}$ be the Euclidean square pillow and $\varphi: \mathbb{C} \to \mathbb{P}$ be its universal orbifold covering maps, as discussed in Section 2.4. Fix a natural number $n \geq 2$. It follows from (2.3) that there is a unique (and well-defined) map $\mathcal{L}_n: \mathbb{P} \to \mathbb{P}$ such that

\[
\mathcal{L}_n(\varphi(z)) = \varphi(nz) \text{ for } z \in \mathbb{C}.
\]

We call $\mathcal{L}_n$ the $(n \times n)$-Lattés map. In fact, $\mathcal{L}_n$ is a rational map under the identification $\mathbb{P} \cong \hat{\mathbb{C}}$ as discussed in Section 2.4

Alternatively, we can describe the map $\mathcal{L}_n$ in a combinatorial fashion as follows. We color the front side of $\mathbb{P}$ white, and the other (back) side black. We considers these sides of $\mathbb{P}$ as 0-tiles, and subdivide each of them into $n^2$ squares of sidelength $1/n$. We refer to these small squares as 1-tiles (with $n$ understood), and color them in a checkerboard fashion black and white so that the 1-tile $S$ in the white side of $\mathbb{P}$ with the vertex $A$ on its boundary is colored white. We map $S$ to the white side of the pillow $\mathbb{P}$ by an orientation-preserving Euclidean similarity (that scales by the factor $n$) so that the vertex $A$ is fixed. If we extend this map by reflection to the whole pillow, we get the $(n \times n)$-Lattés map $\mathcal{L}_n$ (see Figure 1 for $n = 4$). The map $\mathcal{L}_n$ sends each black or white 1-tile homeomorphically (by a similarity) onto the 0-tile in $\mathbb{P}$ of the same color. Illustrations of $\varphi$ and $\mathcal{L}_n$ are given in Figures 5 and 1 respectively.

Based on this combinatorial description, it is easy to see that each critical point of $\mathcal{L}_n$ has local degree 2 and the postcritical set of $\mathcal{L}_n$ coincides with the set of vertices of $\mathbb{P}$, that is, $P_{\mathcal{L}_n} = \{A, B, C, D\} = V$. One can also check that for the ramification function of $\mathcal{L}_n$ we have $\alpha_{\mathcal{L}_n}(p) = 2$ for each $p \in P_{\mathcal{L}_n}$. Substituting this into (3.1), we see that $\chi(\mathcal{O}_{\mathcal{L}_n}) = 0$. Thus, $\mathcal{L}_n$ has a parabolic orbifold.
3.2. Thurston’s characterization of rational maps. Thurston maps are often constructed in a combinatorial fashion as in Section 3.1 (see, for instance, [CFKP03] and [BM17, Chapter 12]). The question whether a given Thurston map $f$ can be realized by a rational map is usually difficult to answer except in some special cases. William Thurston provided a sharp, purely topological criterion that answers this question. The formulation and proof of this celebrated result can be found in [DH93]. In this section, we introduce the necessary concepts and formulate the result only when $\#P_f = 4$, which is the relevant case for this paper.

In the following, let $f: S^2 \to S^2$ be a Thurston map. The map $f$ defines a natural pullback operation on Jordan curves in $(S^2, P_f)$: a pullback of a Jordan curve $\gamma \subset S^2 \setminus P_f$ under $f$ is a connected component $\tilde{\gamma}$ of $f^{-1}(\gamma)$. Since $f$ is a covering map from $S^2 \setminus f^{-1}(P_f)$ onto $S^2 \setminus P_f$, each pullback $\tilde{\gamma}$ of $\gamma$ is a Jordan curve in $(S^2, P_f)$. Moreover, $f[\tilde{\gamma}] \Rightarrow \gamma$ is a covering map. For some $k \in \mathbb{N}$ with $1 \leq k \leq \deg(f)$ each point $p \in \gamma$ has precisely $k$ distinct preimages in $\tilde{\gamma}$. Here $k$ is the (unsigned) mapping degree of $f: \tilde{\gamma} \Rightarrow \gamma$.

Recall that a Jordan curve $\gamma \subset S^2 \setminus P_f$ is called essential if each of the two connected components of $S^2 \setminus \gamma$ contains at least two points from $P_f$, and is called peripheral otherwise.

We will need the following facts. Since they are standard, we will only give an outline of the proofs.

**Lemma 3.3.** Let $f: S^2 \to S^2$ be a Thurston map and let $\gamma$ and $\gamma'$ be Jordan curves in $(S^2, P_f)$ with $\gamma' \sim \gamma$ rel. $P_f$. Then the sets $f^{-1}(\gamma)$ and $f^{-1}(\gamma')$ are isotopic rel. $f^{-1}(P_f) \supset P_f$ and there is a bijection between the pullbacks of $\gamma$ and the pullbacks of $\gamma'$ under $f$ that preserves isotopy classes rel. $P_f$.

In particular, the isotopy classes of curves in $f^{-1}(\gamma)$ rel. $P_f$ only depend on the isotopy class $[\gamma]$ rel. $P_f$ and not on the specific choice of $\gamma$.

**Proof.** Let $\gamma$ and $\gamma'$ be two Jordan curves as in the statement. Then there exists an isotopy $H: S^2 \times [0,1] \to S^2$ rel. $P_f$ such that $H_0 = \text{id}_{S^2}$ and $H_1(\gamma) = \gamma'$. This isotopy can be lifted by $f$ (see [BM17, Proposition 11.1]). More precisely, there exists an isotopy $\overline{H}: S^2 \times I \to S^2$ rel. $f^{-1}(P_f)$ such that $\overline{H}_0 = \text{id}_{S^2}$ and $H_t \circ f = f \circ \overline{H}_t$ for all $t \in I$. Then

$$\overline{H}_1(f^{-1}(\gamma)) = (f \circ \overline{H}^{-1})(\gamma) = (H_1 \circ f)^{-1}(\gamma) = f^{-1}(H_1(\gamma)) = f^{-1}(\gamma').$$

Hence $f^{-1}(\gamma)$ and $f^{-1}(\gamma')$ are isotopic rel. $f^{-1}(P_f) \supset P_f$. Moreover, it is easy to see that the map $\overline{\gamma} \mapsto \overline{\gamma}' \equiv \overline{H}_1(\overline{\gamma})$ is a bijection between the pullbacks of $\gamma$ and the pullbacks of $\gamma'$ that preserves isotopy classes rel. $P_f$. The statement follows.

**Corollary 3.4.** Let $f: S^2 \to S^2$ be a Thurston map and $\gamma$ be Jordan curve in $(S^2, P_f)$.

(i) If $\gamma$ is peripheral, then every pullback of $\gamma$ under $f$ is also peripheral.

(ii) Suppose that $\#P_f = 4$ and let $\overline{\gamma}$ be a pullback of $\gamma$ under $f$. If $\gamma$ and $\overline{\gamma}$ are essential, then the isotopy class $[\overline{\gamma}]$ rel. $P_f$ only depends on the isotopy class $[\gamma]$ rel. $P_f$ and not on the specific choice of $\gamma$ and its essential pullback $\overline{\gamma}$.

**Proof.** (i) Since $\gamma$ is peripheral, $\gamma$ can be isotoped (rel. $P_f$) into a Jordan curve $\gamma'$ inside a small open Jordan region $V \subset S^2$ such that $\#(V \cap P_f) \leq 1$ and $V$ is evenly covered by the branched covering map $f$ as in (2.1).

Then for each component $U_j$ of $f^{-1}(V)$, the map $f|U_j: U_j \to V$ is given by $z \in \mathbb{D} \mapsto z^{d_j} \in \mathbb{D}$ for some $d_j \in \mathbb{N}$ after orientation-preserving homeomorphic coordinate changes in the source and target. This implies that $\#(U_j \cap P_f) \leq 1$ and that each pullback of $\gamma'$ in $U_j$ is peripheral.
Hence all pullbacks of $\gamma'$ under $f$ are peripheral and the same is true for the pullbacks of $\gamma$ as follows from Lemma 3.3.

(ii) Suppose $\tilde{\gamma}$ and $\tilde{\gamma}'$ are two distinct essential pullbacks of $\gamma$ under $f$. Since these are components of $f^{-1}(\gamma)$, the Jordan curves $\tilde{\gamma}$ and $\tilde{\gamma}'$ are disjoint. Then the set $S^2 \setminus (\tilde{\gamma} \cup \tilde{\gamma}')$ is a disjoint union

$$S^2 \setminus (\tilde{\gamma} \cup \tilde{\gamma}') = V \cup U \cup V',$$

where $V, V' \subset S^2$ are Jordan regions and $U \subset S^2$ is an annulus with $\partial U = \tilde{\gamma} \cup \tilde{\gamma}'$. Since $\tilde{\gamma}$ and $\tilde{\gamma}'$ are essential, both $V$ and $V'$ must contain at least two postcritical points. Now $\# P_f = 4$, and thus $U \cap P_f = \emptyset$. Lemma 2.1 then implies that $\tilde{\gamma}$ and $\tilde{\gamma}'$ are isotopic rel. $P_f$.

It follows that the isotopy class $[\tilde{\gamma}]$ rel. $P_f$ does not depend on the choice of the essential pullback $\tilde{\gamma}$ of $\gamma$. At the same time, Lemma 3.3 implies that $[\tilde{\gamma}]$ only depends on the isotopy class $[\gamma]$, as desired.

For a general Thurston map $f$ the concept of an invariant multicurve is important to decide whether $f$ can be realized or is obstructed. By definition, a multicurve is a non-empty finite family $\Gamma$ of essential Jordan curves in $S^2 \setminus P_f$ that are pairwise disjoint and pairwise non-isotopic rel. $P_f$. Suppose now that $\# P_f = 4$. Then any two essential Jordan curves in $S^2 \setminus P_f$ are either isotopic rel. $P_f$ or have a non-empty intersection. Thus, each multicurve $\Gamma$ consists of a single essential Jordan curve $\gamma$ in $S^2 \setminus P_f$. We say that an essential Jordan curve in $S^2 \setminus P_f$ is $f$-invariant if each essential pullback of $\gamma$ under $f$ is isotopic to $\gamma$ rel. $P_f$.

Definition 3.5. Let $f: S^2 \to S^2$ be a Thurston map with $\# P_f = 4$ and let $\gamma \subset S^2 \setminus P_f$ be an essential $f$-invariant Jordan curve. We denote by $\gamma_1, \ldots, \gamma_n$ all the essential pullbacks of $\gamma$ under $f$ and define

$$\lambda_f(\gamma) := \sum_{j=1}^n \frac{1}{\deg(f; \gamma_j \to \gamma)}.$$  

Then $\gamma$ is called a (Thurston) obstruction for $f$ if $\lambda_f(\gamma) \geq 1$.

The following theorem gives a criterion when a Thurston map $f$ with $\# P_f = 4$ can be realized.

Theorem 3.6 (Thurston’s criterion). Let $f: S^2 \to S^2$ be a Thurston map with $\# P_f = 4$ and suppose that $f$ has a hyperbolic orbifold. Then $f$ is realized by a rational map if and only if $f$ has no obstruction.

With a suitable definition of an obstruction (as an invariant multicurve that satisfies certain mapping properties) this statement is also true in the general case; see [DH93] or [BMI17, Section 2.6].

The example of the $(n \times n)$-Lattès map $L_n: \mathbb{P} \to \mathbb{P}$ with $n \geq 2$ shows that Theorem 3.6 is false if $f$ has a parabolic orbifold.

Indeed, let $\gamma$ be any essential Jordan curve in $(\mathbb{P}, P_{L_n})$. Without loss of generality, we may assume that $\gamma = \varphi(\ell_{r/n})$ for some straight line

$$\ell_{r/n} = \{ z_0 + (s + ir) t : t \in \mathbb{R} \} \subset \mathbb{C} \setminus \mathbb{Z}^2$$

with slope $r/s \in \widehat{\mathbb{Q}}$. Using (2.3) and (3.2) one can verify that $\gamma$ has exactly $n$ distinct pullbacks $\gamma_1 = \varphi(\ell), \ldots, \gamma_n = \varphi(\ell^n)$ under $L_n$, where

$$\ell^j = \{ (z_0 + 2j)/n + (s + ir) t : t \in \mathbb{R} \}, \quad j = 1, \ldots, n.$$
It follows that the curve \( \gamma_j \) is isotopic to \( \gamma \) rel. \( P_{\mathcal{L}} \) and \( \deg(\mathcal{L}_n; \gamma_j \to \gamma) = n \) for all \( j = 1, \ldots, n \); see Figure 8 for an illustration. Thus, \( \lambda_{\mathcal{L}_n}(\gamma) = 1 \) and \( \gamma \) is an obstruction.

4. Blowing up arcs

Here, we describe the operation of “blowing up arcs”, originally introduced by Kevin Pilgrim and Tan Lei in [PL98, Section 2.5]. This operation allows us to define and modify various Thurston maps and plays a crucial role in this paper. We will first describe the general construction and then illustrate it for Lattès maps. As we will explain, the blow up of Lattès maps can be viewed as the procedure of “gluing a flap” that we had rather informally introduced in Section 1.1.

The construction of blowing up arcs will involve a finite collection \( E \) of arcs with pairwise disjoint interiors in a 2-sphere \( S^2 \). If \( V \) is the set of endpoints of these arcs, we can consider \((V, E)\) as an embedded graph in \( S^2 \). In the construction we will make various topological choices. The following general statement guarantees that we do not run into topological difficulties. In the formulation we equip \( S^2 \) with a “nice” metric \( d \) so that \((S^2, d)\) is isometric to \( \hat{\mathbb{C}} \) carrying the spherical metric with length element \( ds = 2|dz|/(1 + |z|^2) \).

**Proposition 4.1.** Let \( G = (V, E) \) be a planar embedded graph in \( S^2 \) and \( G \subset S^2 \) be its realization. Then there exists a planar embedded graph \( G' = (V, E') \) in \( S^2 \) with the same vertex set such that its realization \( G' \) is isotopic to \( G \) rel. \( V \) and such that each edge of \( G' \) is a piecewise geodesic arc in \( (S^2, d) \).

An outline of the proof of the is given in [Bo79, Chapter I, §4]; the proposition also follows from [Bus10, Lemma A.8].

4.1. The general construction. Before we provide a formal definition, we give some rough idea of how to “blow up” arcs. In the following, \( f: S^2 \to S^2 \) is a fixed Thurston map. Let \( e \) be an arc in \( S^2 \) such that the restriction \( f|e \) is a homeomorphism onto its image. We cut the sphere \( S^2 \) open along \( e \) and glue in a closed Jordan region \( D \) along the boundary. In this way we obtain a new 2-sphere on which we can define a branched covering map \( \tilde{f} \) as follows: \( \tilde{f} \) maps the complement of \( \text{int}(D) \) in the same way as the original map \( f \) and it maps \( \text{int}(D) \) to the complement of \( f(e) \) by a homeomorphism that matches the map \( f|e \). We say that \( \tilde{f} \) is obtained from \( f \) by blowing up the arc \( e \) with multiplicity 1.

Now we proceed to give a rigorous definition of the blow-up operation in the general case, where several arcs \( e \) are blown up simultaneously with possibly different multiplicities \( m_e \geq 1 \). To this end, let \( E \) be a finite set of arcs in \((S^2, f^{-1}(P_f)) \) with pairwise disjoint
interiors such that the restriction \( f|_e: e \to f(e) \) is a homeomorphism for each \( e \in E \). In this case, we say that \( E \) satisfies the blow-up conditions.

We assume that each arc \( e \in E \) has an assigned multiplicity \( m_e \in \mathbb{N} \). Since each \( e \in E \) is an arc in \((S^2, f^{-1}(P_f))\), its interior \( \text{int}(e) \) is disjoint from \( f^{-1}(P_f) \supset P_f \) and so \( \text{int}(e) \) does not contain any critical or postcritical points of \( f \).

For each arc \( e \in E \), we choose an open Jordan region \( W_e \subset S^2 \) so that the following conditions hold:

(A1) the open Jordan regions \( W_e, e \in E \), are pairwise disjoint.
(A2) for distinct arcs \( e_1, e_2 \in E \), we have \( \text{cl}(W_{e_1}) \cap \text{cl}(W_{e_2}) = \partial e_1 \cap \partial e_2 \).
(A3) \( \text{int}(e) \subset W_e \) and \( \partial e \subset \partial W_e \) for each \( e \in E \).
(A4) \( \text{cl}(W_e) \cap f^{-1}(P_f) = e \cap f^{-1}(P_f) = \partial e \) for each \( e \in E \).
(A5) \( f|\text{cl}(W_e) \) is a homeomorphism onto its image for each \( e \in E \).

The existence of such a choice (and also of the choices below) can easily be justified based on Proposition 4.1 and we will skip the details.

Let \( e \in E \) and \( W_e \) be chosen as above. Then we choose a closed Jordan region \( D_e \) so that \( e \) is a crosscut in \( D_e \) and \( D_e \setminus \partial e \subset W_e \). The two endpoints of \( e \) lie on the Jordan curve \( \partial D_e \) and partition it into two arcs, which we denote by \( \partial D^+_e \) and \( \partial D^-_e \). One can think of \( D_e \) as the resulting region if \( e \) has been “opened up”. This is illustrated in the left and middle part of Figure 9.

In order to define the desired Thurston map, we want to collapse \( D_e \) back to \( e \). For this we choose a continuous map \( h: S^2 \times \mathbb{I} \to S^2 \) with the following properties:

(B1) \( h \) is a pseudo-isotopy, that is, \( h_t := h(\cdot, t) \) is a homeomorphism on \( S^2 \) for each \( t \in [0, 1] \).
(B2) \( h_0 \) is the identity map on \( S^2 \).
(B3) \( h_t \) is the identity map on \( S^2 \setminus \bigcup_{e \in E} W_e \) for each \( t \in [0, 1] \).
(B4) \( h_1 \) is homeomorphism of \( S^2 \setminus \bigcup_{e \in E} D_e \) onto \( S^2 \setminus \bigcup_{e \in E} e \), and \( h_1 \) maps \( \partial D^+_e \) and \( \partial D^-_e \) homeomorphically onto \( e \) for each \( e \in E \).

We can see that the set \( h_t(D_e) \) Hausdorff converges to \( e \), as \( t \to 1^- \). This implies that \( h_1(D_e) = e \). So intuitively, the deformation process described by \( h \) collapses each Jordan region \( D_e \) to \( e \) at time 1 so that the points in \( S^2 \setminus \bigcup_{e \in E} W_e \) remain fixed.

**Figure 9.** Setup for blowing up the arcs \( e_1 \) and \( e_2 \) (in the left sphere) with the multiplicities \( m_{e_1} = 1 \) and \( m_{e_2} = 2 \).
Figure 10. A map $\hat{f}$ obtained from $f$ by blowing up the arcs $e_1$ and $e_2$ with multiplicities $m_{e_1} = 1$ and $m_{e_2} = 2$.

We now make a final choice. For a fixed arc $e \in E$, let $m = m_e$. We choose $m - 1$ crosscuts $e^1, \ldots, e^{m-1}$ in $D_e$ with the same endpoints as $e$ such that these crosscuts have pairwise disjoint interiors. We set $e^0 := \partial D_e^+$ and $e^m := \partial D_e^-$. These arcs subordinate the closed Jordan region $D_e$ into $m$ closed Jordan domains $D^1_e, \ldots, D^m_e$, called components of $D_e$. This is illustrated in the right-hand part of Figure 9.

We may assume that the labeling is such that $\partial D^k_e = e^{k-1} \cup e^k$ for $k = 1, \ldots, m$. For each $k = 1, \ldots, m$, we now choose a continuous map $\varphi_k: D^k_e \to S^2$ with the following properties:

(C1) $\varphi_k$ is a homeomorphism of $\text{int}(D^k_e)$ onto $S^2 \setminus f(e)$ and maps $e^{k-1}$ and $e^k$ homeomorphically onto $f(e)$.

(C2) $\varphi_1|e^0 = f \circ h_1|e^0$, $\varphi_m|e^m = f \circ h_1|e^m$, and $\varphi_k|e^k = \varphi_{k+1}|e^k$ for $k = 1, \ldots, m - 1$.

Note that, by the earlier discussion, $h_1$ maps $e^0 = \partial D^+_e$ and $e^m = \partial D^-_e$ homeomorphically onto $e$ and $f$ is a homeomorphism on $e$. These choices of the maps $\varphi_k$ depend on $e$, but we suppress this in our notation for simplicity.

A map $\tilde{f}: S^2 \to S^2$ can now be defined as follows:

(D1) if $p \in S^2 \setminus \cup_{e \in E} \text{int}(D_e)$, we set $\tilde{f}(p) = f(h_1(p))$;

(D2) if $p \in D_e$ for some $e \in E$, then $p$ lies in one of the components $D^k_e$ of $D_e$ and we set $\tilde{f}(p) = \varphi_k(p)$.

The matching conditions (C2) above immediately imply that $\tilde{f}$ is well-defined and continuous.
Definition 4.2. We say that the map \( \hat{f} : S^2 \rightarrow S^2 \) as described above is obtained from \( f \) by blowing up each arc \( e \in E \) with multiplicity \( m_e \).

