Week 9 Thursday Notes

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Today we will do some more practice in problem solving together by doing past basic qualifying exam problems.

1 Fall 2017 Problem 6

For each of the following three fields F (separately), is it true that every symmetric matrix $A \in M_{2\times 2}(F)$ is diagonalizable?

2 pts For $F = \mathbb{R}$

3 pts For $F = \mathbb{C}$

5 pts For $F = \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$, the field with 3 elements.

Supply proofs/counterexamples (or cite the relevant theorems) for all parts of this problem.

1.1 Part (a)

This is true by the spectral theorem.

1.2 Part (b)

We know that the complex spectral theorem applies for Hermitian matrices, that is, when a matrix is equal to its conjugate transpose. However, a symmetric matrix is one that is equal to its transpose. Then we probably should suspect that this is not true.

This suggests that we should test out a matrix that is equal to its own transpose, but not its conjugate transpose.

Note: We should at least try out a few examples, as a matrix like $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ certainly symmetric and diagonalizable, but is not Hermitian.

Let us test out $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. This has characteristic polynomial $z^2 - i^2 = z^2 + 1$, so it has distinct eigenvalues and is diagonalizable.
Let us instead test out $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. This has characteristic polynomial $(z-1)^2 - i^2 = z^2 - 2z + 2$, which also has two distinct roots. Hmmm, maybe we should try to engineer a counterexample instead.

If we want our matrix to not be diagonalizable, its characteristic polynomial should at the very least have only 1 root with multiplicity 2.

Generically, our matrix looks like $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, so it has characteristic polynomial $(z-a)(z-c) - b^2 = z^2 - (a+c)z + ac - b^2$.
If there is only one eigenvalue λ, then the characteristic polynomial should also be of the form $(z-\lambda)^2 = z^2 - 2\lambda + \lambda^2$.

Perhaps we massage this to get us a characteristic polynomial of $z^2 - 2z + 1$. A reasonable guess to achieve this would be to start by taking $a = c = 1$. This gives us $z^2 - 2z + 1 - b^2$, so we need $b^2 = 0$, but $b = 0$ makes the matrix diagonal.
Perhaps instead we can take $a = 2, c = 0$. Then we get $z^2 - 2z - b^2$, so we want $b^2 = -1$, so we can take $b = i$.

Let us test the matrix $M = \begin{pmatrix} 2 & i \\ i & 0 \end{pmatrix}$.
As we just calculated, this has characteristic polynomial $z^2 - 2z + 1$, so its only eigenvalue is 1. However, observe that if a matrix is diagonalizable and has its only eigenvalue equal to 1, then it is the identity, which M is not.

This is a problem solving technique I like to call "educated guess and check." We make a guess, but not just a completely random guess. As we say, our initial completely random guesses didn't work out. In fact, almost all random guess we could make would not work out.
Instead, we can deduce properties that a counterexample must have and try to find one within that space of possibility.

1.3 Part (c)

Generically, a 2×2 symmetric matrix with entries in $\mathbb{F}_3$, looks like $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

In this case, we can take the following strategy: There are only 27 such matrices, so we can test them all. In fact, when $b = 0$, the matrix is already diagonal, so we only need to check 18 matrices.

However, we can also take an approach like in part (b).

Let us hunt for a counterexample or otherwise exhaust all possibilities:

Assume $b \neq 0$. Then the characteristic polynomial is $(z-a)(z-c)-b^2 = z^2-(a+c)z+ac-b^2$.

As $b \neq 0$, $b = 1$ or 2, which in either case tells us $b^2 = 1$.
This is also a useful fact for another reason: not every polynomial over $\mathbb{F}_3$ has root. In particular, $z^2 - 2$ does not have a root.
If we can arrange for the characteristic polynomial to be equal to $z^2 - 2$, then $\mathbb{F}_3$ will have no eigenvalues, hence no eigenvectors and hence certainly can't have a basis of eigenvectors.

Then we want $a + c = 0$ and $ac - b^2 = ac - 1 = -2$.
Then $a = -c$ and $ac = -1$, so $-a^2 = -1$. Then $a^2 = 1$, so take $a = 1$ or $a = 2$. If we take $a = 1$, we can take $c = 2$ to get the characteristic polynomial as desired (with $b = 1$ or 2)

Hence $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is not diagonalizable.

1.4 Final Thoughts

Here is a potentially useful fact:

Observe that if there is a counterexample, then it must have only 1 eigenvalue. Furthermore, a diagonalizable matrix with only 1 eigenvalue λ is similar to λI, so is equal to $A\lambda I A^{-1} = \lambda I$.

Hence we were really looking for matrices with 1 eigenvalue that were not already diagonal.

2 Fall 2015 Problem 9

Let A be a $n \times n$ real matrix such that $A^T = -A$. Prove that $\det A \geq 0$.

Andrew's hint: Use the fact that $\det A$ is equal to the product of the (complex) eigenvalues of A counting multiplicity.

2.1 Solution

View A as a complex matrix and let v be an eigenvector of A with eigenvalue $\lambda = a + bi$ with $a, b \in \mathbb{R}$.

Then

$$\begin{aligned}
\lambda\langle v, v\rangle &= \langle \lambda v, v\rangle \\
&= \langle Av, v\rangle \\
&= \langle v, A^* v\rangle \\
&= \langle v, A^T v\rangle \quad \text{(As } A \text{ is a real matrix)} \\
&= \langle v, -Av\rangle \\
&= \langle v, -\lambda v\rangle \\
&= -\bar{\lambda}\langle v, v\rangle
\end{aligned}$$

As $v \neq 0, \langle v, v\rangle \neq 0$ so $\lambda = -\bar{\lambda}$.

Hence $a + bi = -a + bi$ so $a = 0$.

Hence the only eigenvalues of AA are 0 or purely imaginary.

As A is a real matrix, the characteristic polynomial of A has real coefficients. Hence it either has 0 as an eigenvalue (in which case $\det A = 0$) or its eigenvalues come in conjugate pairs bi and $-bi$ with $b \in \mathbb{R}$.

The product of one of these conjugate pairs is b^2, so the product of all conjugate pairs (which will be the determinant) is positive.

3 Fall 2014 Problem 11

This is a cute problem:

Let A be a 4×4 matrix with integer entries such that A has 4 distinct real eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. Prove that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \in \mathbb{Z}$.

3.1 Solution

This is a cute problem to which I have a cute solution.

The characteristic polynomial $\chi(zI - A) = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4)$ has integer coefficients.

Expanding out this is: $z^4 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)z^3 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda 2\lambda_3 + \lambda_3\lambda_4)z^2 - (*)z + \lambda_1\lambda_2\lambda_3\lambda_4$ where $(*)$ is whatever the coefficient of that term is.

Hence $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \in \mathbb{Z}$ and $(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda 2\lambda_3 + \lambda_3\lambda_4) \in \mathbb{Z}$.

Then $\mathbb{Z} \ni (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda 2\lambda_3 + \lambda_3\lambda_4)$.

Hence $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda 2\lambda_3 + \lambda_3\lambda_4) - 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda 2\lambda_3 + \lambda_3\lambda_4) \in \mathbb{Z}$.