Today we will be discussing problems from past UCLA qualifying exams. In particular, these are from the linear algebra portion of the so-called “basic” qualifying exam, which is an exam all UCLA graduate students must pass before the start of their second year.

1 Fall 2018 Number 9

Let \( f, g : \mathbb{R}^n \to \mathbb{R} \) be linearly independent elements of the vector space (over \( \mathbb{R} \)) of linear mappings from \( \mathbb{R}^n \to \mathbb{R} \). Show that for any \( v \in \mathbb{R}^n \) there exist \( v_1 \) and \( v_2 \) such that

\[
v = v_1 + v_2, \quad f(v) = f(v_1), \quad \text{and} \quad g(v) = g(v_2).
\]

A good approach when you see a problem like this is to first write down your first observations about the problem.

The first thing I see is to ask what it means if \( f(v) = f(v_1) \)?

Well then \( f(v) = f(v_1 + v_2) = f(v_1) + f(v_2) \), so \( f(v_2) = 0 \). Likewise we see that \( g(v_1) = 0 \).

Then we can rephrase the question as follows: Show that \( \mathbb{R}^n = \ker f + \ker g \).

From the Rank-Nullity theorem, we know that \( \dim \ker f \geq n - 1 \) and \( \dim \ker g \geq n - 1 \). In fact since \( f \) and \( g \) are linearly independent, they are both not zero, so \( \dim \ker f = \dim \ker g = n - 1 \).

Since we know that \( \dim(\ker f + \ker g) = \dim \ker f + \dim \ker g - \dim(\ker f \cap \ker g) = 2n - 2 - \dim(\ker f \cap \ker g) \), it suffices to show that \( \dim(\ker f \cap \ker g) \leq n - 1 \).

Then all we must show is that \( \ker f \neq \ker g \).

We somehow must use the fact that \( f \) and \( g \) are linearly independent. Perhaps we can
try to show that if \( \ker f = \ker g \), then \( f \) and \( g \) are linearly dependent, contradicting our assumptions. Indeed, this will be our plan of attack.

Suppose that \( \ker f = \ker g \) and let \( \{v_1, ..., v_{n-1}\} \) be a basis for \( \ker f = \ker g \) and extend this to a basis \( \{v_1, ..., v_n\} \) for \( \mathbb{R}^n \).

Then \( f(v_i) = 0 \) for \( 1 \leq i \leq n - 1 \) and \( f(v_n) = a \) for some \( a \in \mathbb{R} - \{0\} \). Likewise \( g(v_i) = 0 \) for \( 1 \leq i \leq n - 1 \) and \( g(v_n) = b \) for some \( b \in \mathbb{R} - \{0\} \).

We claim that \( f = \frac{a}{b} g \). Indeed, this is true on every basis element, and is hence true for all vectors \( v \in \mathbb{R}^n \).

Then \( f \) is a multiple of \( g \), so \( f \) and \( g \) are linearly independent.

2 Spring 2014 Problem 3 (Enhanced)

Suppose \( A, B \in M_n(\mathbb{C}) \) satisfy \( AB - BA = A \). Show that \( A \) is not invertible.

Enhancement: Show that \( A \) is nilpotent. That is, for some \( k \in \mathbb{Z}^+ \), \( A^k = 0 \).

Solution (unenhanced version):

Suppose \( A \) is invertible. Then \( ABA^{-1} - B = I \).

Then \( \text{tr}(AB - BA) = \text{tr}(I) = n \).

However \( \text{tr}(ABA^{-1} - B) = \text{tr}(ABA^{-1}) - \text{tr}(B) = \text{tr}(A^{-1}AB) - \text{tr}(B) = \text{tr}(B) - \text{tr}(B) = 0 \) a contradiction.

Hence \( A \) is not invertible.

Solution (Enhanced version):

We have that \( AB = A + BA \).

A natural thing to try is to see if we can get some nice expression for \( A^k \).

Observe that

\[
A^2 = A(AB - BA) \\
= A^2B - ABA \\
= A^2B - (A + BA)A \\
= A^2B - A^2 + BA^2
\]

Then we get that \( 2A^2 = A^2B - BA^2 \).
We can try to multiply again on the left by $A$ to get

$$2A^3 = A(A^2B - BA^3)$$
$$= A^2B - ABA^2$$
$$= A^3B - (A + BA)A^2$$
$$= A^3B - A^3 - BA^3$$

so $3A = A^3B - BA^3$.