Figure 10 illustrates the construction of \( \hat{f} \). Here, we blow up the arcs \( e_1 \) and \( e_2 \) from Figure 9 with multiplicities \( m_{e_1} = 1 \) and \( m_{e_2} = 2 \). Note that the image arcs \( f(e_1) = \hat{f}(e_1) \) and \( f(e_2) = \hat{f}(e_2) \) may in general have more than one point in common (these arcs may even coincide). Nevertheless, we chose to draw them with disjoint interiors for simplicity (the arcs \( f(e_1) \) and \( f(e_2) \) have to have a common endpoint since \( e_1 \) and \( e_2 \) do). By construction, \( \hat{f} \) “acts as \( f \)” outside The Jordan domains \( D_{e_1} = D_{e_1}^1 \) and \( D_{e_2} = D_{e_2}^1 \cup D_{e_2}^2 \). More precisely, \( \hat{f} = f \circ h_1 \) on \( S^2 \setminus \text{int}(D_{e_1}) \cup \text{int}(D_{e_2}) \), where \( h_1 \) collapses the Jordan domains \( D_{e_1} \) and \( D_{e_2} \) onto \( e_1 \) and \( e_2 \), respectively. At the same time, \( \hat{f} \) maps each Jordan domain \( \text{int}(D_{e_j}), j = 1, 2 \), homeomorphically onto \( S^2 \setminus f(e_j) \).

More concretely, let us consider the \((2 \times 2)\)-Lattès map \( f = \mathcal{L}_2 \). We choose two edges \( e_1 \) and \( e_2 \) of a 1-tile in \( P \) as shown in the left pillow in Figure 11. Note that \( f \) sends \( e_1 \) and \( e_2 \) homeomorphically onto the edges \( c \) and \( b \) of \( P \), respectively. Thus the set \( E = \{e_1, e_2\} \) satisfies the blow-up conditions. Figure 11 illustrates the setup for blowing up the arcs \( e_1 \) and \( e_2 \) with the multiplicities \( m_{e_1} = 1 \) and \( m_{e_2} = 2 \). The resulting map \( \hat{f} : P \rightarrow P \) is shown in Figure 12. Here, the marked points on the left pillow \( P \) (the domain of the map) correspond to the preimage points \( \hat{f}^{-1}(V) \). Also, each black closed Jordan region is mapped homeomorphically onto the back side of the pillow \( P \).

The following statement summarizes the main properties of the maps \( \hat{f} \) defined in this way.

Figure 12. The map \( \hat{f} \) obtained from \( f = \mathcal{L}_2 \) by blowing up the arc set \( E = \{e_1, e_2\} \) illustrated in Figure 11 with \( m_{e_1} = 1 \) and \( m_{e_2} = 2 \).
Lemma 4.3. Let \( f : S^2 \to S^2 \) be a Thurston map and \( E \) be a set of arcs in \((S^2, f^{-1}(P_f))\) satisfying the blow-up conditions. Suppose \( \tilde{f} : S^2 \to S^2 \) is the map obtained by blowing up each arc \( e \in E \) \( m_e \)-times, where \( m_e \in \mathbb{N} \).

Then \( \tilde{f} \) is a Thurston map with \( P_{\tilde{f}} = P_f \). Moreover, the map \( \tilde{f} \) is uniquely determined up to Thurston equivalence independent of the choices in the above construction. More precisely, the Thurston equivalence class of \( \tilde{f} \) depends only on the Thurston equivalence class of the original map \( f \), the isotopy classes of the arcs in \( E \) rel. \( f^{-1}(P_f) \), and the multiplicities \( m_e \) for \( e \in E \).

Proof. By construction \( \tilde{f} \) is an orientation-preserving local homeomorphism near each point \( p \in S^2 \setminus f^{-1}(P_f) \). By considering the generic number of preimages of a point in \( S^2 \) we see that the topological degree of \( \tilde{f} \) is equal to \( \deg(f) + \sum_{e \in E} m_e > 0 \). The fact that \( \tilde{f} : S^2 \to S^2 \) is a branched covering map can now be deduced from [BM17, Corollary A.14].

We have \( \deg(\tilde{f}, p) = 1 \) for all \( p \in S^2 \setminus f^{-1}(P_f) \). For \( p \in f^{-1}(P_f) \) we have

\[
\deg(\tilde{f}, p) = \deg(f, p) + \sum_{e \in E : p \in e} m_e.
\]

This implies \( C_f \subset C_{\tilde{f}} \subset f^{-1}(P_f) \). Since on the set \( f^{-1}(P_f) \) the maps \( f \) and \( \tilde{f} \) agree, it follows that \( P_{\tilde{f}} = P_f \). So \( \tilde{f} \) has a finite postcritical set, and we conclude that \( \tilde{f} \) is indeed a Thurston map.

We omit a detailed justification of the second claim that \( \tilde{f} \) is uniquely determined up to Thurston equivalence by \( f \), the isotopy classes of the arcs \( E \), and their multiplicities. A proof can be given along the lines of [PL98, Proposition 2].

4.2. Blowing up the \((n \times n)\)-Lattès map. Let \( \mathbb{P} \) be the Euclidean square pillow and \( \mathcal{L}_n : \mathbb{P} \to \mathbb{P} \) be the \((n \times n)\)-Lattès map for fixed \( n \geq 2 \). We denote by \( \mathcal{C} \subset \mathbb{P} \) the common boundary of the two sides of \( \mathbb{P} \). The set \( \mathcal{C} \) may be viewed as a planar embedded graph with the vertex set \( V = \{A, B, C, D\} \) and the edge set \( \{a, b, c, d\} \) in the notation from Section 2.4.

Let \( \tilde{\mathcal{C}} := \mathcal{L}_n^{-1}(\mathcal{C}) \subset \mathbb{P} \) be the preimage of \( \mathcal{C} \) under \( \mathcal{L}_n \), viewed as a planar embedded graph with the vertex set \( \mathcal{L}_n^{-1}(V) \). In the next section, we will study the question whether a Thurston map is realized by a rational map if it is obtained from \( \mathcal{L}_n \) by blowing up edges of \( \tilde{\mathcal{C}} \). In order to facilitate this discussion, we will provide a more concrete combinatorial model for these maps.

By the definition of the map \( \mathcal{L}_n \), the graph \( \tilde{\mathcal{C}} \) subdivides the pillow \( \mathbb{P} \) into \( 2n^2 \) 1-tiles, which are squares of sidelength \( 1/n \). The edges of the embedded graph \( \tilde{\mathcal{C}} \) are precisely the sides of these squares. The map \( \mathcal{L}_n \) sends each edge \( e \) of \( \tilde{\mathcal{C}} \) to one of the edges \( a, b, c, d \) of \( \mathcal{C} \). We call \( e \) horizontal if \( \mathcal{L}_n \) maps it onto \( a \) or \( c \), and vertical if \( \mathcal{L}_n \) maps it onto \( b \) or \( d \).

We take two disjoint copies of the Euclidean square \([0, 1/n]^2\) and identify the points on three of their sides, say the sides \( \{0\} \times [0, 1/n], [0, 1/n] \times \{1/n\} \), and \( \{1/n\} \times [0, 1/n] \). We call the object obtained a flap \( F \). Note that it is homeomorphic to the closed unit disk and has two “free” sides corresponding to the two copies of \([0, 1/n] \times \{0\}\) in \( F \).

We can cut the pillow \( \mathbb{P} \) open along one of the edges of \( \tilde{\mathcal{C}} \) and glue in a flap \( F \) to the pillow by identifying each copy of \([0, 1/n] \times \{0\}\) in the flap with one side of the slit by an isometry (see Figure [2]). In this way, we get a new polyhedral surface homeomorphic to \( S^2 \). One can also glue multiple copies of the flap to the slit by an isometry and obtain a polyhedral surface \( \hat{\mathbb{P}} \) homeomorphic to \( S^2 \). This can be described more concretely as follows. Let \( e \) be
an edge in $\tilde{C}$ and $F_1, \ldots, F_m$ be $m \geq 1$ copies of the flap. For each $j = 1, \ldots, m$, we denote the two copies of $[0,1/n] \times \{0\}$ in the flap $F_j$ by $e'_j$ and $e''_j$. We now construct a new polyhedral surface $\tilde{P}$ in the following way:

(i) First, we cut the original pillow $P$ open along the edge $e$.

(ii) Then, for each $j = 1, \ldots, m-1$, we identify the edge $e''_j$ of $F_j$ with the edge $e'_j$ of $F_{j+1}$ by an isometry. We get a polyhedral surface $D_e$ homeomorphic to a closed disc, whose boundary consists of two edges $e'_1$ and $e''_m$.

(iii) Finally, we glue the disc $D_e$ to the pillow $P$ cut open along $e$ by identifying the edges $e'_1$ and $e''_m$ in $\partial D_e$ with the two sides of the slit by an isometry so that $e'_1$ and $e''_m$ are identified with different sides of the slit. We obtain a polyhedral surface $\tilde{P}$ that is homeomorphic to a 2-sphere.

More generally, we can cut open $P$ along several edges $e$ of $\tilde{C}$ simultaneously and by the method described, glue $m_e \in \mathbb{N}$ copies of the flap to the slit obtained from $e$. If these edges $e$ of $\tilde{C}$ with their multiplicities $m_e$ are given, then there is essentially only one way of gluing flaps so that the resulting object is a polyhedral surface homeomorphic to $S^2$.

Let $\tilde{P}$ be the polyhedral surface obtained from $P$ by gluing a total number of $n_h \geq 0$ horizontal flaps (i.e., flaps glued along horizontal edges of $\tilde{C}$) and a total number of $n_v \geq 0$ vertical flaps (i.e., flaps glued along vertical edges of $\tilde{C}$). We call this surface a flapped pillow. We denote by $E$ the set of edges in $\tilde{C}$ along which flaps where glued with multiplicities $m_e$, $e \in E$. See the left part of Figure 13 for an example of the flapped pillow $\tilde{P}$ obtained by gluing one horizontal and two vertical flaps at the edges $e_1$ and $e_2$ from Figure 11.

The polyhedral surface $\tilde{P}$ is naturally subdivided into

$$2(n^2 + n_h + n_v) = 2n^2 + 2 \sum_{e \in E} m_e$$

squares of sidelength $1/n$, called the 1-tiles of the flapped pillow $\tilde{P}$. The vertices and the edges of these squares are called the 1-vertices and 1-edges of $\tilde{P}$. There is a natural path metric on $\tilde{P}$ that agrees with the Euclidean metric on each 1-tile. The surface $\tilde{P}$ equipped with this metric is locally Euclidean with conic singularities at some of the 1-vertices. Such a conic singularity arises at a 1-vertex $v \in \tilde{P}$ if $v$ is contained in $k_v \neq 4$ distinct 1-tiles.

We will assume that $\tilde{P}$ has at least one flap, that is, $n_h + n_v \geq 1$. Let $F_j$, $j = 1, \ldots, n_h + n_v$, be the collection of flaps glued to $P$. Each flap $F_j$ consists of two 1-tiles in $\tilde{P}$. We call the four
1-vertices that belong to $F_j$ the vertices of the flap $F_j$. The boundary $\partial F_j$ is a Jordan curve in $\tilde{P}$ composed of two 1-edges $e_j$ and $e'_j$, which we call the base edges of $F_j$. The 1-edge in $F_j$ that is opposite to the base edges is called the top edge of the flap $F_j$. Note that $\partial e_j = \partial e'_j$ consists of two vertices of $F_j$.

We now define the base $B(\tilde{P}) \subset \tilde{P}$ of the flapped pillow as

\begin{equation}
B(\tilde{P}) := \tilde{P} \setminus \left( \bigcup_{j=1}^{n_h+n_v} (F_j \setminus \partial e_j) \right).
\end{equation}

In other words, $B(\tilde{P})$ is obtained from $\tilde{P}$ by removing all flaps $F_j$ from $\tilde{P}$, except that we keep the two vertices in $\partial e_j \subset F_j$ from each flap. There is a natural identification

\begin{equation}
B(\tilde{P}) \cong P \setminus \bigcup_{e \in E} \text{int}(e) \subset P.
\end{equation}

This means that we can consider $B(\tilde{P})$ both as a subset of $\tilde{P}$ and of $P$. Figure 14 illustrates these two viewpoints.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure14.png}
\caption{The base $B(\tilde{P})$ of the flapped pillow $\tilde{P}$ from Figure 13 depicted in two different ways: as a subset of $\tilde{P}$ and as the subset $P \setminus \bigcup_{e \in E} \text{int}(e) \subset P$.}
\end{figure}

This is slightly imprecise, but this point of view will be extremely convenient in the following.

We choose the orientation on $\tilde{P}$ represented by a positively-oriented flag contained in $B(\tilde{P})$ such that the flag is also positively-oriented considered as a subset of the oriented sphere $P \ni B(\tilde{P})$ (see [BM17, Section A.4] for the definition of flags and how to represent orientations on surfaces by flags).

The set $B(\tilde{P}) \cong P \setminus \bigcup_{e \in E} \text{int}(e)$ contains the vertex set $\mathcal{L}_n^{-1}(V) \subset P$ of the graph $\tilde{C} = \mathcal{L}_n^{-1}(C) \subset P$. This means that we can naturally view each vertex of $\tilde{C}$ as a 1-vertex in $\tilde{P}$. Let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ be the 1-vertices of $\tilde{P}$ that correspond to the vertices $A, B, C, D$ of the original pillow, respectively. We set $\tilde{V} := \{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ and call $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ the vertices of $\tilde{P}$.

Recall from Section 3.1 that the faces of $\tilde{C}$ are colored black and white in a checkerboard manner. This coloring induces a checkerboard coloring on the 1-tiles of the flapped pillow $\tilde{P}$. The original map $\mathcal{L}_n : P \to P$ can now be naturally extended to a continuous map $\tilde{\mathcal{L}} : \tilde{P} \to P$ by reflection so that it preserves the coloring: $\tilde{\mathcal{L}}$ maps each 1-tile of $\tilde{P}$ by a Euclidean similarity (scaling distances by the factor $n$) to the 0-tile of $P$ with the same color; see Figure 13 for an illustration. On the base $B(\tilde{P})$ the map $\tilde{\mathcal{L}}$ agrees with the original $(n \times n)$-Lattès map $\mathcal{L}_n$ (if we consider $B(\tilde{P})$ as a subset of $P$ by the identification (4.2)).
It is clear that \( \tilde{\mathcal{C}} : \tilde{P} \to P \) is a branched covering map. This map is essentially the Thurston map obtained from \( \mathcal{L}_n \) by blowing up each arc \( e \in E \) with multiplicity \( m_e \). To make this more precise, we need to identify the source \( \tilde{\mathcal{C}} \) with the target \( P \) of \( \tilde{\mathcal{C}} \) so that we have a self-map on \( P \). For this we choose a natural homeomorphism \( \phi : \tilde{\mathcal{C}} \to \mathcal{P} \), which we will now define.

We view the set \( \tilde{\mathcal{C}} := \tilde{\mathcal{L}}^{-1}(C) \subset \tilde{\mathcal{P}} \) as a planar graph, whose vertices and edges are precisely the 1-vertices and the 1-edges of the flapped pillow \( \tilde{P} \). Each 1-edge of \( \tilde{\mathcal{P}} \) is homeomorphically mapped by \( \tilde{\mathcal{C}} \) onto one of the edges of \( \mathcal{P} \). Similarly as before, the 1-edges of \( \tilde{\mathcal{P}} \) that are mapped by \( \tilde{\mathcal{C}} \) onto \( a \) or \( c \) are called horizontal, while the 1-edges of \( \tilde{\mathcal{P}} \) that are mapped by \( \tilde{\mathcal{C}} \) onto \( b \) or \( d \) are called vertical.

There is a simple path of length \( n \) in the graph \( \tilde{\mathcal{C}} \) that connects the vertices \( \tilde{\mathcal{A}} \) and \( \tilde{\mathcal{B}} \). Clearly, any such path consists only of horizontal 1-edges in \( \tilde{\mathcal{P}} \). We denote by \( \tilde{\alpha} \) the realization of the chosen path in the sphere \( \tilde{\mathcal{P}} \), which is an arc in \( (\tilde{\mathcal{P}}, \tilde{V}) \). The arc \( \tilde{\alpha} \) may not be uniquely determined (namely, if flaps have been glued to slits obtained from edges \( e \in a \)), but any two such arcs are isotopic rel. \( \tilde{\mathcal{V}} \). We define \( \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}} \) in a similar way and call \( \tilde{\alpha}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}} \) the horizontal edges, and \( \tilde{\alpha}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}} \) the vertical edges of \( \tilde{\mathcal{P}} \).

We now choose an orientation-preserving homeomorphism \( \psi : \tilde{\mathcal{P}} \to \mathcal{P} \) that sends \( \tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}} \) to \( A, B, C, D \), and \( \tilde{\alpha}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}} \) to \( a, b, c, d \), respectively. We define \( f := \tilde{\mathcal{C}} \circ \psi^{-1} \), which is a self-map on \( \mathcal{P} \). Clearly, \( f \) is a branched covering map on \( \mathcal{P} \). We will say that \( f : \mathcal{P} \to \mathcal{P} \) is obtained from the \((n \times n)\)-Lattès map \( \mathcal{L}_n \) by gluing \( n_h \) horizontal and \( n_v \) vertical flaps.

A point in \( \tilde{\mathcal{P}} \) is critical for \( \tilde{\mathcal{C}} : \tilde{\mathcal{P}} \to \mathcal{P} \) if and only if it is on the boundary of at least four 1-tiles subdividing \( \tilde{\mathcal{P}} \). This implies that \( C_{\tilde{\mathcal{C}}} \) consists of 1-vertices of \( \tilde{\mathcal{P}} \) and that each critical point of \( \mathcal{L}_n \) remains critical for \( \tilde{\mathcal{C}} \) (recall that we can view each point in \( \mathcal{L}_n^{-1}(V) \supset C_{\mathcal{L}_n} \) as a 1-vertex in \( \tilde{\mathcal{P}} \)). Moreover, if a 1-vertex of \( \tilde{\mathcal{P}} \) is a critical point of \( \tilde{\mathcal{C}} \), but not of \( \mathcal{L}_n \), then it must be one of the points in \( \tilde{\mathcal{V}} \). For example, \( \tilde{\mathcal{A}} \in \tilde{\mathcal{V}} \) is a critical point of \( \tilde{\mathcal{C}} \) if and only if a flap was glued to an edge of \( \tilde{\mathcal{C}} \) incident to \( A \equiv \tilde{\mathcal{A}} \). In any case, \( \tilde{\mathcal{C}} \) sends the 1-vertices of \( \tilde{\mathcal{P}} \) to the vertices of \( \mathcal{P} \), the postcritical set of \( f = \tilde{\mathcal{C}} \circ \psi^{-1} \) coincides with the vertex set \( V \). Thus, \( f \) is a Thurston map.

Since we assumed that \( \tilde{\mathcal{P}} \) contains at least one flap (that is, if \( n_h + n_v > 0 \)), the orbifold of the Thurston map \( f \) is hyperbolic. Indeed, each 1-vertex of \( \tilde{\mathcal{P}} \) that is a critical point of \( \mathcal{L}_n \) remains a critical point of \( \tilde{\mathcal{C}} \) with the same or larger local degree. Since we glued at least one flap, there is at least one 1-vertex \( v \) contained in six or more 1-tiles of \( \tilde{\mathcal{P}} \). Then \( \deg(\tilde{\mathcal{L}}, v) = \deg(f, v') \geq 3 \), where \( v' = \phi(v) \). Now

\[
X := f(v') = \tilde{\mathcal{C}}(v) \in V = \{A, B, C, D\} = \mathcal{P}_f,
\]

and so for the ramification function \( \alpha_f \) of \( f \) we have \( \alpha_f(X) \geq 3 \). On the other hand, for all other points \( Y \in V = \mathcal{P}_f \) we have \( \alpha_f(Y) \geq \alpha_{\mathcal{L}_n}(Y) \geq 2 \). It then follows from (3.1) that \( \chi(O_f) < 0 \), and so \( f \) has indeed a hyperbolic orbifold.

The homeomorphism \( \phi \) chosen in the definition of \( f \) is not unique, but any two such homeomorphisms are isotopic rel. \( \tilde{\mathcal{V}} \) (this easily follows from \([\text{Bus}10, \text{A.5 Theorem}]\)). This implies that \( f \) is uniquely determined up to Thurston equivalence. This map may be viewed (up to Thurston equivalence) as the result of the blowing up operation introduced in Section 4.1 applied to the edges \( e \in E \) with the multiplicities \( m_e \) (compare Figures 12 and 13).
5. Realizing the blown-up Lattès maps

The goal of this section is to determine when a blown-up Lattès map is realized by a rational map. In particular, we will apply Thurston’s theorem to prove Theorem 1.2. The strategies and techniques used in the proof will highlight the main ideas needed for establishing the more general Theorem 1.1.

We fix \( n \geq 2, n_h, n_v \geq 0 \), and follow the notation introduced in Section 4.2. In particular, we denote by \( \widehat{P} \) a flapped pillow with \( n_h \) horizontal and \( n_v \) vertical flaps, by \( \widehat{L} : \widehat{P} \to P \) the respective blown-up \((n \times n)\)-Lattès map, and by \( \phi : \widehat{P} \to P \) the identifying homeomorphism. Then \( f : P \to P \) given as \( f = \widehat{L} \circ \phi^{-1} \) is the Thurston map under consideration. We will assume \( n_h + n_v > 0 \), that is, \( \widehat{P} \) has at least one flap. In this case, \( f \) has a hyperbolic orbifold as we have seen, and so we can apply Thurston’s theorem. For this we consider essential Jordan curves \( \gamma \) in \((P, P_f)\) and study their (essential) pullbacks under \( f \).

