Week 6 Tuesday Notes

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February 9, 2021

More on Dual Spaces will be discussed in the extra lecture this week.

Today I will be relating various topics talked about in lecture to what they mean for matrices and working out some examples.

1 Direct Sums

Recall that we say a vector space V is a direct sum of subspaces U and W if $V = U + W$ and $U \cap W = \{0\}$ and we write $V = U \oplus W$.

For homework you will show that there is a nice isomorphism between this (called the internal direct product) and the external direct product defined by $U \oplus W := U \times W$ with vector addition and scalar multiplication defined term-wise.

We say that a linear operator $T : V \to V$ splits over the internal direct sum $V = U \oplus W$ if $T(U) \subset U$ and $T(W) \subset W$.

Alternatively, given linear operators $S : U \to U$ and $T : V \to V$, there is a split linear operator $S \oplus T : U \oplus V \to U \oplus V$ given by $S \oplus T(u, v) = (S(u), T(v))$.

Furthermore, this operation commutes with the isomorphism between internal and external direct products.

What does this mean for matrices?

Let $\beta = \{u_1, ..., u_m\}$ be a basis for U and $\gamma = \{w_{m+1}, ..., w_n\}$ be a basis for W.
Then $\beta \cup \gamma$ is a basis for $V = U \oplus W$.

Let $T : V \to V$ be split over the direct sum.

What does the matrix $[T]_{\beta\cup\gamma\to\beta\cup\gamma}$ look like?

Well, observe that for any $u_i \in \beta$, $T(u_i) \in U$, so it has non-zero components only in β.

Then the i-th column corresponding to the coordinates of $T(u_i)$ has 0s in the $m+1, ..., n$ coordinates.
Likewise for any $w_i \in \gamma$, $T(w_i) \in W$ so it has non-zero components only in γ. Then the first m coordinates of the i-th column are all 0.

This gives us a so-called "block diagonal" matrix. That is outside of the upper left $m \times m$ block and bottom right $m - n \times m - n$ block of the matrix, it is all 0s. Furthermore, the upper left block is $[T|_U]_{\beta\to\beta}$ and the bottom right block is $[T|_W]_{\gamma\to\gamma}$.

Let's consider an example.

Consider the transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$T\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

(This takes a vector and returns it with its coordinates in reverse order.)

Let $U = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ and $V = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

We see that T exchanges the basis vectors of U so $T(U) = U$ and similarly $T(V) = V$ so T splits over U and V.

Indeed, let $\beta = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right)$

Then $[T]_{\beta\to\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$

2 Eigenvalues and Eigenvectors

From class you were told that an eigenvector of a linear transformation $T : V \to V$ is a non-zero vector $v \in V$ such that $Tv = \lambda v$ for some $\lambda \in F$. λ is called the eigenvalue associated to v.

Equivalently, $Tv - \lambda v = (T - \lambda I)v = 0$. Then $\lambda \in F$ is an eigenvalue if and only if $\ker(T - \lambda I) \neq 0$.

With respect to any basis, λI is always the same matrix. Then given a basis β, we can detect if λ is an eigenvalue of T by seeing if $[T]_{\beta\to\beta} - \lambda I$ is invertible or not, or equivalently check if $\det([T]_{\beta\to\beta} - \lambda I) = 0$.

How do we find these eigenvalues? Well they correspond precisely to the values of z such that $\det(zI - [T]_{\beta\to\beta}) = 0$. Conveniently, by cofactor expansion this is a polynomial in z, so we have reduced checking if λ is an eigenvalue of T to checking if it is the root of a particular polynomial.

We define the characteristic polynomial by

$$\chi_T(z) := \det(zI - T)$$

Note: This is different from the minimal polynomial defined in class.

3 Diagonalizable Transformations

Recall that a linear operator $T : V \to V$ is diagonalizable if T has a basis of eigenvectors.

What does this mean with respect to matrices? Let $T : V \to V$ and $\beta = \{v_1, ..., v_n\}$ be a basis of eigenvectors where $Tv_i = \lambda v_i$.

What does $[T]_{\beta\to\beta}$ look like? As $Tv_i = \lambda v_i$, the (i, i) entry in $[T]_{\beta\to\beta}$ is λ_i and all other entries in the column are 0. Then $[T]_{\beta\to\beta}$.

We saw last week that given some other basis γ (not necessarily of eigenvectors), $[T]_{\beta\to\beta}$ is similar to $[T]_{\gamma\to\gamma}$.

We say that a matrix is diagonalizable if it is similar to a diagonal matrix. Diagonalizable matrices exactly correspond to diagonalizable transformations.

Let us consider an example:

Take $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

We claim that T is diagonalizable.

First we should find the eigenvectors of T.

Let $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\begin{aligned} \det\left(zI - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) &= \det\begin{pmatrix} z & -1 \\ -1 & z \end{pmatrix} \\ &= z^2 - 1 = (z+1)(z-1) \end{aligned}$$

Then $\chi_T(z)$ has roots 1 and -1. We saw in class that as these eigenvalues are distinct, T is diagonalizable, however if we are not convinced, let us find a basis of eigenvectors.

We want to find a vector in $\ker(I - M) = \ker \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and a vector in $\ker(-I - M) = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

For the first case we can take $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the second we can take $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

These are linearly independent, so T is diagonalizable.

Furthermore, observe that $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

This actually shows that the first matrix of today $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ is diagonalizable, because it is similar to $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ whose blocks are diagonalizable.