Week 7 Tuesday Notes

Andrew Sack

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Today we will go over the midterm. I will be presenting solutions, but for a number of problems there were many solutions. In particular, there are many examples one can give for the questions that ask for examples.

1 Part I

In (a), (b), and (c) let V be a nonzero vector space over a field F (not necessarily finite dimensional) and S a nonempty subset of vectors in V. Fully and accurately define what it means for each of the following:

1.1 Part (a)

A vector $v \in V$ is in $\operatorname{Span} S$ if for some non-negative integer n, there exists $a_1, ..., a_n \in F$ and $u_1, ..., u_n \in S$ such that $v = a_1u_1 + \cdots + a_nu_n$ where the sum over 0 elements is taken to be the zero vector.

1.2 Part (b)

S is linearly dependent if for some $n \in \mathbb{Z}^+$ there exists $a_1, ..., a_n \in F$ not all 0 and distinct $v_1, ..., v_n \in S$ such that $a_1v_1 + ... + a_nv_n = 0$.

1.3 Part (c)

S is linearly independent if it is not linearly dependent.

1.4 Part (d)

Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces over a field F.
Let $\beta = (v_1, ..., v_m)$ be an ordered basis for V and $\gamma = (w_1, ..., w_n)$ be an ordered basis for W.

The matrix representation of T with respect to (β, γ), denoted $[T]_{\beta \to \gamma}$ is defined as follows:

For $1 \leq i \leq m$, there is a unique n-tuple $u_i = \begin{pmatrix} a_{1,i} \\ \vdots \\ a_{n,i} \end{pmatrix}$ such that $Tv_i = a_{1,i}w_1 + \cdots + a_{n,i}w_n$.

$[T]_{\beta \to \gamma}$ is an $n \times m$ matrix where for $1 \leq i \leq m$, the i-th column is u_i.

2 Part II

Give examples of each of the following (You do not need to justify):

2.1 Part (a)

Let $M_2(F)$ be the vector space over F of 2×2 matrices. Find a basis for the subspace of skew symmetric matrices $\{A \in M_2(F) | A^t = -A\}$ in each of the following two cases:

2.1.1 Part (i)

F is the field of two elements.

Take the basis: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

2.1.2 Part (ii)

F is the field of three elements.

Take the basis: $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

2.2 Part (b)

Give a specific example of a vector space with 27 elements and subspace of it with 9 elements.

Let F be the field with 3 elements.

Take the vector space F^3 and the subspace $F^2 \times \{0\}$.

2.3 Part (c)

Take $\mathbb{R}[x]$ and $\mathbb{C}[0, 1]$ as real vector space.

2.4 Part (d)

Define $T : M_2(\mathbb{C}) \to \mathbb{C}[0,1]$ by

$$T\begin{pmatrix} a_{1,1} + ib_{1,1} & a_{1,2} + ib_{1,2} \\ a_{2,1} + ib_{2,1} & a_{2,2} + by_{2,2} \end{pmatrix}(x) = a_{1,1} + b_{1,1}x + a_{1,2}x^2 + b_{1,2}x^3 + a_{2,1}x^4 + b_{2,1}x^5 + a_{2,2}x^6 + b_{2,2}x^7$$

2.5 Part (e)

Take $T : \mathbb{R}[x] \to \mathbb{R}[x]$ by $T(p(x)) = xp(x)$.

Take $S : \mathbb{R}[x] \to \mathbb{R}[x]$ by $T(p(x)) = p'(x)$, where $p'(x)$ is the derivative of $p(x)$.

3 Part III

3.1 Part (a)

Fully and accurately state and prove the Replacement Theorem.

Here we use "almost all" to mean "all but finitely many."

Theorem: Suppose that S is a basis for V and x is an element of V . If $x = \sum\limits_{v \in S} \alpha_v v$ with $\alpha_v \in F$ almost all zero and $\alpha_{v_0} \neq 0$, then $(S \setminus \{v_0\}) \cup \{x\}$ is a basis for V .

Proof:

First we show that $(S \setminus \{v_0\}) \cup \{x\}$ spans V.

