Today we will go over the midterm. I will be presenting solutions, but for a number of problems there were many solutions. In particular, there are many examples one can give for the questions that ask for examples.

1 Part I

In (a), (b), and (c) let $V$ be a nonzero vector space over a field $F$ (not necessarily finite dimensional) and $S$ a nonempty subset of vectors in $V$. Fully and accurately define what it means for each of the following:

1.1 Part (a)

A vector $v \in V$ is in $\text{Span} \ S$ if for some non-negative integer $n$, there exists $a_1, ..., a_n \in F$ and $u_1, ..., u_n \in S$ such that $v = a_1u_1 + \cdots + a_nu_n$ where the sum over 0 elements is taken to be the zero vector.

1.2 Part (b)

$S$ is linearly dependent if for some $n \in \mathbb{Z}^+$ there exists $a_1, ..., a_n \in F$ not all 0 and distinct $v_1, ..., v_n \in S$ such that $a_1v_1 + \cdots + a_nv_n = 0$.

1.3 Part (c)

$S$ is linearly independent if it is not linearly dependent.
1.4 Part (d)

Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces over a field $F$.
Let $\beta = (v_1, \ldots, v_m)$ be an ordered basis for $V$ and $\gamma = (w_1, \ldots, w_n)$ be an ordered basis for $W$.

The matrix representation of $T$ with respect to $(\beta, \gamma)$, denoted $[T]_{\beta \to \gamma}$, is defined as follows:

For $1 \leq i \leq m$, there is a unique $n$-tuple $u_i = \begin{pmatrix} a_{1,i} \\ \vdots \\ a_{n,i} \end{pmatrix}$ such that $Tv_i = a_{1,i}w_1 + \cdots + a_{n,i}w_n$.

$[T]_{\beta \to \gamma}$ is an $n \times m$ matrix where for $1 \leq i \leq m$, the $i$-th column is $u_i$. 
2 Part II

Give examples of each of the following (You do not need to justify):

2.1 Part (a)

Let $M_2(F)$ be the vector space over $F$ of $2 \times 2$ matrices. Find a basis for the subspace of skew symmetric matrices $\{A \in M_2(F)|A^t = -A\}$ in each of the following two cases:

2.1.1 Part (i)

$F$ is the field of two elements.

Take the basis: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

2.1.2 Part (ii)

$F$ is the field of three elements.

Take the basis: $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

2.2 Part (b)

Give a specific example of a vector space with 27 elements and subspace of it with 9 elements.

Let $F$ be the field with 3 elements.

Take the vector space $F^3$ and the subspace $F^2 \times \{0\}$.

2.3 Part (c)

Take $\mathbb{R}[x]$ and $\mathbb{C}[0, 1]$ as real vector space.
2.4 Part (d)

Define $T : M_2(\mathbb{C}) \to \mathbb{C}[0,1]$ by

$$T \begin{pmatrix} a_{1,1} + ib_{1,1} & a_{1,2} + ib_{1,2} \\ a_{2,1} + ib_{2,1} & a_{2,2} + ib_{2,2} \end{pmatrix} (x) = a_{1,1} + b_{1,1}x + a_{1,2}x^2 + b_{1,2}x^3 + a_{2,1}x^4 + b_{2,1}x^5 + a_{2,2}x^6 + b_{2,2}x^7$$

2.5 Part (e)

Take $T : \mathbb{R}[x] \to \mathbb{R}[x]$ by $T(p(x)) = xp(x)$.

Take $S : \mathbb{R}[x] \to \mathbb{R}[x]$ by $T(p(x)) = p'(x)$, where $p'(x)$ is the derivative of $p(x)$. 
3 Part III

3.1 Part (a)

Fully and accurately state and prove the Replacement Theorem.

Here we use “almost all” to mean “all but finitely many.”

Theorem: Suppose that \( S \) is a basis for \( V \) and \( x \) is an element of \( V \). If \( x = \sum_{v \in S} \alpha_v v \) with \( \alpha_v \in F \) almost all zero and \( \alpha_{v_0} \neq 0 \), then \( (S \setminus \{v_0\}) \cup \{x\} \) is a basis for \( V \).

Proof:

First we show that \( (S \setminus \{v_0\}) \cup \{x\} \) spans \( V \).

Observe that as \( \alpha_{v_0} \neq 0 \), \( v_0 = x - \sum_{v \in S \setminus \{v_0\}} \frac{\alpha_v}{\alpha_{v_0}} v \)

Let \( y \in V \). Then as \( S \) is a basis for \( V \), \( y = \sum_{v \in S} \beta_v v \) with \( \beta_v \in F \) almost all zero

Then

\[
y = \left( \sum_{v \in S \setminus \{v_0\}} \beta_v v \right) + \beta_{v_0} (x - \sum_{v \in S \setminus \{v_0\}} \frac{\alpha_v}{\alpha_{v_0}} v) \\
= \left( \sum_{v \in S \setminus \{v_0\}} (\beta_v - \beta_{v_0} \frac{\alpha_v}{\alpha_{v_0}}) v \right) + \beta_{v_0} x
\]

Hence \( y \in \text{Span}((S \setminus \{v_0\}) \cup \{x\}) \).

