Week 9 Tuesday Notes

Andrew Sack

March 3, 2021

Today we will be looking at a proof of the real spectral theorem from Axler. This is a slightly different proof than was done in class, although the basic outline is similar.

This theorem states a real symmetric matrix (i.e. one such that $A = A^t$) has an orthonormal basis of (real) eigenvectors.

The basic plan is to first show that A has an eigenvector and then induct.

1 Every real symmetric matrix has an eigenvector.

The most obvious solution to this is as follows: View A as a complex matrix. Then A has an eigenvector v with eigenvalue λ. Then $\langle Av, v\rangle = \langle v, A^t v\rangle = \langle v, Av\rangle$.
Then $\langle \lambda v, v\rangle = \langle v, \lambda v\rangle$.
Then $\lambda\langle v, v\rangle = \bar{\lambda}\langle v, v\rangle$.
As $v \neq 0, \langle v, v\rangle \neq 0$ so $\lambda = \bar{\lambda}$ so λ is real.
Then $\ker(A - \lambda I)$ is non-trivial, so A has a real eigenvector with eigenvalue λ.

Axler has a different approach that never requires us to view A as a complex matrix. First we start with an auxiliary lemma.

Lemma: Let $T \in M_n(\mathbb{C})$ be a real symmetric matrix and $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 < 4\beta$. Then $T^2 + \alpha T + \beta I$ is invertible.
Proof:
Let $v \neq 0$.
Then

$$\begin{aligned}\langle (T^2 + \alpha T + \beta I)v, v\rangle &= \langle T^2 v, v\rangle + \alpha\langle Tv, v\rangle + \beta\langle v, v\rangle \\ &= \langle Tv, T^t v\rangle + \alpha\langle Tv, v\rangle + \beta\langle v, v\rangle \\ &= \langle Tv, Tv\rangle + \alpha\langle Tv, v\rangle + \beta\langle v, v\rangle \\ &= ||Tv||^2 + \alpha\langle Tv, v\rangle + \beta||v||^2\end{aligned}$$

By the Cauchy-Schwarz inequality, $|\langle Tv, v\rangle| \leq ||Tv|| \cdot ||v||$.
Then:

$$\begin{aligned} ||Tv||^2 + \alpha\langle Tv, v\rangle + \beta||v||^2 &\geq ||Tv||^2 + \beta||v||^2 - |\alpha|||Tv|| \cdot ||v||\rangle \\ &= (||Tv|| - \frac{|\alpha| \cdot ||v||}{2})^2 + (\beta - \frac{\alpha^2}{4})||v||^2 \end{aligned}$$

Then as $b - \frac{\alpha^2}{4} > 0$, $\langle (T^2 + \alpha T + \beta I)v, v\rangle > 0$, so $(T^2 + \alpha T + \beta I)v \neq 0$, so $v \notin \ker T^2 + \alpha T + \beta I$.

Now we can prove that every symmetric matrix has an eigenvector.

Let T be a real symmetric matrix.
As you know from your take home exam, there exists a non-zero polynomial with real coefficients $p(x) \in \mathbb{R}[x]$ such that $p(T) = 0$.
Recall from high school algebra that if r is a root of p, then either r is real or r and $\bar{r}$ are both roots of p with the same multiplicity. (Easy exercise: prove this.)
Then we can write $p(x) = c(x^2 + \alpha_1 x + \beta_1) \ldots (x^2 + \alpha_M x + \beta_M)(x - \lambda_1) \ldots (x - \lambda_M)$ where each of the $x^2 + \alpha_i x + \beta$ is irreducible over the reals.
Then $\alpha_i^2 < 4\beta_i$.
Hence $0 = p(T) = c(T^2 + \alpha_1 T + \beta_1 I) \ldots (T^2 + \alpha_M T + \beta_M I)(T - \lambda_1 I) \ldots (T - \lambda_M I)$.

As $T^2 + \alpha_i T + \beta_i I$ is invertible for each I, we have that $(T - \lambda_1 I) \ldots (T - \lambda_M I) = 0$.
At least one $T - \lambda_i I$ must be not invertible as the product of invertible matrices is invertible.
Hence T has a (real) eigenvector.

2 The real spectral theorem

Theorem: Let V be a finite dimensional inner product space over $\mathbb{R}$ and let $T : V \to V$ be symmetric. Then there is an orthonormal basis of eigenvectors of T.

We proceed by induction on $\dim V$.
When $\dim V = 1$, take an eigenvector for T (which exists by the previous section) and normalize.

Now suppose that the theorem holds true for inner product spaces of dimension $\dim V - 1$.

Let u be an eigenvector of T with eigenvalue λ and with norm 1 and let $U = \text{Span}\{u\}$.

We claim that $T|_{U^\perp}$ is an operator on $U^\perp$.

Let $v \in U^\perp$.

Then $\langle u, Tv\rangle = \langle Tu, v\rangle = \lambda\langle u, v\rangle = 0$.
Hence $Tv \in U^\perp$.

Then $T|_{U^\perp}$ is a symmetric operator on a space of dimension $\dim V - 1$, so by the induction hypothesis, there is an orthonormal basis of eigenvectors of $T|_{U^\perp}$.
As each of these is orthogonal to u, adding in u gives an orthonormal basis of eigenvectors for T.