If \( \gamma \) is such a curve, then the homeomorphism \( \phi \) sends the pullbacks of \( \gamma \) under \( \widehat{L} \) to the pullbacks of \( \gamma \) under \( f \). So in order to understand the isotopy types and mapping properties of the pullbacks under \( f \), we will instead look at the pullbacks of \( \gamma \) under \( \widehat{L} \). In particular, if \( \widehat{\gamma} \) is a pullback of \( \gamma \) under \( \widehat{L} \), then \( \deg(\widehat{L} : \widehat{\gamma} \to \gamma) = \deg(f : \phi(\widehat{\gamma}) \to \gamma) \) and \( \phi(\widehat{\gamma}) \) is essential in \((P, P_f)\) if and only if \( \widehat{\gamma} \) is essential in \((\widehat{P}, \widehat{V})\), where \( \widehat{V} \) denotes the vertex set of the flapped pillow \( \widehat{P} \).

Since the mapping \( \widehat{L} : \widehat{P} \to P \) is a similarity map on each 1-tile of \( \widehat{P} \), the preimage \( \widehat{L}^{-1}(\gamma) \) of a Jordan curve \( \gamma \) in \((P, P_f)\) (or of any subset \( \gamma \) of \( P \)), can be obtained in the following intuitive way: we (rescale and) copy the part of \( \gamma \) that belongs to the white side of \( P \) into each white 1-tile of \( \widehat{P} \) and the part of \( \gamma \) that belongs to the black side of \( P \) into each black 1-tile.

5.1. The horizontal and vertical curves. Recall that \( \alpha_h \) and \( \alpha_v \) denote the Jordan curves in \((P, P_f)\) that separate the two horizontal and the two vertical edges of \( P \), respectively. These two curves are invariant under \( f \) and will play a crucial role in the considerations of this section.

![Figure 15. Pullbacks of \( \alpha_h \) for a blown-up \((4 \times 4)\)-Lattès map with \( n_h = n_v = 1 \).](image)

**Lemma 5.1.** Let \( f = \widehat{L} \circ \phi^{-1} : P \to P \) be a Thurston map obtained from the \((n \times n)\)-Lattès map, \( n \geq 2 \), by gluing \( n_h \geq 0 \) horizontal and \( n_v \geq 0 \) vertical flaps. Then the following statements are true:

(i) The Jordan curve \( \alpha_h \) has \( n + n_h \) pullbacks under \( f \) of which exactly \( n \) are essential. Each of these essential pullbacks is isotopic to \( \alpha_h \) rel. \( P_f \).
Theorem 5.3. Let $\alpha$ (this is illustrated in Figure 16 in a special case). Consequently, $n \geq 0$ is the number of distinct vertical flaps in $\overline{P}$ that $\alpha$ meets.

(ii) If $\overline{\alpha}$ is one of the $n$ essential pullbacks of $\alpha^h$, then $\deg(f: \overline{\alpha} \to \alpha^h) = n + n_{\overline{\alpha}}$, where $n_{\overline{\alpha}} \geq 0$ is the number of distinct vertical flaps in $\overline{P}$ that $\overline{\alpha}$ meets.

(iii) We have

$$\lambda_f(\alpha^h) = \sum_{\alpha} \frac{1}{n + n_{\overline{\alpha}}},$$

where the sum is taken over all essential pullbacks $\overline{\alpha}$ of $\alpha^h$ under $f$.

Analogous statements are true for the curve $\alpha^v$.

Proof. Figure 15 illustrates the proof. It is obvious that the Jordan curve $\alpha^h$ has exactly $n + n_h$ distinct pullbacks under $\overline{L}$. Among them, there are $n$ essential pullbacks $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$ that separate the two horizontal edges of $\overline{P}$ and thus, are isotopic to each other relative to the vertex set $\overline{V}$ of $\overline{P}$. For each $j = 1, \ldots, n$, the image $\alpha_j := \phi(\overline{\alpha}_j)$ is isotopic to $\alpha^h$. Moreover, we have $\deg(\overline{L}: \overline{\alpha}_j \to \alpha^h) = n + n_{\overline{\alpha}_j}$. The other $n_h$ pullbacks of $\alpha^h$ under $\overline{L}$ are each contained in one of the horizontal flaps and thus are peripheral in $(\overline{P}, \overline{V})$. Consequently,

$$\lambda_f(\alpha^h) = \sum_{j=1}^n \frac{1}{\deg(f: \alpha_j \to \alpha^h)} = \sum_{j=1}^n \frac{1}{\deg(\overline{L}: \overline{\alpha}_j \to \alpha^h)} = \sum_{j=1}^n \frac{1}{n + n_{\overline{\alpha}_j}}.$$ 

This completes the proof of the lemma for the curve $\alpha^h$. The proof for the curve $\alpha^v$ follow from similar considerations.

The following corollary is an immediate consequence of the previous lemma.

Corollary 5.2. Let $f = \overline{L} \circ \phi^{-1}: \overline{P} \to \overline{P}$ be a Thurston map obtained from the $(n \times n)$-Lattès map, $n \geq 2$, by gluing $n_h \geq 0$ horizontal and $n_v \geq 0$ vertical flaps. Then $\alpha^h$ is an obstruction (for $f$) if and only if $n_v = 0$, and $\alpha^v$ is an obstruction if and only if $n_h = 0$.

Proof. Let us first suppose that $n_v = 0$, that is, $\overline{P}$ does not have any vertical flaps. Then by Lemma 5.1 $\alpha^h$ has $n$ essential pullbacks under $f$ each of which is mapped onto $\alpha^h$ with degree $n$ (this is illustrated in Figure 16 in a special case). Consequently, $\lambda_f(\alpha^h) = n \cdot (1/n) = 1$, which means $\alpha^h$ is an obstruction for $f$.

If $n_v > 0$, the flapped pillow $\overline{P}$ has at least one vertical flap. Then $n_{\overline{\alpha}} > 0$ for at least one essential pullback $\overline{\alpha}$ of $\alpha^h$. Lemma 5.1 implies that $\lambda_f(\alpha^h) \leq (n - 1) \frac{1}{n} + \frac{1}{n+1} < 1$, and so $\alpha^h$ is not an obstruction for $f$.

The proof for the vertical curve $\alpha^v$ is completely analogous.

The above corollary can be read as follows: the obstruction $\alpha^h$ for the $(n \times n)$-Lattès map can be eliminated by gluing a vertical flap. Similarly, the obstruction $\alpha^v$ can be eliminated by gluing a horizontal flap. We will show in the sequel that if both of these obstructions are eliminated (that is, if there are both horizontal and vertical flaps) then no other obstructions are present and so the map $f$ is realized.

5.2. Ruling out other obstructions. Now we discuss what happens with the essential curves in $(\overline{P}, P_f)$ that are not isotopic to the horizontal curve $\alpha^h$ or the vertical curve $\alpha^v$.

Theorem 5.3. Let $f = \overline{L} \circ \phi^{-1}: \overline{P} \to \overline{P}$ be a Thurston map obtained from the $(n \times n)$-Lattès map, $n \geq 2$, by gluing $n_h \geq 0$ horizontal and $n_v \geq 0$ vertical flaps and assume that $n_h + n_v > 0$. If $\gamma \subset \overline{P} \setminus P_f$ is an essential Jordan curve that is not isotopic to either $\alpha^h$ or $\alpha^v$, then $\gamma$ is not an obstruction for $f$. 

Before we turn to the proof of this theorem, we first record how it implies Theorem 1.2 stated in the introduction.

**Proof of Theorem 1.2.** Let \( n, f, n_h \) and \( n_v \) with \( n_h + n_v > 0 \) be as in the statement. We have seen in Section 4.2 that then \( f \) has a hyperbolic orbifold. If \( n_h = 0 \) or \( n_v = 0 \), then by Corollary 5.2 the curve \( \alpha^v \) or the curve \( \alpha^h \) is an obstruction, respectively.

If \( n_h > 0 \) and \( n_v > 0 \), then \( f \) has no obstruction as follows from Corollary 5.2 and Theorem 5.3. Since \( f \) has a hyperbolic orbifold, in this case \( f \) is realized by a rational map according to Theorem 3.6.

Corollary 5.2 and Theorem 5.3 also imply that if \( n_h = 0 \), then \( \alpha^v \) is the only obstruction for \( f \) (up to isotopy rel. \( P_f \)). Similarly, \( \alpha^h \) is the only obstruction if \( n_v = 0 \).

Before we go into the details, we will give an outline for the proof of Theorem 5.3. We argue by contradiction and assume that \( f \) has an obstruction given by an essential Jordan curve \( \gamma \) in \((\mathbb{P}, P_f)\) that is isotopic to neither \( \alpha^h \) nor \( \alpha^v \) rel. \( P_f \). Then \( \lambda_f(\gamma) \geq 1 \). Let \( \gamma_1, \ldots, \gamma_k \) for some \( k \in \mathbb{N} \) be all the essential pullbacks of \( \gamma \) under \( f \), which must be isotopic to \( \gamma \) rel. \( P_f \).

Using facts about intersection numbers and the mapping properties of \( f \), one can show that for the number of essential pullbacks of \( \gamma \) we have \( k \leq n \) and that corresponding mapping degrees satisfy \( \deg(f; \gamma_j \to \gamma) \geq n \) for all \( j = 1, \ldots, k \). Since \( \lambda_f(\gamma) \geq 1 \), it follows that there are exactly \( k = n \) essential pullbacks and that \( \deg(f; \gamma_j \to \gamma) = n \) for \( j = 1, \ldots, n \).

This in turn implies that none of the essential pullbacks \( \tilde{\gamma}_j := \phi^{-1}(\gamma_j) \) of \( \gamma \) under \( \tilde{\mathcal{L}} \) goes over a flap in \( \tilde{\mathbb{P}} \). Then all the \( n \) pullbacks \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \) belong to the base \( B(\tilde{\mathbb{P}}) \) of \( \tilde{\mathbb{P}} \). This means that each \( \tilde{\gamma}_j \) can be thought of as a pullback of \( \gamma \) under the original \((n \times n)\)-Lattès map \( \mathcal{L}_n \). However, there are only \( n \) pullbacks of \( \gamma \) under \( \mathcal{L}_n \), which cross all the edges of the graph \( \mathcal{L}_n^1(\mathcal{C}) \), where \( \mathcal{C} \) is the boundary curve of \( \mathbb{P} \). Consequently, the pullbacks \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \) cross all the 1-edges in the base \( B(\tilde{\mathbb{P}}) \). It follows that one of the pullbacks \( \tilde{\gamma}_j \) must cross one of the base edges of a flap \( F \) in \( \tilde{\mathbb{P}} \), which would necessarily mean that \( \tilde{\gamma}_j \) goes over the flap \( F \). This gives the desired contradiction and Theorem 5.3 follows.

In the remainder of this section we will fill in the details for this outline. First, we establish several general facts about degrees and intersection numbers.

Let \( n \in \mathbb{N} \) and \( f: X \to Y \) be a map between two sets \( X \) and \( Y \). We say that \( f \) is at most \( n \)-to-1 if \( \# f^{-1}(y) \leq n \) for each \( y \in Y \). We say that \( f \) is \( n \)-to-1 if \( \# f^{-1}(y) = n \) for each \( y \in Y \).

**Lemma 5.4.** Let \( f: X \to Y \) be a map between two sets \( X \) and \( Y \). Suppose \( M \subset X \), \( N \subset Y \), and \( f: M \to f(M) \) is at most \( n \)-to-1 for some \( n \in \mathbb{N} \). Then \( \#(M \cap f^{-1}(N)) \leq n \#(f(M) \cap N) \).

**Figure 16.** Pullbacks of \( \alpha^h \) for a blown up \((4 \times 4)\)-Lattès map with \( n_h = 1 \) and \( n_v = 0 \).
Proof. The map $f$ sends each point in $M \cap f^{-1}(N)$ to a point in $f(M) \cap N$. Moreover, each point in $f(M) \cap N$ has at most $n$ preimages in $M \cap f^{-1}(N)$ under $f$. The statement follows. \hfill \Box

**Lemma 5.5.** Let $f : S^2 \to S^2$ be a Thurston map with $\# P_f = 4$ and $\gamma$ be an essential Jordan curve in $(S^2, P_f)$. Suppose that $\tilde{\alpha}$ is an essential Jordan curve or an arc in $(S^2, P_f)$ such that

(i) $f(\tilde{\alpha})$ and $\tilde{\alpha}$ are isotopic rel. $P_f$,
(ii) the map $f|\tilde{\alpha} : \tilde{\alpha} \to f(\tilde{\alpha})$ is at most $n$-to-1, where $n \in \mathbb{N}$,
(iii) $i(f(\tilde{\alpha}), \gamma) = \#(f(\tilde{\alpha}) \cap \gamma) > 0$.

Then $k \leq n$, where $k \in \mathbb{N}_0$ denotes the number of distinct pullbacks of $\gamma$ under $f$ that are isotopic to $\gamma$ rel. $P_f$. Moreover, if $\tilde{\alpha}$ meets a peripheral pullback of $\gamma$, then $k < n$.

Note that since $f(\tilde{\alpha})$ and $\tilde{\alpha}$ are isotopic rel. $P_f$ by assumption, $f(\tilde{\alpha})$ is of the same type as $\tilde{\alpha}$, that is, a Jordan curve or an arc in $(S^2, P_f)$.

**Proof.** Let $\gamma_1, \ldots, \gamma_k$ be the distinct pullbacks of $\gamma$ under $f$ that are isotopic to $\gamma$ rel. $P_f$. Since $f|\tilde{\alpha} : \tilde{\alpha} \to f(\tilde{\alpha})$ is at most $n$-to-1, we can apply Lemma 5.4 and conclude that

$$\#(\tilde{\alpha} \cap f^{-1}(\gamma)) \leq n \cdot \#(f(\tilde{\alpha}) \cap \gamma) = n \cdot i(f(\tilde{\alpha}), \gamma).$$

On the other hand,

$$n \cdot i(f(\tilde{\alpha}), \gamma) \geq \#(\tilde{\alpha} \cap f^{-1}(\gamma)) \geq \sum_{j=1}^k \#(\tilde{\alpha} \cap \gamma_j) \geq \sum_{j=1}^k i(\tilde{\alpha}, \gamma_j) = k \cdot i(f(\tilde{\alpha}), \gamma).$$

Since $i(f(\tilde{\alpha}), \gamma) > 0$, we see that $k \leq n$. If $\tilde{\alpha}$ meets a peripheral pullback of $\gamma$, then the second inequality in (5.1) is strict and we actually have $k < n$. \hfill \Box

The next result will lead to the strict inequality from Lemma 5.5 in the proof of Theorem 5.3

**Lemma 5.6.** As before, let $\widehat{\mathbb{P}} = \mathbb{P}$ be the blown-up $(n \times n)$-Lattès map. Suppose $\gamma := \varphi(\ell_{r/s})$ is an essential Jordan curve in $(\mathbb{P}, V)$, where $\ell_{r/s} \in \mathbb{C} \setminus \mathbb{Z}^2$ is a straight line with slope $r/s \in \hat{\mathbb{Q}} \setminus \{0, \infty\}$. Let $\bar{\gamma}$ be a pullback of $\gamma$ under $\widehat{\mathbb{P}}$. If $\bar{\gamma}$ intersects the interior of a base edge of a flap $F$ in $\widehat{\mathbb{P}}$, then $\bar{\gamma}$ also intersects the top edge of $F$.

**Proof.** Let $\gamma = \varphi(\ell_{r/s}) \subset \mathbb{P}$ and $\bar{\gamma} \subset \hat{\mathbb{P}}$ be as in the statement. As in Section 2.4, we denote by $a$, $b$, $c$, $d$ the edges of the pillow $\mathbb{P}$. Let $e \subset \mathbb{P}$ be a base edge of a flap $F$ in $\mathbb{P}$ such that $\bar{\gamma} \cap \text{int}(e) \neq \emptyset$. We will assume that $F$ is a horizontal flap. Then $\widehat{\mathbb{L}}(e) = a$ or $\widehat{\mathbb{L}}(e) = c$. We will make the further assumption that $\widehat{\mathbb{L}}(e) = a$. The other cases, when $\widehat{\mathbb{L}}(e) = c$ or when $F$ is a vertical flap, can be treated in a way that is completely analogous to the ensuing argument.

Let $e' \subset F$ be the base edge of $F$ different from $e$, and $\bar{e}$ be the top edge of $F$. Then $\widehat{\mathbb{L}}(e') = a$ and $\widehat{\mathbb{L}}(\bar{e}) = c$. Moreover, (5.2)

$$\widehat{\mathbb{L}}^{-1}(a \cup c) \cap F = e \cup e' \cup \bar{e}.$$

Let $p \in \bar{\gamma} \cap \text{int}(e)$. Then $\widehat{\mathbb{L}}(p) \in \bar{\gamma} \cap \text{int}(a)$ and so $\gamma$ meets the edge $a$. Since $\gamma = \varphi(\ell_{r/s})$ with $r/s \in \hat{\mathbb{Q}} \setminus \{0, \infty\}$, the curve $\gamma$ is in the minimal position with each edge of $\mathbb{P}$ (relative to $V$) and we have $i(\alpha^h, \gamma) = 2|r| > 0$. Therefore, we can apply Lemma 2.6 which shows that the
sets $\gamma \cap a$ and $\gamma \cap c$ are non-empty and that the points in these sets alternate on $\gamma$. Since $\widehat{\mathcal{L}}$ is a covering map from $\gamma$ onto $\gamma$, this implies that the points in the non-empty sets $\gamma \cap \widehat{\mathcal{L}}^{-1}(a)$ and $\gamma \cap \widehat{\mathcal{L}}^{-1}(c)$ alternate on $\gamma$.

Since $\gamma = g(\ell_{r/s})$, the curve $\gamma$ has transverse intersections with $\text{int}(a)$. It follows that the pullback $\gamma$ has a transverse intersection with $\text{int}(e)$ at $p$, and so $\gamma$ crosses into the interior of the flap $F$ at $p$. Therefore, if we travel along $\gamma$ starting at $p \in \gamma \cap \widehat{\mathcal{L}}^{-1}(a)$ and traverse into the interior of the flap $F$, we must meet $\widehat{\mathcal{L}}^{-1}(c)$, before we possibly exit $F$ again, which is only possible through the base edges $e \cup e' \subset \widehat{\mathcal{L}}^{-1}(a)$. Since $F \cap \widehat{\mathcal{L}}^{-1}(c) = \bar{e}$ by (5.2), this implies that the pullback $\gamma$ meets the top edge $\bar{e}$; see Figure 17. The statement follows. \hfill \Box

![Figure 17. A pullback $\gamma$ going over a horizontal flap in $\widehat{P}$.](image)

We are now ready to prove the main result of this section.

**Proof of Theorem 5.3.** Let $f: \mathbb{P} \to \mathbb{P}$ be a Thurston map as in the statement, obtained from the $(n \times n)$-Lattès map $\mathcal{L}_n$, $n \geq 2$, by gluing $n_h \geq 0$ horizontal and $n_v \geq 0$ vertical flaps, where we assume $n_h + n_v > 0$. As described in the beginning of this section, then $f = \widehat{\mathcal{L}} \circ \phi^{-1}$, where $\widehat{\mathcal{L}}: \widehat{\mathbb{P}} \to \mathbb{P}$ is a branched covering map defined by the flapped pillow $\widehat{\mathbb{P}}$ and $\phi: \widehat{\mathbb{P}} \to \mathbb{P}$ is an identifying homeomorphism as discussed in Section 4.2. Note that $\widehat{\mathbb{P}}$ has at least one flap, since $n_h + n_v > 0$. Following the notation from Section 4.2, we denote by $\widehat{a}, \widehat{b}, \widehat{c}, \widehat{d}$ the arcs in $(\widehat{\mathbb{P}}, \widehat{V})$ that under $\phi$ correspond to the edges $a, b, c, d$ of $\mathbb{P}$, respectively.

We now argue by contradiction and assume that there exists an essential Jordan curve $\gamma$ in $(\mathbb{P}, P_f) = (\mathbb{P}, V)$ that is not isotopic to $\alpha^h$ or $\alpha^v$ rel. $P_f = V$, but is an obstruction for $f$, that is, $\lambda_f(\gamma) \geq 1$. Without loss of generality, we may assume that $\gamma = \varphi(\ell_{r/s})$ for a straight line $\ell_{r/s} \subset \mathbb{C} \setminus \mathbb{Z}^2$ with slope $r/s \in \mathbb{Q}$. Since $\gamma$ is not isotopic to $\alpha^h$ or $\alpha^v$ rel. $V$, we have $r/s \neq 0$, and so $r, s \neq 0$. Then Lemma 2.5 implies

\begin{equation}
\#(a \cap \gamma) = i(a, \gamma) = |r| = i(c, \gamma) = \#(c \cap \gamma) > 0, \\
\#(b \cap \gamma) = i(b, \gamma) = s = i(d, \gamma) = \#(d \cap \gamma) > 0.
\end{equation}

(5.3)

In particular, $\gamma \subset \mathbb{P} \setminus V$ intersects the interiors of all four edges in $\mathbb{P}$.

