Week 7 Thursday Notes

Andrew Sack

February 18, 2021

1 How to remember the Gram-Schmidt process

Let V be a finite dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$.

In the event that you forget the Gram-Schmidt process like I frequently do, you can reconstruct it fairly easily as long as you remember what it is supposed to do:

The Gram-Schmidt process takes in an ordered basis $\beta = (v_1, ..., v_n)$ for V and returns a basis $\beta' = (u_1, ..., u_n)$ for V such that:

(1) β is orthonormal.

(2) For $1 \leq k \leq n$, $\text{Span}\{v_1, ..., v_k\} = \text{Span}\{u_1, ..., u_k\}$.

The easiest way I think of this as to first make an orthogonal basis with property (2) and then normalize, but you can also normalize along the way.

We build up the basis inductively.

Take $v_1 = u_1$ as $\{u_1\}$ is already an orthogonal set.

Now we want to find u_2 such that $\text{Span}\{v_1, v_2\} = \text{Span}\{u_1, u_2\}$ and is orthogonal to u_1. Certainly we need v_2 to be in the span of u_1 and u_2.

Imagine that we already found u_2. Then we can write $v_2 = a_1u_1 + a_2u_2$.

What we know is that any multiple of u_2 will also work. Then if we find a_1, we can just take $u_2 = v_2 - a_1u_1$.

Taking an inner product with u_1 we get $\langle v_2, u_1 \rangle = \langle a_1u_1, u_1 \rangle + \langle a_2u_2, u_1 \rangle = a_1 \langle u_1, u_1 \rangle$.

Hence $a_1 = \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle}$.

Then simply take $u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$.

In general, if we already have an orthogonal set $(u_1, ..., u_k)$ that satisfies property (2) up to k, and we suppose such that a u_{k+1} exists with the properties that we want, then we can write $v_{k+1} = \sum\limits_{i=1}^{k+1} a_i u_i$.

Then in order to find a_j for $1 \leq i \leq k$, taking an inner product with u_j gives us

$$\begin{aligned}
\langle v_{k+1}, u_j \rangle &= \langle \sum_{i=1}^{k+1} a_i u_i, u_j \rangle \\
&= \sum_{i=1}^{k+1} a_i \langle u_i, u_j \rangle \\
&= a_j \langle u_j, u_j \rangle
\end{aligned}$$

Then $a_j = \frac{\langle v_{k+1}, u_j \rangle}{\langle u_j, u_j \rangle}$

Then we have $a_{k+1} u_{k+1} = v_{k+1} - \sum\limits_{i=1}^{k} \frac{\langle v_{k+1}, u_i \rangle}{\langle u_i, u_i \rangle} u_i$

Any non-zero multiple of u_{k+1} would work, so simply take u_{k+1} to be $v_{k+1} - \sum\limits_{i=1}^{k} \frac{\langle v_{k+1}, u_i \rangle}{\langle u_i, u_i \rangle} u_i$

In order to get an orthonormal basis, normalize each u_i.

The actual proof of Gram-Schmidt is just verifying that the above actually gives an orthonormal basis, which is just a bit of algebra.

A good way to think of the Gram-Schmidt process is as an analog to the basis extension theorem but for orthonormal bases instead.

2 Inner products and Dual Spaces

As we discussed previously, there is no natural isomorphism between a finite dimensional vector space and its dual space. However, if we are working with an inner product space, the scenario is much nicer.

Let V be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$.

Define $\varphi : V \to V^*$ by $\varphi(u) = \langle u, \cdot \rangle$ where $\langle u, \cdot \rangle : V \to F$ is given by $\langle u, \cdot \rangle(v) = \langle u, v \rangle$.

We claim that φ Is an isomorphism.

First observe that $\varphi(u)$ is a a linear functional as the inner product is linear if you fix one variable.

For $a \in F, u, v \in V$, observe that $\varphi(au+v) = \langle au+v, \cdot \rangle = a\langle u, \cdot \rangle + \langle v, \cdot \rangle = a\varphi(u) + v$ so φ is linear.

Furthermore, let $\{v_1, ..., v_n\}$ be an orthonormal basis for V (such a basis exists via the Gram-Schmidt process). Then

$$\varphi(v_i)(v_j) = \langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then $\varphi(v_i) = v_i^*$.

So φ takes a basis to its dual basis and hence is an isomorphism.

We can actually take an orthogonal basis instead of an orthonormal basis (in the case that we are not working in a field with all square roots), and then instead of going to the dual basis we go to a multiple of the dual basis.