Week 8 Thursday Notes

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Today we will be discussing problems from past UCLA qualifying exams. In particular, these are from the linear algebra portion of the so-called "basic" qualifying exam, which is an exam all UCLA graduate students must pass before the start of their second year.

1 Fall 2018 Number 9

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be linearly independent elements of the vector space (over $\mathbb{R}$) of linear mappings from $\mathbb{R}^n \to \mathbb{R}$. Show that for any $v \in \mathbb{R}^n$ there exist v_1 and v_2 such that

$$v = v_1 + v_2, \quad f(v) = f(v_1), \quad \text{and} \quad g(v) = g(v_2).$$

A good approach when you see a problem like this is to first write down your first observations about the problem.

The first thing I see is to ask what it means if $f(v) = f(v_1)$?

Well then $f(v) = f(v_1 + v_2) = f(v_1) + f(v_2)$, so $f(v_2) = 0$. Likewise we see that $g(v_1) = 0$.

Then we can rephrase the question as follows: Show that $\mathbb{R}^n = \ker f + \ker g$.

From the Rank-Nullity theorem, we know that $\dim \ker f \geq n - 1$ and $\dim \ker g \geq n - 1$. In fact since f and g are linearly independent, they are both not zero, so $\dim \ker f = \dim \ker g = n - 1$.

Since we know that $\dim(\ker f + \ker g) = \dim \ker f + \dim \ker g - \dim(\ker f \cap \ker g) = 2n - 2 - \dim(\ker f \cap \ker g)$, it suffices to show that $\dim(\ker f \cap \ker g) \leq n - 1$.

Then all we must show is that $\ker f \neq \ker g$.

We somehow must use the fact that f and g are linearly independent. Perhaps we can try to show that if $\ker f = \ker g$, then f and g are linearly dependent, contradicting our assumptions. Indeed, this will be our plan of attack.

Suppose that $\ker f = \ker g$ and let $\{v_1, ..., v_{n-1}\}$ be a basis for $\ker f = \ker g$ and extend this to a basis $\{v_1, ..., v_n\}$ for $\mathbb{R}^n$.
Then $f(v_i) = 0$ for $1 \leq i \leq n-1$ and $f(v_n) = a$ for some $a \in \mathbb{R} - \{0\}$. Likewise $g(v_i) = 0$ for $1 \leq i \leq n-1$ and $g(v_n) = b$ for some $b \in \mathbb{R} - \{0\}$.

We claim that $f = \frac{a}{b} g$. Indeed, this is true on every basis element, and is hence true for all vectors $v \in \mathbb{R}^n$.
Then f is a multiple of g, so f and g are linearly independent.

2 Spring 2014 Problem 3 (Enhanced)

Suppose $A, B \in M_n(\mathbb{C})$ satisfy $AB - BA = A$. Show that A is not invertible.
Enhancement: Show that A is nilpotent. That is, for some $k \in \mathbb{Z}^+$, $A^k = 0$.

Solution (unenhanced version):

Suppose A is invertible. Then $ABA^{-1} - B = I$.

Then $\mathrm{tr}(ABA^-1 - B) = \mathrm{tr}(I) = n$.

However $\mathrm{tr}(ABA^{-1} - B) = \mathrm{tr}(ABA^{-1}) - \mathrm{tr}(B) = \mathrm{tr}(A^{-1}AB) - \mathrm{tr}(B) = \mathrm{tr}(B) - \mathrm{tr}(B) = 0$ a contradiction.

Hence A is not invertible.

Solution (Enhanced version):

We have that $AB = A + BA$.

A natural thing to try is to see if we can get some nice expression for A^k.

Observe that

$$\begin{aligned} A^2 &= A(AB - BA) \\ &= A^2B - ABA \\ &= A^2B - (A + BA)A \\ &= A^2B - A^2 + BA^2 \end{aligned}$$

Then we get that $2A^2 = A^2B - BA^2$.

We can try to multiply again on the left by A to get

$$\begin{aligned}2A^3 &= A(A^2B - BA^3)\\ &= A^2B - ABA^2\\ &= A^3B - (A + BA)A^2\\ &= A^3B - A^3 - BA^3\end{aligned}$$

so $3A = A^3B - BA^3$.