Let $\gamma_1, \ldots, \gamma_k$ for some $k \in \mathbb{N}$ denote all the pullbacks of $\gamma$ under $f$ that are isotopic to $\gamma$ rel. $P_f$. By construction of the blown-up map, the arc $a = \phi^{-1}(a)$ consists of $n$ 1-edges in $\mathbb{P}$ each of which is homeomorphically mapped onto $a$ by $\widehat{\mathcal{L}}$. This implies that the map $f|_{a}: a \to a$ is at most $n$-to-1. By (5.3) we can apply Lemma 5.5 to $a$ and $\gamma$ and conclude that $k \leq n$.

By Lemma 5.1, the horizontal curve $\alpha^h$ has $n$ distinct pullbacks under $f$ that are isotopic to $\alpha^h$ rel. $P_f$. Since

\begin{equation}
i(\gamma, \alpha^h) = \#(\gamma, \alpha^h) = 2|r| > 0,
\end{equation}

(5.4)
we can apply Lemma 5.5 again, this time to $\gamma_j$ and $\alpha^h$ (in the roles of $\alpha$ and $\gamma$), and conclude that $n \leq \deg(f: \gamma_j \to \gamma)$ for all $j = 1, \ldots, k$.

Then

$$1 \leq \lambda_f(\gamma) = \sum_{j=1}^{k} \frac{1}{\deg(f: \gamma_j \to \gamma)} \leq k/n \leq 1.$$ 

Therefore, $k = n$ and $\deg(f: \gamma_j \to \gamma) = n$ for each $j = 1, \ldots, n$.

This shows that the curve $\gamma$ has exactly $n$ essential pullbacks under $\hat{\varphi}$ given by $\hat{\gamma}_1 := \phi^{-1}(\gamma_1), \ldots, \hat{\gamma}_n := \phi^{-1}(\gamma_n)$. Here, the isotopy classes are considered with respect to the vertex set $\hat{V}$ of $\hat{\mathbb{P}}$. We will now use the second part of Lemma 5.5 to show that none of these pullbacks goes over a flap in $\hat{\mathbb{P}}$.

**Claim.** For each flap $F$ in $\hat{\mathbb{P}}$ and each pullback $\hat{\gamma}_j$, $j = 1, \ldots, n$, we have $F \cap \hat{\gamma}_j = \emptyset$.

The ensuing argument is illustrated in Figure 17 for a horizontal flap. To see that the Claim is true, suppose some pullback $\hat{\gamma}_j$ meets a flap $F$ in $\hat{\mathbb{P}}$. We may assume that $F$ is a horizontal flap; the other case, when $F$ is vertical, can be treated by similar considerations. Then $F$ contains a peripheral pullback $\hat{\alpha}^h$ of $\alpha^h$ under $\hat{\varphi}$, which separates the union $\partial F$ of the two bases edges of $F$ from the top edge of $F$. We will first show that $\hat{\gamma}_j$ intersects $\hat{\alpha}^h$.

Note that $\partial F$ is a Jordan curve and that $\text{int}(F)$ does not contain any point from the vertex set $\hat{V}$ of $\hat{\mathbb{P}}$. It follows that the curve $\hat{\gamma}_j$ must intersect $\partial F$, because $\hat{\gamma}_j$ is essential in $(\hat{\mathbb{P}}, \hat{V})$. Since the curve $\gamma$ does not pass through $P_j = V$, its pullback $\hat{\gamma}_j$ under $\hat{\varphi}$ does not pass through any 1-vertex in $\hat{\mathbb{P}}$. Consequently, $\hat{\gamma}_j$ must meet the interior of one of the two base edges of $F$, which compose the boundary $\partial F$. Lemma 5.6 now implies that $\hat{\gamma}_j$ also meets the top edge of $F$. Therefore, $\hat{\gamma}_j$ meets the peripheral pullback $\hat{\alpha}^h$ in $F$.

It follows that $\gamma_j = \phi(\hat{\gamma}_j)$ meets the peripheral pullback $\phi(\hat{\alpha}^h)$ of $\alpha^h$ under $f$. Lemma 5.5 now implies that $n < \deg(f: \gamma_j \to \gamma) = n$, which is a contradiction. This finishes the proof of the Claim.

The Claim implies that each essential pullback $\hat{\gamma}_j$, $j = 1, \ldots, n$, belongs to the base $B(\hat{\mathbb{P}})$ of $\mathbb{P}$. By (4.2) we can identify $B(\hat{\mathbb{P}})$ with the subset $\mathbb{P} \setminus \bigcup_{e \in E} \text{int}(e)$ of the original pillow $\mathbb{P}$, where $E$ denotes the non-empty subset of edges in the embedded graph $L^{-1}_n(\mathcal{C}) \subset \mathbb{P}$ along which flaps were glued in the construction of $\hat{\mathbb{P}}$ (recall from Section 4.2 that $\mathcal{C} := a \cup b \cup c \cup d$ denotes the boundary of $\mathbb{P}$ and the graph $L^{-1}_n(\mathcal{C})$ has the vertex set $L^{-1}_n(V)$).

Under this identification, the map $\hat{\varphi}$ on $B(\hat{\mathbb{P}})$ coincides with the $(n \times n)$-Lattès map $L_n$. Thus we may view $\hat{\gamma}_1, \ldots, \hat{\gamma}_n$ as pullbacks of $\gamma$ under $L_n$ on the original pillow $\mathbb{P}$. Now $\gamma$ has exactly $n$ pullbacks under $L_n$ (see (3.4)). This implies that

$$L^{-1}_n(\gamma) = \hat{\gamma}_1 \cup \cdots \cup \hat{\gamma}_n.$$ 

Since $\gamma$ meets the interior of every edge of $\mathbb{P}$, the set $L^{-1}_n(\gamma) = \hat{\gamma}_1 \cup \cdots \cup \hat{\gamma}_n$ meets the interior of every edge in the graph $L^{-1}_n(\mathcal{C})$. This is impossible, because

$$\hat{\gamma}_1 \cup \cdots \cup \hat{\gamma}_n \subset B(\hat{\mathbb{P}}) = \mathbb{P} \setminus \bigcup_{e \in E} \text{int}(e)$$

does not meet the interior of any edge in $E \neq \emptyset$. This is a contradiction and the statement follows.
6. Essential circuit length

In order to prove our main result, Theorem \ref{main_theorem}, we need some preparation, in particular a refined version of Lemma \ref{lem:conformal_map}. We will also address the question how blowing up arcs as discussed in Section \ref{sec:blowing-up} modifies the pullbacks of a curve \(\alpha\) under natural restrictions on the blown-up arcs. First, we introduce some terminology and establish some auxiliary facts.

Let \(U \subset S^2\) be a region (that is, an open and connected set), and \(\sigma \subset S^2\) be an arc. We say that \(\sigma\) is an arc \textit{in} \(U\) \textit{ending in} \(\partial U\) if there exists an endpoint \(p\) of \(\sigma\) such that \(\sigma \setminus \{p\} \subset U\) and \(p \in \partial U\).

Let \(\mathcal{G}\) be a connected planar embedded graph in \(S^2\) and \(U\) be one of its faces. Then \(U\) is simply connected, and so we can find a homeomorphism \(\sigma : \mathbb{D} \to U\). Since we want some additional properties of \(\varphi\) here, it is easiest to equip \(S^2\) with a complex structure and choose a conformal map \(\varphi : \mathbb{D} \to U\).

Since \(\partial U\) is a finite union of edges of \(\mathcal{G}\), this set is locally connected and so the conformal map \(\varphi\) extends to a surjective continuous map \(\varphi : \mathbb{D} \to \overline{U}\) \cite[Theorem 2.1]{Pom92}. This extension has the following property: if \(\sigma\) is an arc in \(U\) ending in \(\partial U\), then \(\varphi^{-1}(\sigma)\) is an arc in \(\mathbb{D}\) ending in \(\partial \mathbb{D}\) (see \cite[Proposition 2.14]{Pom92}). Let \((e_1, e_2, \ldots, e_n)\) be a circuit in \(\mathcal{G}\) that traces the boundary \(\partial U\). Recall from Section \ref{sec:circuits} that the number \(n\) is called the circuit length of \(U\) in \(\mathcal{G}\), and each edge \(e \in \partial U\) appears exactly one or two times in the sequence \(e_1, e_2, \ldots, e_n\), depending on whether the face \(U\) lies on one or both sides of \(e\), respectively. Then there is a corresponding decomposition \(\partial \mathbb{D} = \tau_1 \cup \cdots \cup \tau_n\) of the unit circle \(\partial \mathbb{D}\) into non-overlapping subarcs \(\tau_1, \ldots, \tau_n\) of \(\partial \mathbb{D}\) such that \(\varphi\) is a homeomorphism of \(\tau_m\) onto \(e_m\) for each \(m = 1, \ldots, n\). For given \(\mathcal{G}\) and \(U\) we fix, once and for all, such a map \(\varphi = \varphi_{\mathcal{G}, U}\) from \(\mathbb{D}\) onto \(\overline{U}\).

Let \(0 < \epsilon < 1\). We say that a Jordan curve \(\beta \subset U\) is an \(\epsilon\)-boundary of \(U\) with respect to \((\text{wrt.})\) \(\mathcal{G}\) if \(\beta' := \varphi^{-1}(\beta) \subset A_\epsilon := \{z \in \mathbb{C} : 1 - \epsilon < |z| < 1\}\), and \(\beta'\) separates 0 from \(\partial \mathbb{D}\). So \(\beta'\) is a core curve in the annulus \(A_\epsilon\).

Let \(f : S^2 \to S^2\) be a Thurston map and \(e\) be an arc in \((S^2, P_f)\). We can naturally view the set \(\mathcal{G} := f^{-1}(e)\) as a planar embedded graph with the vertex set \(f^{-1}(\partial e)\) and the edges given by the lifts of \(e\) under \(f\). Note that \(\mathcal{G}\) is bipartite.

**Lemma 6.1.** Let \(f : S^2 \to S^2\) be a Thurston map, \(e\) be an arc and \(\gamma\) be a Jordan curve in \((S^2, P_f)\) with \(#(e \cap \gamma) = i(e, \gamma)\). Suppose that \(\mathcal{H}\) is a connected subgraph of \(\mathcal{G} = f^{-1}(e)\) and \(U\) is a face of \(\mathcal{H}\). Let \(2n\) with \(n \in \mathbb{N}\) be the circuit length of \(U\) in \(\mathcal{H}\). Then for each \(0 < \epsilon < 1\) there exists an \(\epsilon\)-boundary \(\beta\) of \(U\) wrt. \(\mathcal{H}\) such that \(#(\beta \cap f^{-1}(\gamma)) = 2n \cdot i(e, \gamma)\).

Note that the circuit length of \(U\) in \(\mathcal{H}\) is even since \(\mathcal{H}\) is a bipartite graph.

**Proof.** Let \(s := i(e, \gamma) \in \mathbb{N}_0\). Then \(e\) and \(\gamma\) have exactly \(s\) distinct points in common, say \(p_1, \ldots, p_s \in e \cap \gamma\). Each point \(p_j\) lies in \(\text{int}(e)\), because \(\partial e \subset P_f\) and \(\gamma \subset S^2 \setminus P_f\). Since \(e\) and \(\gamma\) are in minimal position, they meet transversely (see Lemma \ref{lem:minimal_position}), that is, if we travel along \(\gamma\) towards one of the points \(p_j\) (according to some orientation of \(\gamma\)), then near \(p_j\) we stay on one side of \(e\), but cross over to the other side of \(e\) if we pass \(p_j\).

This implies that we can find disjoint subarcs \(\sigma_1, \ldots, \sigma_s\) of \(\gamma\) such that each arc \(\sigma_j\) contains \(p_j\) in its interior, but contains no other point in \(e \cap \gamma\). Moreover, \(p_j\) splits \(\sigma_j\) into two non-overlapping subarcs \(\sigma_j^L\) and \(\sigma_j^R\) with the common endpoint \(p_j\) so that with some fixed orientation of \(e\), the arc \(\sigma_j^L\) lies to the left and \(\sigma_j^R\) lies to the right of \(e\). Note that if \(\gamma' := \gamma \setminus (\sigma_1 \cup \cdots \cup \sigma_s)\), then \(e \cap \gamma' = \emptyset\).
Let \((e_1, \ldots, e_{2n})\) be a circuit in \(\mathcal{H}\) that traces the boundary \(\partial U\). As we already pointed out, the number of edges in the circuit is even, because \(\mathcal{H}\) is a bipartite graph. With suitable orientation of each arc \(e_m\), the face \(U\) lies on the left of \(e_m\). If an arc appears twice in the list \(e_1, \ldots, e_{2n}\), then it will carry opposite orientations in its two occurrences. We fix a map 
\[ \varphi = \varphi_{\mathcal{H}, U} : \mathbb{D} \to \overline{U} \]
as discussed in the beginning of this section.

We want to investigate the set \(f^{-1}(\gamma) \cap U\) near \(\partial U\). Note that \(f\) is a homeomorphism of \(e_m\) onto \(e\); actually, \(f\) is a homeomorphism on a suitable Jordan region that contains \(e_m\) as a crosscut. This means that we can pullback the local picture near points in \(e \cap \gamma\) to a similar local picture for points in \(e_m \cap f^{-1}(\gamma)\). So if we choose the arcs \(\sigma_j\) small enough, as we may assume, and pull them back by \(f\), then it is clear that \(f^{-1}(\gamma) \cap U\) can be represented in the form
\[ f^{-1}(\gamma) \cap U = K \cup \bigcup_{m=1}^{2n} \bigcup_{j=1}^{s} \sigma_{m,j}, \]
where \(K\) has positive distance to \(\partial U = e_1 \cup \cdots \cup e_{2n}\) (with respect to some base metric on \(S^2\)). Moreover, each \(\sigma_{m,j}\) is an arc in \(U\) ending in \(e_m \in \partial U\) such that \(f\) is a homeomorphism from \(\sigma_{m,j}\) to \(\sigma^L_j\) or \(\sigma^R_j\) depending on whether \(f : e_m \to e\) is orientation-preserving or orientation-reversing. If we remove from each arc \(\sigma_{m,j}\) its endpoint on \(e_m\), then the half-open arcs obtained are all disjoint. Two of these arcs share an endpoint precisely when they arise from edges \(e_m\) and \(e_{m'}\) with the same underlying set, but with opposite orientations. In this case, \(f\) sends one of them to \(\sigma^L_j\), and the other one to \(\sigma^R_j\) for some \(j \in \{1, \ldots, s\}\).

Since \(K\) has positive distance to \(\partial U\), it is clear that if \(\beta\) is an \(\epsilon\)-boundary of \(U\) wrt. \(\mathcal{H}\) for \(\epsilon > 0\) small enough (as we may assume), then \(\beta \cap K = \emptyset\). So in order to control \(#(\beta \cap f^{-1}(\gamma))\), we have to worry only about the intersections of \(\beta\) with the arcs \(\sigma_{m,j}\).

Note that there are exactly \(2n \cdot s = 2n \cdot i(e, \gamma)\) of these arcs. If we pull them back by the map \(\varphi\), then we obtain pairwise disjoint arcs in \(\mathbb{D}\) ending in \(\partial \mathbb{D}\). The statement now follows from the following fact whose precise justification we leave to the reader: if \(\alpha_1, \ldots, \alpha_M\) with \(M \in \mathbb{N}\) are pairwise disjoint arcs in \(\mathbb{D}\) ending in \(\partial \mathbb{D}\), then for each \(0 < \epsilon < 1\) there exists a core curve \(\beta'\) of the annulus \(A_\epsilon = \{ z \in \mathbb{C} : 1 - \epsilon < |z| < 1 \}\) such that \(#(\beta' \cap (\alpha_1 \cup \cdots \cup \alpha_M)) = M\). \(\square\)

Now we are ready to provide a refined version of Lemma 5.5

**Lemma 6.2.** Let \(f : S^2 \to S^2\) be a Thurston map with \(#P_f = 4\), \(\alpha\) and \(\gamma\) be essential Jordan curves in \((S^2, P_f)\), \(c\) be a core arc of \(\alpha\), and assume that \(#(c \cap \gamma) = i(c, \gamma) > 0\).

Let \(\mathcal{H}\) be a connected subgraph of \(\mathcal{G} := f^{-1}(c)\), and \(U\) be a face of \(\mathcal{H}\) such that for small enough \(\epsilon > 0\) each \(\epsilon\)-boundary \(\beta\) of \(U\) wrt. \(\mathcal{H}\) is isotopic to \(\alpha\) rel. \(P_f\). Let \(2n\) with \(n \in \mathbb{N}\) be the circuit length of \(U\) in \(\mathcal{H}\).

Then \(k \leq n\), where \(k\) denotes the number of pullbacks of \(\gamma\) under \(f\) that are isotopic to \(\gamma\) rel. \(P_f\). Moreover, if \(\partial U \subset \mathcal{H}\) meets a peripheral pullback of \(\gamma\) under \(f\), then \(k < n\).

**Proof.** Let \(\gamma_1, \ldots, \gamma_k\) be the pullbacks of \(\gamma\) under \(f\) that are isotopic to \(\gamma\) rel. \(P_f\). Then by Lemma 6.1, for sufficiently small \(\epsilon > 0\), we can find an \(\epsilon\)-boundary \(\beta\) of \(U\) wrt. \(\mathcal{H}\) such that \(\beta \sim \alpha\) rel. \(P_f\) and
\[ #(\beta \cap f^{-1}(\gamma)) \leq 2n \cdot i(c, \gamma) = n \cdot i(\alpha, \gamma), \]
where the last equality follows from Lemma 2.7. Hence, we have
\[ n \cdot i(\alpha, \gamma) \geq #(\beta \cap f^{-1}(\gamma)) \geq \sum_{j=1}^{k} #(\beta \cap \gamma_j) \geq \sum_{j=1}^{k} i(\beta, \gamma_j) = k \cdot i(\alpha, \gamma). \]

Since \(i(\alpha, \gamma) = 2 \cdot i(c, \gamma) > 0\), we conclude that \(k \leq n\), as desired.
To see the second statement, we have to revisit the proof of Lemma 6.1. There we identified $2n \cdot i(c, \gamma) = n \cdot i(\alpha, \gamma)$ distinct arcs $\sigma$ in $U$ ending in $\partial U$. They were subarcs of $f^{-1}(\gamma)$ and accounted for all possible intersections of $\beta$ with $f^{-1}(\gamma)$ for sufficiently small $\epsilon > 0$. With suitable choice of $\beta$ each of these arcs $\sigma$ gave precisely one such intersection point. Now if a peripheral pullback $\tilde{\gamma}$ of $\gamma$ under $f$ meets $\partial U$, then one of these arcs $\sigma$ is a subarc of $\tilde{\gamma}$ (here it is important that the intersection of $c$ and $\gamma$ is necessarily transverse). It follows that the second inequality in (6.1) must be strict and so $k < n$. \qed

For the rest of this section, we fix a Thurston map $f: S^2 \to S^2$ with $\#P_f = 4$, an essential Jordan curve $\alpha$ in $(S^2, P_f)$, and core arcs $a$ and $c$ of $\alpha$ that lie in different components of $S^2 \setminus \alpha$. We can view the set $G := f^{-1}(a \cup c)$ as a planar embedded graph in $S^2$ with the set of vertices $f^{-1}(P_f)$ and the edge set consisting of the lifts of $a$ and $c$. Then $G$ is a bipartite graph.

We illustrate the above setup in Figure 18. Here, the right sphere shows two core arcs $a$ and $c$ of a Jordan curve $\alpha$ in $(S^2, P_f)$. The lifts of $a$ and $c$ under a Thurston map $f$ are shown in blue and magenta colors, respectively, on the left sphere. The points marked in black indicate the four postcritical points of $f$.

\begin{lemma}
Let $\gamma$ be a Jordan curve in $(S^2, P_f)$ with $i(\alpha, \gamma) > 0$. Suppose $\gamma$ is in minimal position with the core arcs $a$ and $c$, and let $\tilde{\gamma}$ be a pullback of $\gamma$ under $f$. Then the sets $f^{-1}(a) \cap \tilde{\gamma}$ and $f^{-1}(c) \cap \tilde{\gamma}$ are non-empty and their points alternate on $\tilde{\gamma}$. \qed
\end{lemma}

\begin{proof}
We may assume that our sphere $S^2$ is a square pillow $P$ and $a$ and $c$ are the horizontal edges of the pillow. Since $f: \tilde{\gamma} \to \gamma$ is a covering map of degree $d = \deg(f: \tilde{\gamma} \to \gamma) > 0$, the statement now follows from Lemma 2.6. \qed
\end{proof}

Let $U$ be the unique connected component of $S^2 \setminus (a \cup c)$. Then $U$ is a topological annulus and $\alpha$ is its core curve. The connected components $\tilde{U}$ of $f^{-1}(U) = S^2 \setminus G$ are precisely the complementary components of $\tilde{G} := f^{-1}(a \cup c)$ in $S^2$. It easily follows from the Riemann-Hurwitz formula, that each $\tilde{U}$ is a topological annulus, and that $f: \tilde{U} \to U$ is a covering map. Each such annulus contains precisely one pullback $\tilde{\alpha}$ of $\alpha$ under $f$; see Figure 18 for an illustration. We call $\tilde{U}$ essential or peripheral, depending on whether $\tilde{\alpha}$ is essential or peripheral in $(S^2, P_f)$.