Next we show that \( (S \setminus \{v_0\}) \cup \{x\} \) is linearly independent.

Suppose that \( \sum_{v \in (S \setminus \{v_0\}) \cup \{x\}} \beta_v v = 0 \) with \( \beta_v \in F \) almost all zero.

Then
\[ 0 = \sum_{v \in (S \setminus \{v_0\})} \cup \{x\} \beta_v v \]
\[ = \beta_x x + \sum_{v \in (S \setminus \{v_0\})} \beta_v v \]
\[ = \beta_x \left( \sum_{v \in S} \alpha_v v \right) + \sum_{v \in (S \setminus \{v_0\})} \beta_v v \]
\[ = \beta_x \alpha_{v_0} v_0 + \sum_{v \in (S \setminus \{v_0\})} \left( \beta_v + \beta_x \alpha_v \right) v \]

As \( S \) is linearly independent, \( \beta_x \alpha_{v_0} = 0 \) and for \( v \in S - \{v_0\}, \beta_v + \beta_x \alpha_v = 0 \).

As \( \alpha_{v_0} \neq 0, \beta_x = 0 \). Hence \( \beta_v = 0 \) for each \( v \). Hence \( (S \setminus \{v_0\}) \cup \{x\} \) is linearly independent.
Then \( (S \setminus \{v_0\}) \cup \{x\} \) is a basis.

### 3.2 Part (b)

See pages 5 and 6 of Lecture 10 notes.

### 4 Part IV

Define \( T : V \to W \) by \( T(w_1, w_2, w_3) = w_1 + w_2 + w_3 \).

We claim that \( T \) is a surjective linear transformation.

Indeed, observe that for \( \alpha \in F, (w_1, w_2, w_3), (w_1', w_2', w_3') \in V \) we have
\[
T(\alpha(w_1, w_2, w_3) + (w_1', w_2', w_3')) = T(\alpha w_1 + w_1', \alpha w_2 + w_2', \alpha w_3 + w_3')
\]
\[= \alpha w_1 + w_1' + \alpha w_2 + w_2' + \alpha w_3 + w_3'
\]
\[= \alpha(w_1 + w_2 + w_3) + w_1' + w_2' + w_3'
\]
\[= \alpha T(w_1, w_2, w_3) + T(w_1', w_2', w_3')
\]

Hence \( T \) is linear. Furthermore as \( W = W_1 + W_2 + W_3 \), for \( w \in W \), there exists \( w_1 \in W_1, w_2 \in W_2, w_3 \in W_3 \) such that \( w = w_1 + w_2 + w_3 \).

Hence \( w = T(w_1, w_2, w_3) \), so \( T \) is surjective.

Now suppose that \( T(w_1, w_2, w_3) = 0 \).

Then \( w_1 + w_2 + w_3 = 0 \), so \( \ker T = \{0\} \) if and only if for all \( w_1 \in W_1, w_2 \in W_2, w_3 \in W_3, w_1 + w_2 + w_3 = 0 \) if and only if \( T \) is injective if and only if \( T \) is an isomorphism.
Then suppose that $T$ is an isomorphism. and let $w \in W_i$ and $w \in W_j + W_k$ for $i,j,k$ all distinct.

Then $w \in W_i$, and $w = w_j + w_k$ for some $w_j \in W_j, w_k \in W_k$.

Then $w - w_j - w_k = 0$, hence $w = -w_j = -w_k = 0$. Hence $w = 0$ so $W_i \cap (W_j + W_k) = \{0\}$.

Now suppose that $W_i \cap (W_j + W_k) = 0$ whenever $i,j,k$ are all distinct.

Then let $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3$ such that $w_1 + w_2 + w_3 = 0$.

Hence $w_1 = -(w_2 + w_3) \in (W_2 + W_3)$.

Hence $w_1 \in W_1 \cap (W_2 + W_3) = \{0\}$, so $w_1 = 0$.

Then $w_2 + w_3 = 0$. But then $w_2 = -w_3 \in W_3 \subset W_1 + W_3$.

Hence $w_2 \in W_2 \cap (W_1 + W_3) = \{0\}$ so $w_2 = 0$.

Hence $w_3 = 0$.

Hence $w_1 = w_2 = w_3 = 0$, so $T$ is an isomorphism.