Then it seems like a reasonable guess is that $kA = A^kB - BA^k$.

Indeed, we proceed by induction. If $kA = A^kB - BA^k$ then:

$$\begin{aligned}kA^{k+1} &= A(A^kB - BA^{k+1})\\ &= A^kB - ABA^k\\ &= A^{k+1}B - (A + BA)A^k\\ &= A^{k+1}B - A^{k+1} - BA^{k+1}\end{aligned}$$

Then $(k+1)A^{k+1} = A^{k+1}B - BA^{k+1}$.

What can we do with this information? There are a few approaches here. In the vein of the solution to the unenhanced version, we can observe that $\mathrm{tr}(A^k) = 0$ for every k. There is a complicated argument you can make using how the eigenvalues of A^k relate to the eigenvalues of A and the fact that $\mathrm{tr}(X)$ is equal to the sum of the eigenvalues of X counting multiplicity.

Here is a very elegant solution:

Consider the linear transformation $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $T(X) = XB - BX$.

Suppose that $A^k \neq 0$ for any k.

We have shown that A^k is an eigenvector of T for every $k \in \mathbb{Z}^+$ with eigenvalue k.

However T can have at most n eigenvalues, a contradiction.

3 Fall 2019 Problem 1

Let A be an invertible $n \times n$ matrix with real entries and let e_1 denote the unit vector with a 1 in the first position and zeros elsewhere. Show that for each $\lambda \in \mathbb{R}$, the linear transformation A_λ defined by

$$A_\lambda x = Ax + \lambda\langle e_1, x\rangle e_1$$

is invertible if and only if $1 + \lambda\langle e_1, A^{-1}e_1\rangle \neq 0$.

When you first look at this problem, a natural thing to do is to see that $\lambda\langle e_1, A^{-1}e_1\rangle$ term and try to make it appear using A_λ.

Indeed, observe that

$$\begin{aligned} A_\lambda(A^{-1}(e_1)) &= AA^{-1}e_1 + \lambda\langle e_1, A^{-1}e_1\rangle e_1 \\ &= e_1 + \lambda\langle e_1, A^{-1}e_1\rangle e_1 \\ &= (1 + \lambda\langle e_1, A^{-1}e_1\rangle)e_1 \end{aligned}$$

Then if A_λ is invertible, as $A^{-1}e_1 \neq 0$, $(1 + \lambda\langle e_1, A^{-1}e_1\rangle)e_1 \neq \vec{0}$ so $(1 + \lambda\langle e_1, A^{-1}e_1\rangle) \neq 0$.

It remains to show that if $1 + \lambda\langle e_1, A^{-1}e_1\rangle \neq 0$ then A_λ is invertible.

The following is my solution, however I welcome other solutions.

Suppose that $1 + \lambda\langle e_1, A^{-1}e_1\rangle \neq 0$
We know that A_λ is invertible if it maps a basis to a basis.
As A is invertible, $\{A^{-1}e_1, ..., A^{-1}e_n\}$ is a basis.

$A_\lambda(A^{-1}e_1) = (1 + \lambda\langle e_1, A^{-1}e_1\rangle)e_1$

For $i > 1$, set $a_i = \lambda\langle e_1, x\rangle$.
Then $A_\lambda(A^{-1}e_i) = e_i + a_ie_1$.

Now we use the fact that $\text{Span}\{v_1, ..., v_n\} = \text{Span}\{v_1, v_2 + c_2v_1, v_3 + c_3v_1, \ldots, v_n + c_nv_1\}$ for any $c_i \in F$.

Then $A_\lambda A^{-1}e_1, ..., A_\lambda A^{-1}e_n\}$ has the same span as if we add $-\frac{a_i}{1+\lambda\langle e_1, A^{-1}e_1\rangle}A_\lambda A^{-1}e_1$ to each $A_\lambda A^{-1}e_i$ for $i > 1$.

This gives us $\{(1+\lambda\langle e_1, A^{-1}e_1\rangle)e_1, e_2, e_3, ..., e_n\}$, and rescaling the first element by $\frac{1}{(1+\lambda\langle e_1, A^{-1}e_1\rangle)}$ gives us the standard basis.

Hence this is a basis so A_λ is invertible.