Each boundary $\partial \tilde{U}$ has exactly two connected components. One of them is mapped to $a$ and the other to $c$ by $f$; accordingly, we denote them by $\partial_a\tilde{U}$ and $\partial_c\tilde{U}$, respectively. Then we have $\partial \tilde{U} = \partial_a\tilde{U} \cup \partial_c\tilde{U}$, $\partial_a\tilde{U} = f^{-1}(a) \cap \partial \tilde{U}$, and $\partial_c\tilde{U} = f^{-1}(c) \cap \partial \tilde{U}$.

The sets $\partial_a\tilde{U}$ and $\partial_c\tilde{U}$ are subgraphs of $\tilde{G}$. Since $\tilde{U}$ is a connected subset of $S^2 \setminus \tilde{G} \subset S^2 \setminus \partial_a\tilde{U}$, there exists a unique face $V_a$ of $\partial_a\tilde{U}$ (considered as a subgraph of $\tilde{G}$) such that $\tilde{U} \subset V_a$. Similarly, there exists a unique face $V_c$ of $\partial_c\tilde{U}$ with $\tilde{U} \subset V_c$. By definition, the circuit length of $\partial_a\tilde{U}$ or of $\partial_c\tilde{U}$ is the circuit length of $V_a$ in $\partial_a\tilde{U}$ or $V_c$ in $\partial_c\tilde{U}$, respectively.

Then the following statement is true.

\begin{lemma}
The circuit lengths of $\partial_a\tilde{U}$ and $\partial_c\tilde{U}$ are both equal to $2 \cdot \deg(f: \tilde{U} \to U)$.
\end{lemma}

We call the identical circuit lengths of $\partial_a\tilde{U}$ and $\partial_c\tilde{U}$ the circuit length of $\tilde{U}$ (for fixed $f$, $\alpha$, $a$, and $c$).
**Figure 18.** A Thurston map $f$. The right sphere shows a Jordan curve $\alpha$ in $(S^2, P_f)$ and two core arcs $a$ and $c$. The right sphere shows the pullbacks of $\alpha$ under $f$ and the planar embedded graph $\mathcal{G} = f^{-1}(a \cup c)$.

**Proof.** Let $2n$ with $n \in \mathbb{N}$ be the circuit length of $\partial_a \widetilde{U}$ and $d := \deg(f; \widetilde{U} \to U)$. We will prove that $2n = 2d$, which implies the first part of the statement.

Let $W$ be a Jordan region that contains $a$ as a crosscut and satisfies $W \cap c = \emptyset$. Denote by $W^+$ and $W^-$ the two components of $W \setminus a$. Then $W^+, W^- \subset U$.

Each connected component $\widetilde{W}$ of $f^{-1}(W)$ is a Jordan region and $f|\widetilde{W}$ is a homeomorphism of $\widetilde{W}$ onto $W$. It follows that each component $\widetilde{W}$ contains exactly one lift $\widetilde{a}$ of $a$ under $f$ as a crosscut. Moreover, the two components of $\widetilde{W} \setminus \widetilde{a}$ are mapped homeomorphically onto the components $W^+$ and $W^-.$

Since $f; \widetilde{U} \to U$ is a $d$-to-$1$ covering map, $\widetilde{U}$ contains exactly $2d$ components of the set $f^{-1}(W^+ \cup W^-)$, namely $d$ components of $f^{-1}(W^+)$ and $d$ components of $f^{-1}(W^-)$. Each such component $\widetilde{W}^\pm$ contains exactly one edge of $\partial_a \widetilde{U}$ its boundary. Let $(e_1, \ldots, e_{2n})$ be a circuit in $\partial_a \widetilde{U}$ that traces the boundary $\partial V_a$, where $V_a$ is the unique face of $\partial_a \widetilde{U}$ with $\widetilde{U} \subset V_a$. Note that each edge $e \in \partial_a \widetilde{U}$ appears twice in the list $e_1, \ldots, e_{2n}$ if and only if $e$ is on the boundary of a component $\widetilde{W}^+$ of $\widetilde{U} \cap f^{-1}(W^+)$ and a component $\widetilde{W}^-$ of $\widetilde{U} \cap f^{-1}(W^-)$. Otherwise, the edge $e$ appears in the list $e_1, \ldots, e_{2n}$ only once, and it is on the boundary of exactly one component of $\widetilde{U} \cap f^{-1}(W^+ \cup W^-)$. Consequently, the circuit length $2n$ coincides with the total number of components of $f^{-1}(W^+ \cup W^-)$ in $\widetilde{U}$, which is $2d$. The statement follows. □

Suppose $\widetilde{U}$ is an essential component of $f^{-1}(U)$. We consider a circuit $\mathcal{H}$ in $\partial \widetilde{U} \subset \mathcal{G}$. Since $\widetilde{U}$ is a connected set in $S^2 \setminus \mathcal{G} \subset S^2 \setminus \mathcal{H}$, there exists a unique face $V$ of $\mathcal{H}$ (considered as a subgraph of $\mathcal{G}$) such that $\widetilde{U} \subset V$. By definition, for $0 < \epsilon < 1$, an $\epsilon$-boundary $\beta$ of $\widetilde{U}$ wrt. $\mathcal{H}$ is an $\epsilon$-boundary of $V$ wrt. $\mathcal{H}$. This is an abuse of terminology, because even for small $\epsilon > 0$ such an $\epsilon$-boundary $\beta$ may not lie in $\widetilde{U}$, but it is convenient in the following. Note that for small enough $\epsilon > 0$ such $\epsilon$-boundaries for fixed $\mathcal{H}$ have the same isotopy type rel. $P_f$.

By definition the essential circuit length of $\widetilde{U}$ is the minimal length of all circuits $\mathcal{H}$ in $\partial \widetilde{U}$ such that for all small enough $\epsilon > 0$ each $\epsilon$-boundary of $\widetilde{U}$ wrt. $\mathcal{H}$ is isotopic to a core curve of $\widetilde{U}$ rel. $P_f$. As we will see momentarily, $\partial_a \widetilde{U}$ and $\partial_e \widetilde{U}$ are such circuits, and so the essential circuit length of $\widetilde{U}$ is well-defined. We call a circuit $\mathcal{H}$ in $\partial \widetilde{U}$ that realizes the essential circuit length an essential circuit of $\widetilde{U}$.

**Lemma 6.5.** We have the inequality

\[
\text{circuit length of } \widetilde{U} \geq \text{essential circuit length of } \widetilde{U}.
\]
For example, consider the annulus $\tilde{U}$ containing the pullback $\tilde{\alpha}$ in Figure 18. Then the circuit length of $\tilde{U}$ equals 6, while the essential circuit length of $\tilde{U}$ equals 4.

**Proof.** Consider $\partial_a\tilde{U}$ as a circuit in $\partial \tilde{U}$. Let $V_a$ be the unique face of $\partial_a\tilde{U}$ that contains $\tilde{U}$. Then, for sufficiently small $\epsilon > 0$, each $\epsilon$-boundary $\beta$ of $V_a$ wrt. $\partial_a\tilde{U}$ necessarily separates $\partial_a\tilde{U}$ and $\partial_c\tilde{U}$, and is thus a core curve of $\tilde{U}$. The statement follows. □

Let $\tilde{f}$ be a Thurston map obtained from $f$ by blowing up some set of arcs $E$ in $S^2 \setminus f^{-1}(P_f)$. Under certain natural assumptions on the arcs in $E$, we want to describe the components of $\tilde{f}^{-1}(U)$ and their properties in terms of the components of $f^{-1}(U)$. We first formulate suitable conditions that allow such a comparison.

**Definition 6.6** ($\alpha$-restricted blow-up conditions). Let $f: S^2 \to S^2$ be a Thurston map with $\#P_f = 4$, $\alpha$ be an essential Jordan curve in $(S^2, P_f)$, and $a$ and $c$ be core arcs of $\alpha$ that lie in different components of $S^2 \setminus \alpha$. Suppose $E \neq \emptyset$ is a finite set of arcs in $(S^2, f^{-1}(P_f))$ satisfying the blow-up conditions, that is, the interiors of the arcs in $E$ are disjoint and $f: e \to f(e)$ is a homeomorphism for each $e \in E$.

We say that $E$ satisfies the $\alpha$-restricted blow-up conditions if

\[
i(f(e), \alpha) = \#(f(e) \cap \alpha) = 1 \quad \text{and} \quad f(\text{int}(e)) \cap a = \emptyset = f(\text{int}(e)) \cap c\]

for each $e \in E$.

In other words, for each $e \in E$ the arc $f(e)$ is in minimal position with respect to $\alpha$ and intersects $\alpha$ only once, and $f(\text{int}(e)) = \text{int}(f(e))$ belongs to the annulus $U = S^2 \setminus (a \cup c)$. Note that the endpoints of $f(e)$ lie in $P_f \subset a \cup c = \partial U$; see Figure 19 for an illustration.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure19.png}
\caption{The Thurston map $f$ from Figure 18 and a set $E = \{e_1, e_2\}$ of arcs in $(S^2, f^{-1}(P_f))$ satisfying the $\alpha$-restricted blow-up conditions.}
\end{figure}

Condition (6.2) is somewhat artificial, because it depends on the choices of $a$ and $c$. One can show that up to isotopy it can be replaced by the more natural condition $i(f(e), \alpha) = 1$ for all $e \in E$. Since the justification of this claim involves some topological machinery that is beyond the scope of the paper, we prefer to work with (6.2).

**Lemma 6.7.** Let $f: S^2 \to S^2$ be a Thurston map with $\#P_f = 4$, $\alpha$ be an essential Jordan curve in $(S^2, P_f)$, and $a$ and $c$ be core arcs of $\alpha$ that lie in different components of $S^2 \setminus \alpha$. 

\[
\text{Figure 19. The Thurston map } f \text{ from Figure 18 and a set } E = \{e_1, e_2\} \text{ of arcs in } (S^2, f^{-1}(P_f)) \text{ satisfying the } \alpha\text{-restricted blow-up conditions.}
\]
Figure 20. A Thurston map $\tilde{f}$ obtained from the Thurston map $f$ in Figures 18 and 19 by blowing up the arcs $e_1$ and $e_2$ with multiplicities $m_{e_1} = 2$ and $m_{e_2} = 1$, respectively.

Suppose the set of arcs $E$ in $(S^2, f^{-1}(P_f))$ satisfies the $\alpha$-restricted blow-up conditions and we are given multiplicities $m_e \in \mathbb{N}$ for $e \in E$.

Then the Thurston map $\tilde{f}$ obtained from $f$ by blowing up each arc $e \in E$ with multiplicity $m_e$ can be constructed so that it satisfies the following conditions:

(i) $\mathcal{G} = f^{-1}(a \cup c)$ is a subgraph of $\tilde{\mathcal{G}} := \tilde{f}^{-1}(a \cup c)$.

(ii) Each complementary annulus $\tilde{U}$ of $\tilde{\mathcal{G}}$ is contained in a unique complementary annulus $\tilde{\tilde{U}}$ of $\tilde{\mathcal{G}}$. Moreover, the assignment $\tilde{U} \mapsto \tilde{\tilde{U}}$ is a bijection between the complementary annuli of $\tilde{\mathcal{G}}$ and of $\mathcal{G}$.

Let $\tilde{U}$ and $\tilde{\tilde{U}}$ be corresponding annuli as in (ii). Then the following statements are true:

(iii) The core curves of $\tilde{U}$ and of $\tilde{\tilde{U}}$ are isotopic rel. $P_f = P_{\tilde{f}}$. In particular, $\tilde{U}$ is essential if and only if $\tilde{\tilde{U}}$ is essential.

(iv) The essential circuit lengths of $\tilde{U}$ and $\tilde{\tilde{U}}$ are the same. Moreover, if $\mathcal{H}$ is an essential circuit for $\tilde{U}$, then $\mathcal{H}$ is an essential circuit for $\tilde{\tilde{U}}$. In particular, $\mathcal{H} \subset \partial \tilde{\tilde{U}} \subset \partial \tilde{U}$.

To visualize the lemma consider the Thurston map $\tilde{f}$, see Figure 20, obtained from the Thurston map $f$ from Figure 18 by blowing up the arcs $e_1$ and $e_2$ in Figure 19. Here, the lifts of $a$ and $c$ under the blown-up map $\tilde{f}$ are shown in blue and magenta colors, respectively, on the left sphere. Comparing Figures 18 and 20 we can verify the conditions in the lemma.

Proof. Under the given assumptions, for each $e \in E$ the set $\text{int}(e)$ belongs to a unique annulus $\tilde{U}$ obtained as a complementary component of $\mathcal{G} = f^{-1}(a \cup c)$. Then one endpoint of $e$ is in $\partial_a \tilde{U} = f^{-1}(a) \cap \partial \tilde{U}$ and the other one in $\partial_c \tilde{U} = f^{-1}(c) \cap \partial \tilde{U}$. In the blow-up construction described in Section 4.1 we can choose the open Jordan region $W_e$ so that $W_e \subset \tilde{U}$ for each $e \in E$ (of course, here the face $\tilde{U}$ depends on $e$). Now we make choices of the subsequent ingredients in the blow-up construction as discussed in Section 4.1. That is, for each fixed arc $e \in E$, we choose a closed Jordan region $D_e$ inside $W_e$, which is subdivided into $m = m_e$ components $D_{e1}^1, \ldots, D_{e1}^m$. In addition, we also choose a pseudo-isotopy $h: S^2 \times [0,1] \to S^2$ satisfying conditions (B1)–(B4) as well as homeomorphisms $\varphi_k: D_k^e \to S^2$, $k = 1, \ldots, m = m_e$, satisfying conditions (C1) and (C2). Let $\tilde{f}$ be the Thurston map obtained by blowing up the arcs $e \in E$ with the multiplicities $m_e$ according to these choices. We claim that $\tilde{f}$ satisfies all the conditions in the statement.
It immediately follows from \([B3]\) and the definition of \(\tilde{f}\) that \(G = f^{-1}(a \cup c)\) is a subgraph of \(\tilde{G} := \tilde{f}^{-1}(a \cup c)\), and so statement (i) is true.

We have \(\tilde{G} \setminus G \subset \bigcup_{e \in E} D_e\). Condition \([C1]\) now implies that, for each \(e \in E\) and each \(k = 1, \ldots, m = m_e\), the set \(\tilde{G} \cap D_{e}^{k}\) consists of two disjoint edges, one of which is homeomorphically mapped onto \(a\) and the other one onto \(c\) by \(\tilde{f}\). We will call these edges a-sticks and c-sticks, respectively. Each disc \(D_e\) contains exactly \(m_e\) a-sticks, which have a common endpoint in \(\partial e \cap f^{-1}(a)\), and exactly \(m_e\) c-sticks with a common endpoint in \(\partial e \cap f^{-1}(c)\). The edge set of \(\tilde{G}\) consists of all the edges of \(G\) and all the a-sticks and c-sticks.

Each complementary component \(\tilde{U}\) of \(\tilde{G}\) is equal to a unique complementary component \(\tilde{U}\) of \(G\) with all the a- and c-sticks removed that are contained in \(\text{cl}(\tilde{U})\). Statement (ii) follows. Furthermore, since \(P_f \cap \tilde{U} = \emptyset\), statement (iii) follows as well.

To prove (iv) let \(\tilde{U}\) and \(\tilde{U}\) be corresponding annuli as in (ii). Viewing \(\partial \tilde{U}\) and \(\partial \tilde{U}\) as planar embedded graphs, we see that \(\partial \tilde{U}\) is a subgraph of \(\partial \tilde{U}\). The additional edges of \(\partial \tilde{U}\) are exactly the a- and c-sticks contained in \(\text{cl}(\tilde{U})\). It follows from the definition that the essential circuit length of \(\tilde{U}\) is greater than or equal to the essential circuit length of \(\tilde{U}\).

Let \(\mathcal{H}\) be an essential circuit for \(\tilde{U}\) and suppose \(\mathcal{H}\) contains an a- or c-stick \(\sigma\). Then one of the endpoints of \(\sigma\) has vertex degree 1 in \(\partial \tilde{U}\), and so \(\sigma\) must appear in two consecutive positions in the circuit \(\mathcal{H}\). Omitting these two occurrences of \(\sigma\) from \(\mathcal{H}\) we get a shorter circuit \(\mathcal{H}'\) in \(\partial \tilde{U}\) such that for all small enough \(\epsilon > 0\) each \(\epsilon\)-boundary of \(\tilde{U}\) is shown to be a core curve of \(\tilde{U}\) rel. \(P_f = P_f\). This contradicts the choice of \(\mathcal{H}\), and it follows that \(\mathcal{H}\) does not contain any a- or c-sticks. Consequently, \(\mathcal{H} \subset \partial \tilde{U} \subset \partial \tilde{U}\), and the definition of the essential circuit length together with (iii) imply that \(\mathcal{H}\) is an essential circuit for \(\tilde{U}\). Statement (iv) follows.

7. Eliminating obstructions by blowing up arcs

The goal of this section is to show that the blow-up surgery can be applied to an obstructed Thurston map \(f\) with four postcritical points in such a way that the resulting map \(\tilde{f}\) is realized by a rational map. The precise formulation is given in Theorem 1.1. We will prove this statement by contradiction. For this we assume that \(\tilde{f}\) has an obstruction, and will carefully analyze some related mapping degrees. This leads to a very tight situation, where in some inequalities we actually have equality. From this we want to conclude that \(f\) has a parabolic orbifold, in contradiction to our hypothesis in Theorem 1.1. We first formulate two related criteria for parabolicity.

**Lemma 7.1.** Let \(f:S^2 \to S^2\) be a Thurston map with \(#P_f \geq 4\). If \(f^{-1}(P_f) \subset C_f \cup P_f\), then \(f\) has a parabolic orbifold.

**Proof.** This follows from \([DH93\), Lemma 2]. For the convenience of the reader we provide the simple proof. Let \(d = \deg(f) \geq 2\). Note that \(f^{-1}(P_f) \subset (C_f \setminus P_f) \cup P_f\) by our hypotheses.

Each point \(p \in S^2\) has precisely \(d\) preimages counting multiplicities, that is,

\[
d = \sum_{q \in f^{-1}(p)} \deg(f, q).
\]
Furthermore, since \( C_f \subset f^{-1}(P_f) \) and \( \deg(f, q) = 1 \) if and only if \( q \notin C_f \), the Riemann-Hurwitz formula implies
\[
\#(C_f \setminus P_f) \leq \#C_f \leq \sum_{c \in C_f} (\deg(f, c) - 1) = \sum_{q \notin f^{-1}(P_f)} (\deg(f, q) - 1)
= 2d - 2.
\]

It follows that
\[
d \cdot \#P_f = \sum_{q \notin f^{-1}(P_f)} \deg(f, q) = \sum_{q \notin f^{-1}(P_f)} (\deg(f, q) - 1) + \#f^{-1}(P_f)
= (2d - 2) + \#f^{-1}(P_f) \leq (2d - 2) + \#(C_f \setminus P_f) + \#P_f
\leq 4(d - 1) + \#P_f.
\]

Hence \((d - 1) \cdot \#P_f \leq 4(d - 1)\), and so \(4 \leq \#P_f \leq 4\). This implies \(\#P_f = 4\) and that all the previous inequalities must be equalities. In particular, \(\#(C_f \setminus P_f) = 2d - 2\), which shows that all critical points of \(f\) must be simple and none of them belongs to \(P_f\).

Now define a function \(\beta : S^2 \to \mathbb{N}\) by setting \(\beta(p) = 2\) for \(p \in P_f\) and \(\beta(p) = 1\) for \(p \in S^2 \setminus P_f\). For the ramification function \(\alpha_f\) of \(f\) (see Section 3), we have \(\alpha_f(p) = 1\) for \(p \in S^2 \setminus P_f\) and \(\alpha_f(p) \geq 2\) for \(p \in P_f\). Hence \(\beta \leq \alpha_f\).

On the other hand, it follows from our considerations that \(\deg(f, p) \cdot \beta(p)\) divides \(\beta(f(p))\) for all \(p \in S^2\). Hence we also have \(\alpha_f \leq \beta\) (see [BM17, Proposition 2.8 (iii)]). We conclude that \(\alpha_f = \beta\). This implies that the Euler characteristic (see (3.1)) of the orbifold \(\mathcal{O}_f\) associated with \(f\) is equal to
\[
\chi(\mathcal{O}_f) = 2 - \sum_{p \in P_f} \left(1 - \frac{1}{\beta(p)}\right) = 2 - (1/2 + 1/2 + 1/2 + 1/2) = 0.
\]

Hence \(f\) has a parabolic orbifold. \(\square\)

We will apply the previous lemma to deduce the following more technical result.

**Lemma 7.2.** Let \(f : S^2 \to S^2\) be a Thurston with \(\#P_f = 4\). Suppose there exists a Jordan curve \(\alpha\) in \((S^2, P_f)\) such that the following conditions are true:

(i) \(\alpha\) is an obstruction for \(f\).

