Today we will be looking at a proof of the real spectral theorem from Axler. This is a slightly different proof than was done in class, although the basic outline is similar.

This theorem states a real symmetric matrix (i.e. one such that \( A = A^t \)) has an orthonormal basis of (real) eigenvectors.

The basic plan is to first show that \( A \) has an eigenvector and then induct.

1 Every real symmetric matrix has an eigenvector.

The most obvious solution to this is as follows: View \( A \) as a complex matrix. Then \( A \) has an eigenvector \( v \) with eigenvalue \( \lambda \). Then \( \langle Av, v \rangle = \langle v, A^t v \rangle = \langle v, Av \rangle \).

Then \( \langle \lambda v, v \rangle = \langle v, \lambda v \rangle \).

Then \( \lambda \langle v, v \rangle = \bar{\lambda} \langle v, v \rangle \).

As \( v \neq 0 \), \( \langle v, v \rangle \neq 0 \) so \( \lambda = \bar{\lambda} \) so \( \lambda \) is real.

Then \( \ker(A - \lambda I) \) is non-trivial, so \( A \) has a real eigenvector with eigenvalue \( \lambda \).

Axler has a different approach that never requires us to view \( A \) as a complex matrix. First we start with an auxiliary lemma.

Lemma: Let \( T \in M_n(\mathbb{C}) \) be a real symmetric matrix and \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha^2 < 4\beta \). Then \( T^2 + \alpha T + \beta I \) is invertible.

Proof:

Let \( v \neq 0 \).

Then

\[
\langle (T^2 + \alpha T + \beta I)v, v \rangle = \langle T^2v, v \rangle + \alpha \langle Tv, v \rangle + \beta \langle v, v \rangle
\]

\[
= \langle Tv, T^tv \rangle + \alpha \langle Tv, v \rangle + \beta \langle v, v \rangle
\]

\[
= \langle Tv, Tv \rangle + \alpha \langle Tv, v \rangle + \beta \langle v, v \rangle
\]

\[
= ||Tv||^2 + \alpha \langle Tv, v \rangle + \beta ||v||^2
\]
By the Cauchy-Schwarz inequality, $|\langle Tv, v \rangle| \leq ||Tv|| \cdot ||v||$.

Then:

$$||Tv||^2 + \alpha \langle Tv, v \rangle + \beta ||v||^2 \geq ||Tv||^2 + \beta ||v||^2 - |\alpha||Tv|| \cdot ||v||$$

$$= (||Tv||^2 - \frac{|\alpha| \cdot ||v||}{2})^2 + (\beta - \frac{\alpha^2}{4}) ||v||^2$$

Then as $b - \frac{\alpha^2}{4} > 0$, $\langle (T^2 + \alpha T + \beta I)v, v \rangle > 0$, so $(T^2 + \alpha T + \beta I)v \neq 0$, so $v \notin \ker T^2 + \alpha T + \beta I$.

Now we can prove that every symmetric matrix has an eigenvector.

Let $T$ be a real symmetric matrix.

As you know from your take home exam, there exists a non-zero polynomial with real coefficients $p(x) \in \mathbb{R}[x]$ such that $p(T) = 0$.

Recall from high school algebra that if $r$ is a root of $p$, then either $r$ is real or $r$ and $\bar{r}$ are both roots of $p$ with the same multiplicity. (Easy exercise: prove this.)

Then we can write $p(x) = c(x^2 + \alpha_1 x + \beta_1) \cdots (x^2 + \alpha_M x + \beta_M)(x - \lambda_1) \cdots (x - \lambda_M)$ where each of the $x^2 + \alpha_i x + \beta_i$ is irreducible over the reals.

Then $\alpha_i^2 < 4 \beta_i$.

Hence $0 = p(T) = c(T^2 + \alpha_1 T + \beta_1 I) \cdots (T^2 + \alpha_M T + \beta_M I)(T - \lambda_1 I) \cdots (T - \lambda_M I)$.

As $T^2 + \alpha_i T + \beta_i I$ is invertible for each $I$, we have that $(T - \lambda_1 I) \cdots (T - \lambda_M I) = 0$.

At least one $T - \lambda_i I$ must be not invertible as the product of invertible matrices is invertible. Hence $T$ has a (real) eigenvector.

\section{The real spectral theorem}

Theorem: Let $V$ be a finite dimensional inner product space over $\mathbb{R}$ and let $T : V \to V$ be symmetric. Then there is an orthonormal basis of eigenvectors of $T$.

We proceed by induction on $\dim V$.

When $\dim V = 1$, take an eigenvector for $T$ (which exists by the previous section) and normalize.

Now suppose that the theorem holds true for inner product spaces of dimension $\dim V - 1$.

Let $u$ be an eigenvector of $T$ with eigenvalue $\lambda$ and with norm 1 and let $U = \text{Span}\{u\}$.

We claim that $T|_{U^\perp}$ is an operator on $U^\perp$.

Let $v \in U^\perp$.

Then $\langle u, Tv \rangle = \langle Tu, v \rangle = \lambda \langle u, v \rangle = 0$.

Hence $Tv \in U^\perp$. 

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Then \( T|_{U^\perp} \) is a symmetric operator on a space of dimension \( \dim V - 1 \), so by the induction hypothesis, there is an orthonormal basis of eigenvectors of \( T|_{U^\perp} \).

As each of these is orthogonal to \( u \), adding in \( u \) gives an orthonormal basis of eigenvectors for \( T \).