Observe that as $\alpha_{v_0} \neq 0$, $v_0 = x - \sum\limits_{v \in S - \{v_0\}} \frac{\alpha_v}{\alpha_{v_0}} v$

Let $y \in V$. Then as S is a basis for V, $y = \sum\limits_{v \in S} \beta_v v$ with $\beta_v \in F$ almost all zero

Then

$$\begin{aligned} y &= \left(\sum_{v \in S - \{v_0\}} \beta_v v \right) + \beta_{v_0}(x - \sum_{v \in S - \{v_0\}} \frac{\alpha_v}{\alpha_{v_0}} v) \\ &= \left(\sum_{v \in S - \{v_0\}} (\beta_v - \beta_{v_0} \frac{\alpha_v}{\alpha_{v_0}}) v \right) + \beta_{v_0} x \end{aligned}$$

Hence $y \in \text{Span}((S \setminus \{v_0\}) \cup \{x\})$.

Next we show that $(S \setminus \{v_0\}) \cup \{x\}$ is linearly independent.

Suppose that $\sum_{v \in (S \setminus \{v_0\}) \cup \{x\}} \beta_v v = 0$ with $\beta_v \in F$ almost all zero.

Then

$$
\begin{aligned}
0 &= \sum_{v\in(S\setminus\{v_0\})} \cup\{x\}\beta_v v \\
&= \beta_x x + \sum_{v\in(S\setminus\{v_0\})} \beta_v v \\
&= \beta_x \left(\sum_{v\in S} \alpha_v v\right) + \sum_{v\in(S\setminus\{v_0\})} \beta_v v \\
&= \beta_x \alpha_{v_0} v_0 + \sum_{v\in(S\setminus\{v_0\})} (\beta_v + \beta_x\alpha_v) v
\end{aligned}
$$

As S is linearly independent, $\beta_x\alpha_{v_0} = 0$ and for $v \in S - \{v_0\}$, $(\beta_v + \beta_x\alpha_v) = 0$.

As $\alpha_{v_0} \neq 0, \beta_x = 0$. Hence $\beta_v = 0$ for each v. Hence $(S \setminus \{v_0\}) \cup \{x\}$ is linearly independent.

Then $(S \setminus \{v_0\}) \cup \{x\}$ is a basis.

3.2 Part (b)

See pages 5 and 6 of Lecture 10 notes.

4 Part IV

Define $T : V \to W$ by $T(w_1, w_2, w_3) = w_1 + w_2 + w_3$.

We claim that T is a surjective linear transformation.

Indeed, observe that for $\alpha \in F, (w_1, w_2, w_3), (w_1', w_2', w_3') \in V$ we have

$$
\begin{aligned}
T(\alpha(w_1, w_2, w_3) + (w_1', w_2', w_3')) &= T(\alpha w_1 + w_1', \alpha w_2 + w_2', \alpha w_3 + w_3') \\
&= \alpha w_1 + w_1' + \alpha w_2 + w_2' + \alpha w_3 + w_3' \\
&= \alpha(w_1 + w_2 + w_3) + w_1' + w_2' + w_3' \\
&= \alpha T(w_1, w_2, w_3) + T(w_1', w_2', w_3')
\end{aligned}
$$

Hence T is linear. Furthermore as $W = W_1 + W_2 + W_3$, for $w \in W$, there exists $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3$ such that $w = w_1 + w_2 + w_3$.

Hence $w = T(w_1, w_2, w_3)$, so T is surjective.

Now suppose that $T(w_1, w_2, w_3) = 0$.

Then $w_1 + w_2 + w_3 = 0$, so $\ker T = \{0\}$ if and only if for all $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3$, $w_1 + w_2 + w_3 = 0$ if and only if T is injective if and only if T is an isomorphism.

Then suppose that T is an isomorphism. and let $w \in W_i$ and $w \in W_j + W_k$ for i, j, k all distinct.

Then $w \in W_i$, and $w = w_j + w_k$ for some $w_j \in W_j, w_k \in W_k$.

Then $w - w_j - w_k = 0$, hence $w = -w_j = -w_k = 0$. Hence $w = 0$ so $W_i \cap (W_j + W_k) = \{0\}$.

Now suppose that $W_i \cap (W_j + W_k) = 0$ whenever i, j, k are all distinct.
Then let $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3$ such that $w_1 + w_2 + w_3 = 0$.
Hence $w_1 = -(w_2 + w_3) \in (W_2 + W_3)$.
Hence $w_1 \in W_1 \cap (W_2 + W_3) = \{0\}$, so $w_1 = 0$.
Then $w_2 + w_3 = 0$. But then $w_2 = -w_3 \in W_3 \subset W_1 + W_3$.
Hence $w_2 \in W_2 \cap (W_1 + W_3) = \{0\}$ so $w_2 = 0$.
Hence $w_3 = 0$.

Hence $w_1 = w_2 = w_3 = 0$, so T is an isomorphism.