(ii) \(\alpha\) has no peripheral pullbacks under \(f\).

(iii) If choose core arcs \(a\) and \(c\) of \(\alpha\) in different components of \(S^2 \setminus \alpha\) and consider the graph \(\mathcal{G} = f^{-1}(a \cup c)\), then \(\mathcal{G}\) has precisely \(n \in \mathbb{N}\) essential complementary components \(U_1, \ldots, U_n\) with core curves isotopic to \(\alpha\) rel. \(P_f\). Moreover, we assume the essential circuit length of \(U_j\) is equal to \(2n\) for each \(j = 1, \ldots, n\).

Then \(f\) has a parabolic orbifold.

**Proof.** Each \(U_j\) contains precisely one pullback \(\alpha_j\) of \(\alpha\) under \(f\). The curves \(\alpha_1, \ldots, \alpha_n\) are all the pullbacks of of \(\alpha\) under \(f\). Then it follows from our assumptions that
\[
2 \deg(f; \alpha_j \to \alpha) = 2 \deg(f; U_j \to U)
= \text{circuit length of } U_j
\geq \text{essential circuit length of } U_j
= 2n,
\]
and so \( \deg(f: \alpha_j \to \alpha) \geq n \) for \( j = 1, \ldots, n \). On the other hand, \( \alpha \) is an obstruction for \( f \), and so
\[
1 \leq \lambda_f(\alpha) \leq \sum_{j=1}^{n} \frac{1}{\deg(f: \alpha_j \to \alpha)} \leq n/n = 1.
\]
It follows that we have equality in all previous inequalities. In particular,
\[
\text{circuit length of } U_j = \text{essential circuit length of } U_j = 2n
\]
for \( j = 1, \ldots, n \).

Now let \( v \in f^{-1}(P_f) \) be a vertex of \( \mathcal{G} \). If \( v \) is incident with two or more edges in \( \mathcal{G} \), then \( v \in C_f \).

Otherwise, \( v \) is the endpoint of precisely one edge \( e \) in \( \mathcal{G} \), and so \( \deg_{\mathcal{G}}(v) = 1 \). We claim that then \( v \in P_f \); to see this, we argue by contradiction and assume that \( v \notin P_f \). Since \( \alpha \) has no peripheral preimages, we have
\[
e \in \mathcal{G} = \bigcup_{j=1}^{n} \partial U_j,
\]
and so \( e \in \partial U_j \) for some \( U_j \). Then \( e \) is contained in a circuit \((e_1, \ldots, e_{2n})\) of length \( 2n \) that traces one of the boundary components of \( \partial U_j \). Since \( \deg_{\mathcal{G}}(v) = 1 \), the circuit must traverse \( e \) twice with opposite orientation, that is, the edge \( e \) appears precisely in two consecutive entries in the cycle \((e_1, \ldots, e_{2n})\). Erasing these two occurrences from the cycle, we obtain a new circuit in \( \partial U_j \in \mathcal{G} \) with length \( 2n - 2 \). Let \( \mathcal{H} \) denote the underlying subgraph of \( \mathcal{G} \) corresponding to this shortened circuit and let \( U \) be the face of \( \mathcal{H} \) that contains \( U_j \). Since the endpoint \( v \) of \( e \) does not belong to \( P_f \), the curve \( \alpha_j \) is isotopic rel. \( P_f \) to the \( \epsilon \)-boundary of \( \mathcal{H} \) wrt. \( U \) for small enough \( \epsilon > 0 \). Then the essential circuit length of \( U_j \) would be \( \geq 2n - 2 \), contradicting the fact that \( 2n \) is the essential circuit length of \( U_j \). So we must have \( v \in P_f \).

It follows that \( f^{-1}(P_f) \subset C_f \cup P_f \), and so \( f \) has a parabolic orbifold by Lemma 7.1.

We are now ready to prove our main result.

**Proof of Theorem 1.1.** As in the statement, suppose \( f: S^2 \to S^2 \) is a Thurston map with \#\( P_f = 4 \) and a hyperbolic orbifold. We assume that \( f \) has an obstruction given by a Jordan curve \( \alpha \) in \((S^2, P_f)\). We choose core arcs \( a \) and \( c \) for \( \alpha \) that lie in different components of \( S^2 \setminus \alpha \), and assume that \( E \neq \emptyset \) is a finite set of arcs in \((S^2, f^{-1}(P_f))\) that satisfies the \( \alpha \)-restricted blow-up conditions as in Definition 6.6.

We assume that we obtained a Thurston map \( \tilde{f}: S^2 \to S^2 \) by blowing up arcs in \( E \) (with some multiplicities) such that \( \lambda_{\tilde{f}}(\alpha) < 1 \). Then \( P_{\tilde{f}} = P_f \) and \( \tilde{f} \) has a hyperbolic orbifold. Up to replacing \( \tilde{f} \) with a Thurston equivalent map, we may also assume that the statements in Lemma 6.7 are true for the map \( \tilde{f} \). We now argue by contradiction and suppose that \( \tilde{f} \) is not realized by a rational map. Then \( \tilde{f} \) has an obstruction given by a Jordan curve \( \gamma \) in \((S^2, P_f)\).

We set \( U = S^2 \setminus (a \cup c) \). Since \( E \) satisfies the \( \alpha \)-restricted blow-up conditions, we have \#\((f(e) \cap \alpha) = 1 \) and \( f(\text{int}(e)) \cap a = f(\text{int}(e)) \cap c = \emptyset \) for each \( e \in E \). In other words, \( f(e) \) intersects \( \alpha \) only once and \( \text{int}(f(e)) \) belongs to \( U \). Then each arc in \( E \) intersects only one pullback of \( \alpha \) and only once.

Since \( \gamma \) is an an obstruction for \( \tilde{f} \), but \( \alpha \) is not, the curves \( \alpha \) and \( \gamma \) are not isotopic rel. \( P_{\tilde{f}} = P_f \). So we have \( i(\alpha, \gamma) > 0 \). We may assume that \#\((a \cap \gamma) = i(a, \gamma) \) and \#\((c \cap \gamma) = i(c, \gamma) \). Note that
\[
i(a, \gamma) = i(c, \gamma) = \frac{1}{2}i(\alpha, \gamma) > 0.
\]
We denote by \( \alpha_1, \ldots, \alpha_n \) the pullbacks of \( \alpha \) under \( f \) that are isotopic to \( \alpha \) rel. \( P_f \). We consider the graphs \( \mathcal{G} = f^{-1}(a \cup c) \) and \( \widehat{\mathcal{G}} = \widehat{f}^{-1}(a \cup c) \). By Lemma 6.7, \( \mathcal{G} \) is a subgraph of \( \widehat{\mathcal{G}} \). Moreover, the following facts are true for their complementary components. Each \( \alpha_j \) is a core curve in an essential annulus \( U_j \) that is a component of \( S^2 \setminus \mathcal{G} \). Each \( U_j \) contains precisely one component \( \widehat{U}_j \) of \( S^2 \setminus \widehat{\mathcal{G}} \). This component is essential and contains precisely one essential pullback \( \widehat{\alpha}_j \) of \( \alpha \) under \( \widehat{f} \). The essential circuit length of \( U_j \) is the same as the essential circuit length of \( \widehat{U}_j \). The curves \( \widehat{\alpha}_1, \ldots, \widehat{\alpha}_n \) are precisely all the distinct essential pullbacks of \( \alpha \) under \( \widehat{f} \). They are isotopic to \( \alpha \) rel. \( P_f \).

Let \( \gamma_1, \ldots, \gamma_k \) the pullbacks of \( \gamma \) under \( \widehat{f} \) that are isotopic to \( \gamma \) rel. \( P_{\widehat{f}} = P_f \). Then Lemmas 6.4 and 6.2 imply that

\[
\deg(f; \alpha_j \rightarrow \alpha) = \deg(f; U_j \rightarrow U) \\
= \frac{1}{2} \cdot \text{circuit length of } U_j \\
\geq \frac{1}{2} \cdot \text{essential circuit length of } U_j \\
= \frac{1}{2} \cdot \text{essential circuit length of } \widehat{U}_j \\
\geq k
\]

for \( j = 1, \ldots, n \).

On the other hand, \( \alpha \) has the \( n \) distinct essential pullbacks \( \widehat{\alpha}_1, \ldots, \widehat{\alpha}_n \) under \( \widehat{f} \), and so Lemma 5.5 implies that \( \deg(\widehat{f}; \gamma_m \rightarrow \gamma) \geq n \) for \( m = 1, \ldots, k \). Since \( \alpha \) and \( \gamma \) are obstructions for \( f \) and \( \widehat{f} \), respectively, we conclude that

\[
1 \leq \lambda_f(\alpha) = \frac{1}{\deg(\widehat{f}; \alpha_j \rightarrow \alpha)} \leq \frac{n}{k}
\]

and

\[
1 \leq \lambda_{\widehat{f}}(\gamma) = \sum_{m=1}^{k} \frac{1}{\deg(\widehat{f}; \gamma_m \rightarrow \gamma)} \leq \frac{k}{n}.
\]

It follows that \( k = n \), which forces \( \deg(f; \alpha_j \rightarrow \alpha) = \deg(\widehat{f}; \gamma_j \rightarrow \gamma) = n \) for \( j = 1, \ldots, n \). If we combine this with (7.1), then we also see that

\[
\text{circuit length of } U_j = \text{essential circuit length of } U_j = \text{essential circuit length of } \widehat{U}_j = 2n
\]

for \( j = 1, \ldots, n \).

We now want to apply Lemma 7.2 to our map \( f \) and the obstruction \( \alpha \). In order to verify the hypotheses of Lemma 7.2, it remains to show that \( \alpha \) has no peripheral pullbacks under \( f \), or equivalently, no peripheral pullbacks under \( \widehat{f} \).

We argue by contradiction and assume that \( \alpha \) has some peripheral pullbacks under \( \widehat{f} \). Then there exists at least one peripheral annulus in the complement of \( \widehat{\mathcal{G}} \). Such an annulus is disjoint from each annulus \( \widehat{U}_j \). We can travel from an interior point \( p \) of such a peripheral annulus to the closure \( M \) of the set \( \widehat{U}_1 \cup \cdots \cup \widehat{U}_n \) along an arc \( \sigma \) in \( S^2 \setminus \widehat{f}^{-1}(P_{\widehat{f}}) \) that crosses each edge in the graph \( \widehat{\mathcal{G}} \) transversely. Then there is a first point \( q \in \sigma \cap M \) on the arc when we enter \( M \). The point \( q \) is necessarily an interior point of an edge \( e \) of \( \widehat{\mathcal{G}} \) contained in the boundary \( \partial \widehat{U}_j \) for some \( j \in \{1, \ldots, n\} \). Interior points of the subarc of \( \sigma \) between \( p \) and \( q \) that
are close to \( q \) do not lie in \( M \cup \hat{\mathcal{G}} \). Hence such points must belong to a peripheral annulus \( \hat{U} \) of \( \hat{\mathcal{G}} \). Then necessarily \( e \in \partial \hat{U} \).

In other words, there exists an edge \( e \) in the graph \( \hat{\mathcal{G}} \) that belongs to the boundary of an essential annulus \( \hat{U}_j \) and a peripheral annulus \( \hat{U} \). Clearly, \( \hat{f}(e) = a \) or \( \hat{f}(e) = c \). In the following, we will assume that \( \hat{f}(e) = c \), that is, \( e \subset \partial \hat{U}_j \cap \partial \hat{U} \); the other case, \( \hat{f}(e) = a \), is completely analogous.

Since \( i(c, \gamma) = \#(c \cap \gamma) > 0 \), there exists a pullback \( \hat{\gamma} \) of \( \gamma \) under \( \hat{f} \) that meets \( e \) transversely. Consequently, this pullback \( \hat{\gamma} \) meets both \( \hat{U}_j \) and \( \hat{U} \). By Lemma 6.3, the points in \( \hat{f}^{-1}(a) \cap \hat{\gamma} \) and \( \hat{f}^{-1}(c) \cap \hat{\gamma} \) alternate on \( \hat{\gamma} \). This implies that the curve \( \hat{\gamma} \) also meets the sets \( \partial_a \hat{U}_j \) and \( \partial_a \hat{U} \), and hence both components of the boundary of \( \hat{U}_j \) and of \( \hat{U} \). We conclude that \( \hat{\gamma} \) meets the core curve \( \hat{\alpha}_j \) of \( \hat{U}_j \) and the core curve \( \hat{\alpha} \) of \( \hat{U} \). Note that \( \hat{\alpha} \) is a peripheral pullback of \( \alpha \) under \( \hat{f} \). In order to show that this is impossible, we consider two cases.

Case 1: \( \hat{\gamma} \) is an essential pullback of \( \gamma \) under \( \hat{f} \), say \( \hat{\gamma} = \gamma_m \) for some \( m \in \{1, \ldots, n\} \). Then Lemma 5.5 (for the map \( \hat{f} \) and the roles of \( \alpha \) and \( \gamma \) reversed) shows \( n < \deg(\hat{f}; \hat{\gamma} \to \gamma) \), because \( \alpha \) has \( n \) essential pullbacks and \( \hat{\gamma} \) meets a peripheral pullback of \( \alpha \). On the other hand, we know that \( \deg(\hat{f}; \hat{\gamma} \to \gamma) = \deg(\hat{f}; \gamma_m \to \gamma) = n \). This is a contradiction.

Case 2: \( \hat{\gamma} \) is a peripheral pullback of \( \gamma \) under \( \hat{f} \). Let \( \mathcal{H} := \partial U_j \). Then it follows from Lemma 6.7 that \( \mathcal{H} \subset \partial \hat{U}_j \). Moreover, (7.2) implies that \( \mathcal{H} \) (considered as a circuit) realizes the essential circuit lengths of \( U_j \) and \( \hat{U}_j \), which are both equal \( 2n \).

Now \( e \subset \partial_e U_j = \mathcal{H} \), and so \( \mathcal{H} \) meets the peripheral pullback \( \hat{\gamma} \) of \( \gamma \) under \( \hat{f} \). The second part of Lemma 6.2 applied to \( \mathcal{H} \) implies that the number \( k \) of essential pullbacks of \( \gamma \) under \( \hat{f} \) is less than \( n \), contradicting \( k = n \).

To summarize, these contradictions show that \( \alpha \) has no peripheral pullbacks under \( \hat{f} \), and hence no peripheral pullbacks under \( f \) as follows from Lemma 6.7. So we can apply Lemma 7.2 and conclude that \( f \) has a parabolic orbifold. This is a yet another contradiction, because \( f \) has a hyperbolic orbifold by our hypotheses. This shows that our initial assumption that \( \hat{f} \) has an obstruction is false. Hence \( \hat{f} \) is realized by a rational map. \qed

8. Global curve attractors

In this section we will prove Theorem 1.4. We consider the pillow \( \mathbb{P} \) with its vertex set \( V = \{A, B, C, D\} \). For the remainder of this section, \( f: \mathbb{P} \to \mathbb{P} \) is a Thurston map obtained from the \((2 \times 2)\)-Lattès map by gluing \( n_h \geq 1 \) horizontal and \( n_v \geq 1 \) vertical flaps to \( \mathbb{P} \). Then \( f \) is Thurston equivalent to a rational map by Theorem 1.2. In the following, all isotopies on \( \mathbb{P} \) are considered relative to \( \partial_f = V \).

In order to prove Theorem 1.4, we want to show that Jordan curves in \((\mathbb{P}, V)\) are getting “less twisted” under taking preimages under \( f \). To formalize this, we define the complexity \( \|x\| \) of \( x \in \hat{\mathcal{Q}} \cup \{\emptyset\} \) as \( \|\emptyset\| := 0 \) for \( x = \emptyset \) and \( \|r/s\| := |r| + s \) for \( x = r/s \in \hat{\mathcal{Q}} \). Recall that \( \emptyset \) represent the isotopy classes of all peripheral curves and that we use the convention that \( r \in \mathbb{Z} \) and \( s \in \mathbb{N}_0 \) here are relatively prime and that \( r = 1 \) if \( s = 0 \). Note that \( \|x\| = 0 \) for \( x \in \hat{\mathcal{Q}} \cup \{\emptyset\} \) if and only if \( x = \emptyset \).

The complexity admits a natural interpretation in terms of intersection numbers. To see this, recall that \( \alpha^h \) and \( \alpha^v \) (see (2.5)) represent simple closed geodesics in \((\mathbb{P}, V)\) that separate the two horizontal and the two vertical edges of \( \mathbb{P} \), respectively. Suppose the slope \( r/s \in \hat{\mathcal{Q}} \) corresponds to the isotopy class \([\gamma]\) of a (necessarily essential) Jordan curve \( \gamma \) in \((\mathbb{P}, V)\).
Then
\[ \|r/s\| = |r| + s = \frac{1}{2}i(\gamma, \alpha^h) + \frac{1}{2}i(\gamma, \alpha^v). \]
Moreover, if \( \gamma \) is peripheral, then \( i(\gamma, \alpha^h) + i(\gamma, \alpha^v) = 0 \) which agrees with the fact that \( \|\circ\| = 0 \).
As we will see, under the slope map \( \mu_f \) (as defined in the Introduction) complexities do not increase, and actually strictly decrease unless \( x \in \hat{Q} \cup \{\circ\} \) belongs to a certain finite set. More precisely, we will show the following statement.

**Proposition 8.1.** Let \( f: \mathbb{P} \to \mathbb{P} \) be a Thurston map obtained from the \((2 \times 2)\)-Lattès map by gluing \( n_h \geq 1 \) horizontal and \( n_v \geq 1 \) vertical flaps to the pillow \( \mathbb{P} \). Then the following statements are true:

(i) \( \|\mu_f(x)\| \leq \|x\| \) for all \( x \in \hat{Q} \cup \{\circ\} \).

(ii) \( \|\mu_f(x)\| < \|x\| \) for all \( x \in \hat{Q} \cup \{\circ\} \) with \( \|x\| > 8 \).

Since the set \( \{x \in \hat{Q} \cup \{\circ\} : \|x\| \leq 8\} \) is finite, we actually have the strict inequality in (i) with at most finitely many exceptions. The proof of the proposition will show that \( \|\mu_f(x)\| = \|x\| \) if and only if \( \mu_f(x) = x \) (see Remark 8.6). As we will see below, Theorem 1.4 easily follows from Proposition 8.1.

Before we proceed with the proof of this proposition, we will establish several auxiliary results. As in Section 2.4, \( a, b, c, d \) are the sides of the pillow \( \mathbb{P} \). As before, the Weierstrass function \( \varphi: \mathbb{C} \to \mathbb{P} \) is doubly periodic with respect to the lattice \( 2\mathbb{Z}^2 \), \( T = \mathbb{C}/2\mathbb{Z}^2 \) is the Euclidean torus obtained as the quotient of \( \mathbb{C} \) by the lattice \( 2\mathbb{Z}^2 \), and \( \pi: \mathbb{C} \to T \) is the corresponding quotient map. We know that \( \varphi \) descends to a double branched covering map \( \overline{\varphi}: T \to \mathbb{P} \) such that \( \varphi = \overline{\varphi} \circ \pi \).

We are interested in simple closed geodesics and geodesic arcs \( \tau \) in \((\mathbb{P}, V)\). Every such geodesic has the form \( \tau = \varphi(\ell_{r/s}) \) for a line \( \ell_{r/s} \subset \mathbb{C} \) with slope \( r/s \in \hat{Q} \). If \( \ell_{r/s} \subset \mathbb{C} \setminus 2\mathbb{Z}^2 \), then \( \tau = \varphi(\ell_{r/s}) \) is a simple closed geodesic in \((\mathbb{P}, V)\), that is, \( \tau \subset \mathbb{P} \setminus V \). If \( \ell_{r/s} \) contains a point in \( 2\mathbb{Z}^2 \), then \( \tau = \varphi(\ell_{r/s}) \) is a geodesic arc in \((\mathbb{P}, V)\), that is, its interior lies in \( \mathbb{P} \setminus V \) and its endpoints are in \( V \).

**Lemma 8.2.** Let \( \tau \) be a closed geodesic or a geodesic arc in \((\mathbb{P}, V)\) with slope \( r/s \in \hat{Q} \). We consider 1-edges on \( \mathbb{P} \) with respect to the \((n \times n)\)-Lattès map \( \mathcal{L}_n \), \( n \geq 2 \), that is, the lifts of the edges \( a, b, c, d \) of \( \mathbb{P} \) under \( \mathcal{L}_n \). Then the following statements are true:

(i) if \( |r| > 2n \) then \( \tau \) intersects the interior of every horizontal 1-edge of \( \mathbb{P} \).

(ii) if \( s > 2n \) then \( \tau \) intersects the interior of every vertical 1-edge of \( \mathbb{P} \).