Then it seems like a reasonable guess is that $kA = A^kB - BA^k$.

Indeed, we proceed by induction. If $kA = A^kB - BA^k$ then:

$$kA^{k+1} = A(A^kB - BA^{k+1})$$
$$= A^kB - ABA^k$$
$$= A^{k+1}B - (A + BA)A^k$$
$$= A^{k+1}B - A^{k+1} - BA^{k+1}$$

Then $(k + 1)A^{k+1} = A^{k+1}B - BA^{k+1}$.

What can we do with this information? There are a few approaches here. In the vein of the solution to the unenhanced version, we can observe that tr$(A^k) = 0$ for every $k$. There is a complicated argument you can make using how the eigenvalues of $A^k$ relate to the eigenvalues of $A$ and the fact that tr$(X)$ is equal to the sum of the eigenvalues of $X$ counting multiplicity.

Here is a very elegant solution:

Consider the linear transformation $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $T(X) = XB - BX$.

Suppose that $A^k \neq 0$ for any $k$.

We have shown that $A^k$ is an eigenvector of $T$ for every $k \in \mathbb{Z}^+$ with eigenvalue $k$.

However $T$ can have at most $n$ eigenvalues, a contradiction.

3 Fall 2019 Problem 1

Let $A$ be an invertible $n \times n$ matrix with real entries and let $e_1$ denote the unit vector with a 1 in the first position and zeros elsewhere. Show that for each $\lambda \in \mathbb{R}$, the linear transformation $A_\lambda$ defined by

$$A_\lambda x = Ax + \lambda(e_1, x)e_1$$

is invertible if and only if $1 + \lambda(e_1, A^{-1}e_1) \neq 0$. 

When you first look at this problem, a natural thing to do is to see that $\lambda \langle e_1, A^{-1}e_1 \rangle$ term and try to make it appear using $A_\lambda$.

Indeed, observe that

$$A_\lambda(A^{-1}(e_1)) = AA^{-1}e_1 + \lambda \langle e_1, A^{-1}e_1 \rangle e_1$$
$$= e_1 + \lambda \langle e_1, A^{-1}e_1 \rangle e_1$$
$$= (1 + \lambda \langle e_1, A^{-1}e_1 \rangle)e_1$$

Then if $A_\lambda$ is invertible, as $A^{-1}e_1 \neq 0$, $(1 + \lambda \langle e_1, A^{-1}e_1 \rangle)e_1 \neq \vec{0}$ so $(1 + \lambda \langle e_1, A^{-1}e_1 \rangle) \neq 0$.

It remains to show that if $1 + \lambda \langle e_1, A^{-1}e_1 \rangle \neq 0$ then $A_\lambda$ is invertible.

The following is my solution, however I welcome other solutions.

Suppose that $1 + \lambda \langle e_1, A^{-1}e_1 \rangle \neq 0$

We know that $A_\lambda$ is invertible if it maps a basis to a basis.

As $A$ is invertible, $\{A^{-1}e_1, ..., A^{-1}e_n\}$ is a basis.

$$A_{\lambda}(A^{-1}e_1) = (1 + \lambda \langle e_1, A^{-1}e_1 \rangle)e_1$$

For $i > 1$, set $a_i = \lambda \langle e_1, x \rangle$.

Then $A_{\lambda}(A^{-1}e_i) = e_i + a_i e_1$.

Now we use the fact that $\text{Span}\{v_1, ..., v_n\} = \text{Span}\{v_1, v_2 + c_2v_1, v_3 + c_3v_1, ..., v_n + c_nv_1\}$ for any $c_i \in F$.

Then $A_{\lambda}A^{-1}e_1, ..., A_{\lambda}A^{-1}e_n$ has the same span as if we add $-\frac{a_i}{1 + \lambda \langle e_1, A^{-1}e_1 \rangle} A_{\lambda}A^{-1}e_1$ to each $A_{\lambda}A^{-1}e_i$ for $i > 1$.

This gives us $\{(1 + \lambda \langle e_1, A^{-1}e_1 \rangle)e_1, e_2, e_3, ..., e_n\}$, and rescaling the first element by $\frac{1}{(1 + \lambda \langle e_1, A^{-1}e_1 \rangle)}$ gives us the standard basis.

Hence this is a basis so $A_\lambda$ is invertible.