Week 5 Tuesday Notes

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1 Determinants and Trace

Let us start by recalling determinants and some properties of them.

As you are probably familiar, the determinant of an $n \times n$ matrix can be calculated by a recursive method called "cofactor expansion" and you have heard that the determinant of a matrix is non-zero if and only if the matrix is invertible.

I am going to state several properties of determinants without proof. If you are interested in seeing proofs, they are covered in a section of your textbook which you can read through. If you would like help working your way through any of the proofs in that section, please let me know.

Let $A, B \in M_n(F)$.

1. $\det I = 1$.
2. $\det A^t = \det A$
3. If $\det A \neq 0$ then A is invertible.
4. $\det A^{-1} = \frac{1}{\det A}$.
5. $\det(AB) = \det(A)\det(B)$
6. For $c \in F, \det(cA) = c^n \det(A)$.

Let $A = [v_1, ..., v_n]$ where $v_1, .., v_n$ are the columns of A and $c \in F$. Then:

1. $\det[v_1, \ldots, av_i, \ldots, v_n] = a \det[v_1, \ldots, v_n]$.
2. $\det[v_1, \ldots, v_i + av_j, \ldots, v_n] = \det[v_1, \ldots, v_n]$

3. $\det[v_1, \ldots, v_i + v_i', \ldots, v_n] = \det[v_1, \ldots, v_i, \ldots, v_n] + \det[v_1, \ldots, v_i', \ldots, v_n]$
4. $\det[v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n] = -\det[v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n]$

There are far more properties, but these are worth knowing.

We will see a bit later today that while at first it may seem that the definition of determinant depends on our choice of basis, we will see that it does not so is defined for a linear function on a finite dimensional vector space regardless of basis. Similarly, let us discuss the trace map.

Given an $n \times n$ matrix M, $\operatorname{tr} M = \sum_{i=1}^{n} M_{ii}$. That is, the trace is the sum of the diagonal elements of your matrix.

Let $A, B, C \in M_n(F)$ and $c \in F$. We have the following properties (again discussed without proof):

1. $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$
2. $\operatorname{tr}(cA) = c \operatorname{tr}(A)$.
3. $\operatorname{tr}(A) = \operatorname{tr}(A^t)$
4. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. NOTE: This does not mean that trace is invariant under any permutation of the matrices.
5. We do have the more general property that trace is invariant under cyclic permutations. For example, $\operatorname{tr}(ABC) = \operatorname{tr}(CAB) = \operatorname{tr}(BCA)$. However in general we do not have that $\operatorname{tr}(ABC) = \operatorname{tr}(ACB)$.
6. We also do not generally have that $\operatorname{tr}(AB) = \operatorname{tr}(A) \operatorname{tr}(B)$.

As I said, this is also defined regardless of choice of basis which we will see later.

2 Matrix Representation of a Linear Map

Let V and W be finite dimensional vector spaces over a field F with $\dim V = m$ and $\dim W = n$ and $T : V \to W$ be linear.

Let $\beta = (v_1, ..., v_m)$ be an ordered basis for V and $\gamma = (w_1, ..., w_n)$ be an ordered basis for W.

Then for each v_i, we can write $T(v_i) = \sum_{j=1}^{n} a_j w_j$ for some unique $a_1, ..., a_n \in F$.

Take the column vector $c_i := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$.

The matrix $[M]_{\alpha \to \gamma} := [c_1, ..., c_m]$ where $c_1, ..., c_m$ are the columns of the matrix.

That is, this is the matrix representation of the linear map from the basis α to the basis γ.

2.1 An example

Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_1 \\ x_1 + x_2 + x_3 \end{pmatrix}$$

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Let us find the matrix representation for T with respect to the bases

$$\beta = \left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \text{ for } \mathbb{R}^3$$

and

$$\gamma = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right) \text{ for } \mathbb{R}^4$$

Then $T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_\beta = T \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}_\gamma$

$T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_\beta = T \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}_\gamma$

$$T\begin{pmatrix}0\\0\\-1\end{pmatrix}_\beta = T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\2\\2\\3\end{pmatrix} = \begin{pmatrix}0\\-1\\-1\\3\end{pmatrix}_\gamma$$

Then $[T]_{\beta\to\gamma} = \begin{pmatrix}1 & 1 & 0\\ -1 & 0 & -1\\ 1 & -1 & -1\\ 0 & 0 & 3\end{pmatrix}$

You might ask, what if we wanted to find this from the standard basis σ_3 for $\mathbb{R}^3$ to the standard basis σ_4 for $\mathbb{R}^4$ instead?

Well our first option is to do this process over again. Alternatively, consider the identity transformation $I : \mathbb{R}^3 \to \mathbb{R}^3$ and the identity transformation $I : \mathbb{R}^4 \to \mathbb{R}^4$. (It will be clear from context which I is which.)

Then $[T]_{\sigma_3\to\sigma_4} = [I]_{\gamma\to\sigma_4}[T]_{\beta\to\gamma}[I]_{\sigma_3\to\beta}$ and we can simply multiply the matrices.

We have that $[I]_{\sigma_3\to\beta} = \begin{pmatrix}1 & 0 & 1\\ -1 & 1 & 1\\ 0 & -1 & 1\end{pmatrix}$ and that $[I]_{\gamma\to\sigma_4} = \begin{pmatrix}1 & -1 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{pmatrix}$.

You can check for yourself that this process gives us the same ending matrix.

2.2 Change of basis matrix

Let V be a finite dimensional vector space over a field F and let β and γ be bases for V.

Given a transformation $T : V \to V$ we ask what is the relationship between $[T]_{\beta\to\beta}$ and $[T]_{\gamma\to\gamma}$.

As we did in the previous example, we have a so-called "change of basis matrix" which is the identity I with the input coordinates given with respect to β and the output coordinates given with respect to γ, which we denote $[I]_{\beta\to\gamma}$ and conversely we have $[I]_{\gamma\to\beta}$.

As we should expect, $[I]_{\beta\to\gamma}[I]_{\gamma\to\beta}$ should be the identity matrix and vice versa.

Furthermore, we have that $[T]_{\beta\to\beta} = [I]_{\gamma\to\beta}[T]_{\gamma\to\gamma}[I]_{\beta\to\gamma}$.

Then the two matrix representations of T are called <u>similar</u>. That is, two matrices A and B are called similar if there is another invertible matrix S such that $A = S^{-1}BS$. As we have seen, S can be viewed as a change of basis matrix. Two matrices are similar if and only if they are the same transformation with respect to a different basis.

With respect to determinants and the trace map, we see that if $A = S^{-1}BS$ then

$\det A = \det S^{-1} \det B \det S = \frac{\det S}{\det S} \det B = \det B$.

Similarly, $\operatorname{tr}(A) = \operatorname{tr}(S^{-1}BS) = \operatorname{tr}(SS^{-1}B) = B$.

So determinants and trace maps are actually independent of which basis you choose for your vector space.