**Proof.** We will only show the first part of the statement. The proof of the second part is completely analogous.

Let \( \tau \) be a closed geodesic or a geodesic arc in \((\mathbb{P}, V)\) with slope \( r/s \in \hat{Q} \) where \( |r| > 2n \). Suppose that \( e \) is a horizontal 1-edge. Then \( e \) belongs to a connected component \( \alpha \) of \( \mathcal{L}_n^{-1}(a \cup c) \). Let \( \mathcal{P} \) be a connected component of \( \overline{\varphi}^{-1}(\tau) \) and \( \mathcal{P} \) be a connected component of \( \overline{\varphi}^{-1}(\alpha) \). Then \( \mathcal{P} \) and \( \mathcal{P} \) are closed geodesics of slopes \( r/s \) and 0 on the torus \( T \), respectively. Note that the length of \( \mathcal{P} \) equals 2 and \( \#(\mathcal{P} \cap \alpha) = |r| \). So by Lemma 2.4 the points in the intersection \( \mathcal{P} \cap \mathcal{P} \) subdivide the curve \( \alpha \) into arcs of length \( 2/|r| \). Now let \( \overline{e} \subset \overline{\mathcal{P}} \) be a lift of the 1-edge \( e \) under \( \overline{\varphi} \). Then \( \overline{e} \) has length equal to the length of \( e \), that is, \( 1/n \). Out hypothesis shows that \( 2/|r| < 1/n \), which implies that \( \mathcal{P} \) must intersect \( \overline{e} \subset \overline{\mathcal{P}} \) in its interior. Consequently, \( \tau \) intersects \( e \) in its interior as well. Since the horizontal 1-edge \( e \) was arbitrary, the statement follows. \( \square \)
We now want to see what happens to a geodesic arc $\tau$ in $(\mathbb{P}, V)$ if we take preimages under a map $f$ as in Proposition 8.1. Unless $\tau$ has slope in a finite exceptional set, suitable sets $\mathcal{H}$ in the preimage $f^{-1}(\tau)$ will meet the interior of a flap glued to the pillow $\mathbb{P}$, and consequently a peripheral preimage of the horizontal curve $\alpha^h \subset \mathbb{P}$ or the vertical curve $\alpha^v \subset \mathbb{P}$. We will formulate a relevant statement in a slightly more general situation. We first introduce some terminology.

Let $M, N, K$ be subsets of a topological 2-sphere $S^2$. We say that $K$ separates $M$ and $N$ if every path in $S^2$ joining $M$ and $N$ meets $K$. Note that here $K$ is not necessarily disjoint from $M$ or $N$.

Suppose $Z \subset S^2$ consists of four distinct points. We say that $K \subset S^2$ essentially separates $Z$ if we can split $Z$ into two pairs (that is, into disjoint subsets $Z_1, Z_2 \subset Z$ consisting of two points each) such that $K$ separates $Z_1$ and $Z_2$. Note that $K$ trivially has this property if $K \cap Z$ consists of two or more points.

Now let $n \in \mathbb{N}$, $n \geq 2$, and consider the $(n \times n)$-Lattès map $L_n : \mathbb{P} \to \mathbb{P}$. If $\tau$ is a geodesic arc in $(\mathbb{P}, V)$, then the preimage $L_n^{-1}(\tau)$ is a union of simple closed geodesics and geodesic arcs in $(\mathbb{P}, V)$. Note that each connected component of $L_n^{-1}(\tau)$ essentially separates $V$, but no proper subset of such a component does. It follows that if $K \subset L_n^{-1}(\tau)$ is a connected set, then it essentially separates $V$ if and only if $K$ contains a simple closed geodesic or a geodesic arc in $(\mathbb{P}, V)$.

Let $\widehat{L}, \widehat{\mathbb{P}} \to \mathbb{P}$ be a branched covering map obtained from the $(n \times n)$-Lattès map by adding flaps to $\mathbb{P}$. As in Section 4.2, we denote by $\widehat{\nu}$ the vertex set and by $B(\widehat{\mathbb{P}})$ the base pillow of the flapped pillow $\widehat{\mathbb{P}}$. By construction, $\widehat{L}$ maps each 1-tile in $\widehat{\mathbb{P}}$ by a Euclidean similarity (with scaling factor $n$) onto a side of $\mathbb{P}$. We also recall that we can naturally view the base pillow $B(\widehat{\mathbb{P}})$ as a subset of $\mathbb{P}$ (see (4.2)) and, with such identification, the map $\widehat{L}$ coincides with $L_n$ on $B(\widehat{\mathbb{P}})$.

Suppose that the geodesic arc $\xi \subset (\mathbb{P}, V)$ joins two distinct points $X, Y \in V$. We consider $\widehat{\mathcal{G}} := \widehat{L}^{-1}(\xi)$ as an embedded graph in $\widehat{\mathbb{P}}$ with the set of vertices $\widehat{L}^{-1}((X,Y))$ and the edges given by lifts of $\xi$ under $\widehat{L}$.

**Lemma 8.3.** Let $\widehat{L}, \widehat{\mathbb{P}} \to \mathbb{P}$ be a branched covering map obtained from the $(n \times n)$-Lattès map with $n \geq 2$ by gluing $n_h \geq 1$ horizontal and $n_v \geq 1$ vertical flaps to $\mathbb{P}$. Suppose $\xi$ is a geodesic arc in $(\mathbb{P}, V)$ with slope $r/s \in \mathbb{Q} \setminus \{0, \infty\}$ and $\widehat{\xi}$ is a lift of $\xi$ under $\widehat{L}$.

Let $F$ be a flap in $\widehat{\mathbb{P}}$ with the base edges $e'$ and $e''$. If

$$\widehat{\xi} \cap (\text{int}(F) \cup \text{int}(e') \cup \text{int}(e'')) \neq \emptyset,$$

then $\widehat{\xi}$ meets a base edge and the top edge of the flap $F$.

**Proof.** The proof is similar to the proof of Lemma 5.6. Recall that $a, b, c, d$ denote the sides of the pillow $\mathbb{P}$. Suppose $\xi \subset \mathbb{P}$, $\xi \subset \widehat{\mathbb{P}}$, and $e', e'' \subset F$ are as in the statement of the lemma. Let $\widehat{e} \subset F$ be the top edge of $F$.

Without loss of generality we will assume that $F$ is a horizontal flap. Then $\widehat{L}(e') = a$ or $\widehat{L}(e') = c$. We will make the further assumption that $\widehat{L}(e') = a$. The other cases, when $\widehat{L}(e') = c$ or when $F$ is a vertical flap, can be treated in a way that is completely analogous to the ensuing argument. Then $\widehat{L}(e'') = a$ and $\widehat{L}(\widehat{e}) = c$. Moreover,

$$\widehat{L}^{-1}(a \cup c) \cap F = e' \cup e'' \cup \widehat{e}.$$
Since \( r/s \neq 0 \), the arc \( \xi \) must meet both edges \( a \) and \( c \). We claim that there is a point \( p \in \hat{\xi} \cap \text{int}(F) \). This can only fail if \( \hat{\xi} \) meets either \( \text{int}(e') \) or \( \text{int}(e'') \) in a point \( q \). In either case, \( \xi \) has a transverse intersection with \( \text{int}(a) \) at \( \hat{L}(q) \), and so \( \xi \) has a transverse intersection with \( \text{int}(e') \) or \( \text{int}(e'') \) at \( q \). Then \( \hat{\xi} \) meets \( \text{int}(F) \) in a point \( p \).

Since \( \xi \) is a geodesic arc, it is in minimal position with each side \( a, b, c, d \) of \( \mathbb{P} \). Thus, by Lemma [2.6], the points in the non-empty sets \( a \cap \xi \) and \( c \cap \xi \) alternate on \( \xi \). This implies that the points in \( \hat{\xi} \cap \hat{L}^{-1}(a) \) and \( \hat{\xi} \cap \hat{L}^{-1}(c) \) alternate on \( \hat{\xi} \). Note that \( F \cap \hat{L}^{-1}(c) = \partial F \) and \( F \cap \hat{L}^{-1}(a) = \partial F = e' \cup e'' \). So, if we trace the arc \( \hat{\xi} \) starting from \( p \) in two different directions, we must meet a base edge for one direction and the top edge for the other direction. The statement follows.

Remark 8.4. Suppose that we are in the setup of Lemma 8.3 with the extra assumption that \( F \) is a horizontal flap. Then there exists a peripheral preimage \( \hat{\alpha} \) of the horizontal curve \( \alpha^b \) under that map \( \hat{L} \) that is contained in \( F \). This curve \( \hat{\alpha} \) separates \( \partial F = e' \cup e'' \) from the top edge \( \partial F \) of the flap. Since \( \hat{\xi} \) is connected and meets both \( \partial F \) and \( e' \cup e'' \), we conclude that \( \xi \cap \hat{\alpha} \neq \emptyset \). If \( \beta \) is a connected set that traces \( \hat{\tau} \) closely, then it will also have points close to \( \partial F \) and close to \( e' \cup e'' \). Again this will imply that \( \beta \cap \hat{\alpha} \neq \emptyset \). This remark will become important in the proof Proposition 8.1.

Now the following fact is true.

Lemma 8.5. Let \( \hat{L}: \mathbb{P} \rightarrow \mathbb{P} \) be a branched covering map obtained from the \((n \times n)\)-Lattès map with \( n \geq 2 \) by gluing \( n_h \geq 1 \) horizontal and \( n_v \geq 1 \) vertical flaps to \( \mathbb{P} \). Suppose that \( \xi \) is a geodesic arc in \((\mathbb{P}, V)\) with slope \( r/s \in \hat{Q} \) and \( \mathcal{H} \) is any connected subgraph of \( \mathcal{G} = \hat{L}^{-1}(\xi) \) that essentially separates \( \hat{V} \subset \mathbb{P} \). If \( |r| + s > 4n \), then \( \mathcal{H} \) meets a base edge and the top edge of a flap in \( \mathbb{P} \).

Proof. Suppose \( \xi \) is a geodesic arc in \((\mathbb{P}, V)\) with slope \( r/s \in \hat{Q} \), where \( |r| + s > 4n \), and \( \mathcal{H} \) is a connected subgraph of \( \mathcal{G} = \hat{L}^{-1}(\xi) \) that essentially separates the vertex set \( \hat{V} \) of \( \mathbb{P} \). We now argue by contradiction and suppose that there is no flap \( F \) in \( \mathbb{P} \) such that \( \mathcal{H} \) meets both a base edge and the top edge of \( F \). By (4.1) and Lemma 8.3 each edge of \( \mathcal{H} \), and thus the graph \( \mathcal{H} \) itself, is contained in \( B(\hat{\mathbb{P}}) \). This means we can consider \( \mathcal{H} \) as a connected subset of \( \mathbb{P} \supset B(\hat{\mathbb{P}}) \). On \( B(\hat{\mathbb{P}}) \) the maps \( \hat{L} \) and \( L_n \) are identical. This means that we can also regard \( \mathcal{H} \) as a connected subset of \( L_n^{-1}(\xi) \).

The set \( \mathcal{H} \) now considered as a subset of \( \mathbb{P} \), essentially separates \( V \subset \mathbb{P} \). Indeed, let \( \hat{V}_1, \hat{V}_2 \subset \hat{V} \subset \mathbb{P} \) be two pairs of vertices separated by \( \mathcal{H} \) in \( \mathbb{P} \). We can identify \( \hat{V}_1 \) and \( \hat{V}_2 \) with two pairs \( V_1 \) and \( V_2 \) of vertices of \( \mathbb{P} \). They are separated by \( \mathcal{H} \) in \( \mathbb{P} \). Indeed, if this was not the case, then we could find a path \( \beta \) in \( \mathbb{P} \) that joins \( V_1 \) and \( V_2 \) without meeting \( \mathcal{H} \). This path can be modified to a path \( \hat{\beta} \) in \( \hat{L}(\mathbb{P}) \) that joins \( \hat{V}_1 \) and \( \hat{V}_2 \) and does not meet \( \mathcal{H} \subset \mathbb{P} \). To see this, we replace the parts of \( \beta \) between a first entry \( p \) and a last exit point \( q \) to a 1-edge \( e \) to which a flap \( F \) is attached, by a path that connects \( p \) and \( q \), but travels on the flap and does not meet \( \mathcal{H} \). But such a path \( \hat{\beta} \) cannot exist, because \( \mathcal{H} \) separates \( \hat{V}_1 \) and \( \hat{V}_2 \) in \( \hat{L}(\mathbb{P}) \).

We see that \( \mathcal{H} \subset L_n^{-1}(\xi) \) indeed essentially separates \( V \). Since \( \mathcal{H} \) is connected, the discussion above (after the definition of essential separation) implies that \( \mathcal{H} \) contains a simple closed geodesic or a geodesic arc in \( \xi' \) in \((\mathbb{P}, V)\) with slope \( r/s \). Since \( |r| + s > 4n \), either \( |r| > 2n \) or \( s > 2n \). Thus, by Lemma 8.2, the geodesic \( \xi' \subset \mathcal{H} \) meets each horizontal 1-edge in the first case or each vertical 1-edge in the second case. Since \( n_h \geq 1 \) and \( n_v \geq 1 \), in either case, \( \xi' \subset \mathcal{H} \) must meet the interior of a 1-edge along which a flap \( F \) is glued and hence it also must meet
the top edge of $F$ by Lemma 8.3. This contradicts our initial assumption on $\mathcal{H}$, and the lemma follows. \qed

A completely analogous statement to Lemma 8.5 is true (with a very similar proof) if we assume that $\xi$ is a simple closed geodesic in $(\mathbb{P}, V)$ and $\mathcal{H}$ is an essential preimage of $\xi$ under $\mathcal{L}$.

We now turn to the proof of Proposition 8.1.

**Proof of Proposition 8.1.** Let $f: \mathbb{P} \to \mathbb{P}$ be a Thurston map obtained from the $(2 \times 2)$-Lattès map by gluing $n_h \geq 1$ horizontal and $n_v \geq 1$ vertical flaps to $\mathbb{P}$. Then $P_f = V$, where $V = \{A, B, C, D\}$ is the set of vertices of $\mathbb{P}$, and $A$ is the unique point in $P_f = V$ that is fixed by $f$.

To prove the first statement, let $x \in \mathfrak{Q} \cup \{\ominus\}$ be arbitrary. If $x = \ominus$, then $\mu_f(\ominus) = \ominus$ and $\|\mu_f(\ominus)\| = \|\ominus\| = 0$. So in the following, we will assume that $x = r/s \in \mathfrak{Q}$. Let $\gamma \in \mathbb{P} \setminus V$ be a simple closed geodesic with rational slope $r/s \in \mathfrak{Q}$. Then $\gamma$ is an essential Jordan curve and so each of the two complementary components of $\gamma$ in $\mathbb{P}$ contains precisely two postcritical points of $f$. Let $\tau$ and $\tau'$ be core arcs of $\gamma$ belonging to the two components of $\mathbb{P} \setminus \gamma$, respectively. Here we may assume that $\tau$ and $\tau'$ are geodesic arcs in $(\mathbb{P}, V)$ with slope $r/s$.

As before, we denote by $\alpha^h$ and $\alpha^v$ simple closed geodesics in $(\mathbb{P}, V)$ that separate the two horizontal and the two vertical edges of $\mathbb{P}$, respectively. Then, by Lemma 2.5, we have:

$$i(\gamma, \alpha^h) = 2i(\tau, \alpha^h) = \#(\gamma \cap \alpha^h) = 2|r|,$$
$$i(\gamma, \alpha^v) = 2i(\tau, \alpha^v) = \#(\gamma \cap \alpha^v) = 2s,$$
$$\|x\| = |r| + s = \frac{1}{2}i(\gamma, \alpha^h) + \frac{1}{2}i(\gamma, \alpha^v).$$

Recall that a point $p \in \mathbb{P}$ is called a 1-vertex if $f(p) \in P_f = \{A, B, C, D\}$. We say that a 1-vertex is of type $A$, $B$, $C$, or $D$ if it is a preimage of $A$, $B$, $C$, or $D$ under $f$, respectively.

Without loss of generality, we may assume that the core arc $\tau$ connects the point $A$ with a point $X \in \{B, C, D\}$. Then $\tau'$ joins the two points in $\{B, C, D\} \setminus X$. Let $\mathcal{G} = f^{-1}(\tau \cup \tau')$, which we view as a planar embedded graph with the set of vertices $f^{-1}(V)$. Note that the degree of a vertex $p$ in $\mathcal{G}$ is equal to the local degree of the map $f$ at $p$. In addition, the graph $\mathcal{G}$ has the following properties:

(P1) $\mathcal{G}$ is a bipartite graph. In particular, 1-vertices of type $A$ are connected only to 1-vertices of type $X$ and vice versa.

(P2) Each postcritical point is a 1-vertex of type $A$. If a 1-vertex of type $A$ has degree $\geq 2$ in $\mathcal{G}$, then it must be a postcritical point.

The analog of (P1) is valid for arbitrary Thurston maps with four postcritical points. To see that (P2) is true, note that the $(2 \times 2)$-Lattès map sends each of the four vertices of $\mathbb{P}$ to $A$. This remains true if we glue any number of flaps to $\mathbb{P}$. Moreover, gluing additional flaps only creates additional preimages of $A$ of degree 1 in $\mathcal{G}$.

If every pullback of $\gamma$ under $f$ is peripheral, then $\mu_f(x) = \ominus$, and so

$$\|\mu_f(x)\| = \|\ominus\| = 0 < |r| + s = \|x\|.$$  

Suppose $\gamma$ has an essential pullback $\gamma'$ under $f$. Then $\mu_f(x) \in \mathfrak{Q}$ is the slope corresponding to the isotopy class of $\gamma'$. By the discussion in Section 6 the pullback $\gamma'$ belongs to a unique component $\widetilde{U}$ of $\mathbb{P} \setminus \mathcal{G}$. We use the notation $\partial_v \widetilde{U} := f^{-1}(\tau) \cap \partial \widetilde{U}$. Then $\partial_v \widetilde{U}$ is a subgraph of $\mathcal{G}$ that only contains 1-vertices of type $A$ and $X$. Since $\gamma'$ is essential, $\partial_v \widetilde{U}$ satisfies:
(P3) \( \#(\partial_r \widehat{U} \cap P_f) \leq 2 \).

Our goal now is to simplify the pullback \( \widetilde{\gamma} \) using an isotopy depending on the combinatorics of \( \partial_r \widehat{U} \). More precisely, we will construct a curve \( \beta \) that is isotopic to \( \widetilde{\gamma} \), but has fewer intersections with \( \alpha^h \) and \( \alpha^n \). In order to obtain a suitable curve \( \beta \), we now distinguish several cases that exhaust all possibilities.

**Case 1:** \( \partial_r \widehat{U} \) does not contain any simple cycle. Then \( \partial_r \widehat{U} \) is a tree and, since \( \widetilde{\gamma} \) is essential, there are exactly two postcritical points in \( \partial_r \widehat{U} \). These are 1-vertices of type A by (P1). Let \( \mathcal{H} \subset \partial_r \widehat{U} \) be the unique simple path that joins these two postcritical points in \( \partial_r \widehat{U} \). By (P1) the path \( \mathcal{H} \) must have length \( \geq 2 \), because the endpoints of \( \mathcal{H} \) have type A and the vertices of type A and X alternate on \( \mathcal{H} \).

If the length of \( \mathcal{H} \) was \( \geq 3 \), then \( \mathcal{H} \) would contain at least one additional point \( p \) of type A apart from its endpoints. Then \( \deg_{\mathcal{H}}(p) = 2 \), so \( \deg_{\mathcal{G}}(p) \geq 2 \), which means \( p \) must be a postcritical point by (P2). But then \( \mathcal{H} \subset \partial_r \widehat{U} \) contains at least three postcritical points, which contradicts (P3). We conclude that \( \mathcal{H} \) has length 2; see Figure 21 (Case 1).

![Figure 21](image)

**Case 2:** \( \partial_r \widehat{U} \) contains a simple cycle. Then one of the vertices of such a cycle must be of type A. Since this vertex has degree equal to 2 in the cycle, and hence degree \( \geq 2 \) in \( \mathcal{G} \), it must be a postcritical point by (P2). It follows that \( \#(\partial_r \widehat{U} \cap P_f) \geq 1 \). So by (P3), either \( \#(\partial_r \widehat{U} \cap P_f) = 1 \) or \( \#(\partial_r \widehat{U} \cap P_f) = 2 \).
Case 2a: \(\#(\partial_r \bar{U} \cap P_f) = 1\). Since \(\bar{\gamma}\) is essential, there are exactly two postcritical points in the component of \(S^2 \setminus \bar{\gamma}\) that contains \(\partial_r \bar{U}\). One of them belongs to \(\partial_r \bar{U}\), while the other one belongs to a face of \(\partial_r \bar{U}\) disjoint from \(\bar{U}\). This postcritical point then necessarily belongs to a face of a simple cycle \(H\) in \(\partial_r \bar{U}\).

This simple cycle \(H\) then necessarily contains the unique postcritical point in \(\partial_r \bar{U}\) as we have seen above. Moreover, \(H\) must have length 2, because otherwise \(H\) has an even length \(\geq 4\) by (P1). But then \(H\) contains another 1-vertex of type \(A\) with degree \(\geq 2\), which is necessarily a postcritical point by (P2). Then \(H \subset \partial_r \bar{U}\) contains at least two postcritical points which contradicts our assumption for this case. So \(H\) has indeed length 2; see Figure 21 (Case 2a).

Let \(\bar{U}\) denote the face of \(H\) that contains \(\bar{U}\). Then again the annulus between \(\bar{\gamma}\) and \(H\) contains no postcritical points of \(f\), and hence each \(\epsilon\)-boundary \(\beta\) of \(\bar{U}\) wrt. \(H\) is isotopic to \(\bar{\gamma}\) for sufficiently small \(\epsilon\).

Case 2b: \(\#(\partial_r \bar{U} \cap P_f) = 2\). Let \(H\) be a simple path in \(\partial_r \bar{U}\) that joins these two postcritical points. By the same reasoning as in Case 1, \(H\) has length 2; see Figure 21 (Case 2b). Let \(\bar{U} = S^2 \setminus H\). Since \(\bar{\gamma}\) is essential, there are no postcritical points in the annulus between \(\bar{\gamma}\) and \(H\). Thus, each \(\epsilon\)-boundary \(\beta\) of \(\bar{U}\) wrt. \(H\) is isotopic to \(\bar{\gamma}\) for sufficiently small \(\epsilon\).

Note that in all cases \(H\) essentially separates \(V = P_f\), because in all cases \(H\) separates the pairs of points in \(V\) contained in different complementary components of \(\bar{\gamma}\). Moreover, by our choice the circuit length of \(\bar{U}\) is equal to 4 in Cases 1 and 2b, and equal to 2 in Case 2a. So in each case it is \(\leq 4\). Now, by applying Lemma 6.1 to the face \(\bar{U}\) of \(H\), we can choose \(\beta\) so that \(\#(\beta \cap f^{-1}(\alpha^h)) \leq 4i(\tau, \alpha^h)\).

Let \(\alpha_1^h\) and \(\alpha_2^h\) be the two pullbacks of \(\alpha^h\) that are isotopic to \(\alpha^h\) (there are exactly two such pullbacks by Lemma 6.7). Then in all cases we have

\[
\begin{align*}
2i(\bar{\gamma}, \alpha^h) & = 2i(\beta, \alpha^h) = i(\beta, \alpha_1^h) + i(\beta, \alpha_2^h) \\
& \leq \#(\beta \cap \alpha_1^h) + \#(\beta \cap \alpha_2^h) \\
& \leq \#(\beta \cap f^{-1}(\alpha^h)) \\
& \leq 4i(\tau, \alpha^h) \\
& = 2i(\gamma, \alpha^h).
\end{align*}
\]

Thus, \(i(\bar{\gamma}, \alpha^h) \leq i(\gamma, \alpha^h)\). The same reasoning (with a possibly different choice of \(\beta\)) also shows \(i(\bar{\gamma}, \alpha^v) \leq i(\gamma, \alpha^v)\). Combining these inequalities, we conclude:

\[
\|\mu_f(x)\| = \frac{1}{2}i(\bar{\gamma}, \alpha^h) + \frac{1}{2}i(\bar{\gamma}, \alpha^v) \leq \frac{1}{2}i(\gamma, \alpha^h) + \frac{1}{2}i(\gamma, \alpha^v) = \|x\|.
\]

This completes the proof of the first part of statement.

Note that the second inequality in (8.3) is strict if \(\beta\) intersects a peripheral pullback of \(\alpha^h\). A similar statement is also true for the analogous inequality for the curve \(\alpha^v\). We now assume that \(x = r/s \in \mathbb{Q}\) satisfies \(\|x\| > 8\) and will argue that then indeed either inequality (8.3) or the analogous inequality for \(\alpha^h\) is strict which will lead to \(\|\mu_f(x)\| < \|x\|\).

To see this, first note that \(f = \tilde{\phi} \circ \phi^{-1}\), where \(\tilde{\phi} : \mathbb{P} \to \mathbb{P}\) is the associated branched covering map obtained by blowing up the \((2 \times 2)\)-Lattès map and \(\phi : \mathbb{P} \to \mathbb{P}\) is a suitable homeomorphism (see Section 4.2 for the details). We proceed as in the first part of the proof and represent \(x\) by a geodesic \(\gamma\) in \((\mathbb{P}, V)\) of slope \(x = r/s\). We may assume that \(\gamma\) has an essential pullback \(\bar{\gamma}\) under \(f\), because otherwise we have the desired strict inequality by (8.2).
We choose $\mathcal{H}$ as before and define $\widehat{\mathcal{H}} := \phi^{-1}(\mathcal{H})$. Then $\widehat{\mathcal{H}}$ is a connected subset of $\widehat{\mathcal{L}}^{-1}(\tau)$ that essentially separates $\widehat{\mathcal{V}}$, where $\widehat{\mathcal{V}} = \phi^{-1}(V)$ is the set of vertices of the flapped pillow $\widehat{P}$. Since $\|x\| = |r| + s > 8$, we can apply Lemma 8.5 (with $n = 2$) and conclude that the set $\widehat{\mathcal{H}}$ will meet a base edge and the top edge of some flap $F$ in $\widehat{P}$. We will assume that $F$ is a horizontal flap, the case of a vertical flap being completely analogous.

If $\epsilon$ is small enough, then the $\epsilon$-boundary $\beta$ constructed above traces $\mathcal{H}$ very closely in the sense that for each point in $\mathcal{H}$, there is a nearby point in $\beta$. The same is true for $\widehat{\beta} := \phi^{-1}(\beta)$ and $\widehat{\mathcal{H}}$. Using Remark 8.4, this implies that if $\epsilon$ is sufficiently small (as we may assume), then $\widehat{\beta}$ will meet the peripheral pullback $\widehat{\alpha}$ of $\alpha^h$ under $\widehat{\mathcal{L}}$ contained in the horizontal flap $F$. Consequently, $\beta$ meets the peripheral pullback $\phi(\widehat{\alpha})$ of $\alpha^h$ under $f$. As we already pointed out, this leads to a strict inequality in (8.3) and consequently also in (8.4). The statement follows. □

Remark 8.6. The proof of Theorem 1.4 shows that the global curve attractor for the slope map $\mu_f$ is a subset of the finite set $S = \{x \in \widehat{\mathcal{Q}} \cup \{\circ\} : \|x\| \leq 8\}$. Even more, (8.3) and (8.4) imply that $\|\mu_f(x)\| = \|x\|$ if and only if $\mu_f(x) = x$. Therefore, the attractor for the slope map is given by the set $\{x \in \widehat{\mathcal{Q}} \cup \{\circ\} : \mu_f(x) = x\} \subset S$. In other words, the global curve attractor $A(f)$ consists of essential curves that are invariant under $f$ and peripheral curves.

The proof of Theorem 1.4 is now easy.

Proof of Theorem 1.4. Let $f : P \to P$ be a Thurston map as in the statement. Then Proposition 8.1 implies that if $x \in \widehat{\mathcal{Q}} \cup \{\circ\}$ is arbitrary, then the complexities of the elements $x$, $\mu_f(x)$, $\mu_f^2(x)$, ... of the orbit of $x$ under iteration of $\mu_f$ strictly decrease until this orbit eventually reaches the set $S := \{u \in \widehat{\mathcal{Q}} \cup \{\circ\} : \|u\| \leq 8\}$. From this point on, the orbit of $x$ stays in $S$. The statement follows. □

In principle, the global curve attractor for a map $f$ as in Theorem 1.4 depends on the locations of the flaps. By Remark 8.6 for each special case one can easily determine the exact attractor by checking if a slope $x \in \widehat{\mathcal{Q}}$ with $\|x\| \leq 8$ is invariant. For example, by using a computer program written by Darragh Glynn, we verified that for the map $f$ corresponding to flapped pillow in Figure 22 (with one horizontal flap and one vertical flap added at the two 1-edges incident to the vertex $B$) the invariant slopes are $0$, $\infty$, $1$, $-1$. 

![Figure 22. A flapped pillow.](image-url)
9. Further discussion

In this section, we briefly discuss some additional topics related to the investigations in this paper.

9.1. Julia sets of blown-up Lattès maps. An obvious question is what we can say about the Julia sets of the rational maps provided by Theorem [1.2].

Proposition 9.1. Let \( g: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a rational map that is Thurston equivalent to a map \( f: \mathbb{P} \to \mathbb{P} \) obtained from the \((n \times n)\)-Lattès map with \( n \geq 2 \) by gluing \( n_h \geq 1 \) horizontal and \( n_v \geq 1 \) vertical flaps to pillow \( \mathbb{P} \). Then the following statements are true:

(i) the Julia set of \( g \) is equal to \( \hat{\mathbb{C}} \) if \( n \) is even and the vertex \( A \) is not contained in a flap, or if \( n \) is odd and none of the points in \( V \) is contained in a flap.

(ii) the Julia set of \( g \) is equal to a Sierpiński carpet in \( \hat{\mathbb{C}} \) if \( n \) is even and \( A \) is contained in a flap, or if \( n \) is odd and one of the points in \( V \) is contained in a flap.

Obviously, these cases cover all possibilities and so the Julia set of \( g \) is either the whole Riemann sphere \( \mathbb{P} \) or a Sierpiński carpet, i.e., a subset of \( \mathbb{P} \) homeomorphic to the standard 1/3-Sierpiński carpet fractal. As we will see, in the first case the map \( g \) has no periodic critical points, while it has periodic critical points (namely critical fixed points) in the second case.

Proof. Let \( g \) be a rational map as in the statement. To see what the Julia set of \( g \) is, we will check whether \( g \) has periodic critical points or not, and verify in the second case that \( g \) has no Levy arcs (see below for the definition). These conditions are invariant under Thurston equivalence and therefore it is enough to consider the map \( f \). Then \( P_f = V \), where \( V = \{A, B, C, D\} \) is the set of vertices of \( \mathbb{P} \). By definition of the \((n \times n)\)-Lattès map, for each \( X \in V \) we have \( L_n(X) = A \) if \( n \) is even and \( L_n(X) = X \) if \( n \) is odd. Since \( f|V \) agrees with \( L_n|V \), this implies that for each \( X \in V \) we also have \( f(X) = A \) if \( n \) is even and \( f(X) = X \) if \( n \) is odd.

Since the orbit of each critical point under iteration of \( f \) passes through the set \( P_f = V \), this shows that the only possible periodic critical points of \( f \) is the point \( A \) if \( n \) is even or the points in \( V \) if \( n \) is odd. Now for \( X \in V \) we have \( \deg_{L_n}(X) = 1 \) and so

\[
\deg_f(X) = n_X + \deg_{L_n}(X) = n_X + 1,
\]

where \( n_X \in \mathbb{N}_0 \) is the number of flaps that contain \( X \). These considerations show that \( f \), and hence also \( g \), has no periodic critical points in case (i). Hence the Julia set of \( g \) is the whole sphere \( \hat{\mathbb{C}} \) in this case.

In case (ii), the map \( f \), and hence also \( g \), has a critical fixed point, and so the Fatou set of \( g \) is non-empty. To show that its Julia set is a Sierpiński carpet, we use the following criterion that follows from [BD18, Lemma 4.16]: the Julia set of \( g \) is a Sierpiński carpet if and only if \( g \), or equivalently \( f \), has no Levy arcs. Here a Levy arc of \( f \) is a path \( \alpha \) in \( \mathbb{P} \) satisfying the following conditions:

(L1) \( \alpha \) is an arc in \((\mathbb{P}, V)\), or \( \alpha \) is a simple loop based at a point \( X \in V \) such that \( \alpha \setminus \{X\} \subset \mathbb{P} \setminus V \) and each component of \( \mathbb{P} \setminus \alpha \) contains at least one point in \( V \).

(L2) there exist \( k \in \mathbb{N} \) and a lift \( \tilde{\alpha} \) of \( \alpha \) under \( f^k \) such that \( \alpha \) and \( \tilde{\alpha} \) are isotopic rel. \( V \).

Now suppose that \( f \) has a Levy arc \( \alpha \) with \( \tilde{\alpha} \) and \( k \in \mathbb{N} \) as in (L2). Then \( f^k|\tilde{\alpha} \) is a 1-to-1 map and either \( i(\alpha, \alpha^h) > 0 \) or \( i(\alpha, \alpha^h) > 0 \). Without loss of generality, we may assume that \( i(\alpha, \alpha^h) = \#(\alpha \cap \alpha^h) > 0 \). If \( \alpha \) is an arc in \((\mathbb{P}, V)\), then we can apply Lemma [5.5] with \( \gamma := \alpha^h \),
\( f := f^k \), and \( \tilde{\alpha} := \tilde{\alpha} \) and conclude that the number of distinct pullbacks of \( \alpha^h \) under \( f^k \) that are isotopic to \( \alpha^h \) is at most 1. This is also true if \( \alpha \) is a simple loop as in (L1) by the argument in the proof of Lemma 5.5.

We reach a contradiction, because it follows from Lemma 5.4 that \( \alpha^h \) has \( n^k > 1 \) such pullbacks. Consequently, \( f \) and \( g \) do not have any Levy arcs and so the Julia set of \( g \) is a Sierpiński carpet. \( \square \)

9.2. The global curve attractor problem. We were able to prove the existence of a finite global curve attractor only for blown-up \((n \times n)\)-Lattès maps for \( n = 2 \). The proof of Theorem 1.4 crucially relies on Proposition 8.1 which says that the (naturally defined) complexity of curves does not increase under the pullback operation. The latter statement is false for blown-up \((n \times n)\)-Lattès maps with \( n \geq 3 \).

Numerical computations by Darragh Glynn suggest that for some blown-up \((3 \times 3)\)-Lattès map \( f \) one can have infinitely many slopes \( x \in \hat{\mathbb{Q}} \) such that \( \| \mu_f(x) \| > \| x \| \). For example, consider the map \( f \) obtained from the \((3 \times 3)\)-Lattès map by blowing once the horizontal and vertical edges incident to the vertex \( B \) of \( \mathbb{P} \). Then one can prove the following general relation for the slope map \( \mu_f \):

\[
\mu_f(r/s) = r'/s' \Rightarrow \mu_f(r/(s + 24r)) = r'/(s' + 22r').
\]

Based on this one can show that \( \| \mu_f(x) \| > \| x \| \) for all

\[
x \in \{ 1/(m + 24k) : m \in \{ 7, 8, 9, 15, 16, 17 \}, k \in \mathbb{N}_0 \}.
\]

Actually, it seems that in this case the slope map \( \mu_f \) has orbits with arbitrarily many strict increases of complexity. For instance, we have two jumps of complexity for the orbit of slope 1/9 under \( \mu_f \):

\[
1/9 \rightarrow 3/25 \rightarrow 3/23 \rightarrow 1/7 \rightarrow 3/19 \rightarrow 3/17 \rightarrow 1/5 \rightarrow 1/5.
\]

Note that this orbit stabilizes at the fixed point 1/5 of \( \mu_f \). The numerical computations by Darragh Glynn also show that there are examples of blown-up \((n \times n)\)-Lattès maps with \( n \geq 5 \) for which the slope map has periodic cycles of length \( \geq 2 \).

It is natural to ask what one can say about the behavior of the slope map \( \mu_f \) for an obstructed Thurston map \( f \) (with \( \# P_f = 4 \)). It was already observed in [KPS16] that for a blown-up \((2 \times 2)\)-Lattès map \( f \) with only vertical flaps, there are infinitely many (non-isotopic) invariant essential Jordan curves. Indeed, for such a map \( f \) the curve \( \alpha^v \) is \( f \)-invariant and satisfies \( \lambda_f(\alpha^v) = 1 \). One can use this to show that \( f \) commutes with \( T^2 \) (up to isotopy rel. \( P_f \)), where \( T \) is a Dehn twist about \( \alpha^v \). This implies that each curve \( T^{2n}(\alpha^h) \) is \( f \)-invariant. In fact, it is easy to verify directly that each essential Jordan curve with slope \( x \in \mathbb{Z} \cup \{ \infty \} \) is \( f \)-invariant, or equivalently, that \( \mu_f(x) = x \) for \( x \in \mathbb{Z} \cup \{ \infty \} \).

However, for such a blown-up \((2 \times 2)\)-Lattès map \( f \) with only vertical flaps the general behavior of the slope map \( \mu_f \) under iteration has not been analyzed before. The considerations in the proof of the first part of Proposition 8.1 also apply in this situation. In particular, (8.3) and (8.4) are still true and show that the orbit of an arbitrary \( x \in \hat{\mathbb{Q}} \cup \{ \infty \} \) under \( \mu_f \) eventually lands in a fixed point of \( \mu_f \). Moreover, results in Section 8 provide a method to determine all fixed slopes for \( \mu_f \).

The easiest case is the map \( f \) obtained from the \((2 \times 2)\)-Lattès map by adding at least one vertical flap to each of the four vertical 1-edges in the “middle” of the pillow. Then each closed geodesic and each geodesic arc with slope \( x \in \mathbb{Q} \setminus \mathbb{Z} \) must pass through the interior of one of these four vertical 1-edges. Consequently, we can apply the discussion in the second
part of the proof of Proposition 8.1 to and conclude that \( \| \mu_f(x) \| < \| x \| \) for \( x \in \mathbb{Q} \setminus \mathbb{Z} \). Thus, the orbit of each \( x \in \mathbb{Q} \cup \{ \bigcirc \} \) under \( \mu_f \) eventually lands in \( \mathbb{Z} \cup \{ \infty, \bigcirc \} \) (that is, in a fixed point of \( \mu_f \)). Since the map \( f \) is easily seen to be expanding (see [BM17, Definition 2.2 and Theorem 14.1]), it provides an answer to a question by Kevin Pilgrim (see [Pil18, Question 4.4]).

9.3. Twisting problems. To this date, many natural problems related to Thurston equivalence remain rather mysterious and are often very difficult to solve. Twisting problems are examples of this nature.

To explain this, suppose we are given a rational Thurston map \( f : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}} \). Let \( \phi : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}} \) be an orientation-preserving homeomorphism that fixes the postcritical set \( P_f \) pointwise. We now consider the branched covering map \( g := \phi \circ f \) on \( \tilde{\mathbb{C}} \), called the \( \phi \)-twist of \( f \). Then \( C_g = C_f \) and \( g \) has the same dynamics on \( C_f \) as \( f \). In particular, \( P_g = P_f \); so \( g \) has a finite postcritical set and is a Thurston map.

This leads us to the natural questions: Is \( g \) realized? And if yes, to which rational map is \( g \) equivalent depending on the isotopy type of \( \phi \)? In fact, there are only finitely many rational maps \( g \) (up to Möbius conjugation) that can arise in this way from a fixed map \( f \). A famous instance of this question, called the “twisted rabbit problem”, was solved by Laurent Bartholdi and Volodymyr Nekrashevych in [BN06] (see also [Lod13, BLMW19]).

In our context we can ask which twists of maps as in Theorem 1.2 are realized. We do not have an answer to this questions, but it seem that this leads to non-trivial and difficult problems. For example, consider the blown-up \((2 \times 2)\)-Lattès map \( f : \mathbb{P} \to \mathbb{P} \) corresponding to the flapped pillow \( \hat{\mathbb{P}} \) as in Figure 22. Then \( \mathbb{P} \) has one horizontal and one vertical flap, and so \( f \) is realized by Theorem 1.2. One can check that the Jordan curve \( \gamma := \varphi(\ell_{1/3}) \) has exactly two essential pullbacks \( \gamma_1, \gamma_2 \sim \varphi(\ell_{1/3}) \) with \( \deg(f : \gamma_1 \to \gamma) = 1 \) and \( \deg(f : \gamma_2 \to \gamma) = 2 \). We now choose an orientation-preserving homeomorphism \( \phi : \mathbb{P} \to \mathbb{P} \) that maps \( \gamma \) onto \( \gamma_1 \), while fixing each point in \( V = P_f \). Then the curve \( \gamma_1 \) is an obstruction for the twisted map \( g := \phi \circ f \) with \( \lambda_g(\gamma_1) = 3/2 \). An analogous construction applies to some other essential Jordan curves, for instance, with slopes \( 23/3 \) and \( 49/3 \), and gives twists of \( f \) with an obstruction.

It follows from this discussion that the mapping class biset associated with \( f \) is not contracting (see [BD17, BD18] for the definitions). Thus, the algebraic methods for solving the global curve attractor problem developed in [Pil12] (see, specifically, Theorem 1.4) do not apply in general for the maps considered in Theorem 1.2.

9.4. More than four postcritical points. While in this paper we only discuss the case of Thurston maps \( f : S^2 \to S^2 \) with \( \# P_f = 4 \), it is natural to ask if one can adapt Theorem 1.1 to the case when \( \# P_f > 4 \). The main difficulty is that an obstruction in this case is in general not given by a unique essential Jordan curve in \((S^2, P_f)\), but by a multicurve. Of course, this fact complicates the analysis of pullback properties of curves and their intersection numbers. However, we expect that one can naturally generalize our result for an arbitrary Thurston map: given an obstructed Thurston map \( f \) one can eliminate all possible multicurve obstructions by successively applying the blow-up operation and obtain a map that is realized.

9.5. Other combinatorial constructions of rational maps. The dynamical behavior of curves under the pullback operation is an important topic in holomorphic dynamics. While in this paper we only study the realization and the global curve attractor problems, one is led
to similar considerations in the study of iterated monodromy groups, for example. For these investigations it is important to have a class of rational maps constructed in combinatorial fashion against which conjectures can be tested or which lead to the discovery of general phenomena. The maps provided by Theorem 1.1 may be useful in this respect. Another interesting class of maps worthy of further investigation are Thurston maps constructed from tilings of the Euclidean or hyperbolic plane as in [BM17 Example 12.25].

References


