Week 6 Tuesday Notes

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In the extra lecture yesterday you defined dual spaces.

That is, given a vector space V over F, we define $V^* := L(V, F)$, the set of linear transformations from V to F.

If V is finite dimensional, you saw that given a basis $\beta = \{v_1, ..., v_n\}$ for F, there exists the so-called dual basis β^* defined as follows:

Given $v \in V$, for a unique $a_1, ..., a_n \in F$ write $v = \sum_{i=1}^{n} a_i v_i$. Define $v_i^* : V \to F$ by $v_i^*(v) = a_i$. Then as shown in the extra lecture, $\{v_1^*, ..., v_n^*\}$ is a basis for V^*.

Alternatively we can define v_i^* using the universal property as follows:

$$v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then we see that V^* and V have the same dimension.

Moreover, V^* and V^{**} also have the same dimension, so $V, V^*,$ and V^{**} are all isomorphic. Interestingly, if V is not finite dimensional, then the dimension of V^* is strictly greater than the dimension of V.

You saw that (back to the finite dimensional case) not only are V and V^{**} isomorphic, but there is an isomorphism $\varphi : V \to V^{**}$ that is defined without using a basis.
This isomorphism is given by $\varphi(v) = e_v$ where $e_v : V^* \to F$ is given by $e_v(u*) = u^*(v)$.

This is called the evaluation map.

Understandably, double dual spaces can be a little strange to wrap your head around. Dual spaces themselves are already a bit strange, as your vectors are functions that take in vectors. Double dual spaces are even stranger as the vectors are functions that take in functions. If you have a computer science background, you can think of these as an example of a functional programming concept called higher order functions.

Duality is a tremendously useful concept in math. Generally speaking, duality involves transforming objects via an operation that preserves or transforms some properties. This operation is usually an involution. That is, the dual of a dual object should get back to the original object. A great example is taking a set complement.

Given $A, B \subset U$ define $A^c = U - A$. Indeed, $(A^c)^c = A$. Set complements also interact with unions and intersections. We have that $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. So complements exchange intersection with union.

We should investigate if something similar happens with dual spaces.

At least in finite dimensional vector spaces, the double dual gets you back to your original vector space, or at least gets you to something with an extremely nice isomorphism to the original vector space.

Here's an interesting question, what happens to linear transformations under the dual map?

Given $T : V \to W$, is there a nice map S related to T in some way with $S : V^* \to W^*$. In particular, can we define such an S without regards to a basis?

It turns out, there is no such nice map. Our problem is that S is pointing the wrong way.
Define $T^* : W^* \to V^*$ by $T^*(w^*)(v) = w^*(T(v))$.
As an exercise, check that T^* is linear.

We should hope that $(T^*)^* = T$, at least in some sense. This seems like a problem at first as $T : V \to W$ but $(T^*)^* : V^{**} \to W^{**}$. However, we can view $v \in V$ and $e_v \in V^{**}$ as being "the same elements", so our real hope is that $(T^*)^*(e_v) = e_{T(v)}$.

Indeed, let $v \in V$. Then

$$\begin{aligned}(T^*)^*(e_v)(f) &= e_v(T^*(f)) \\ &= (T^*(f))(v) \\ &= f(T(v)) \\ &= e_{T(v)}(f)\end{aligned}$$

Hence $(T^*)^*(e_v) = e_{T(v)}$, so at least in a very reasonable sense, the double dual of a linear transformation gets back to the original linear transformation.

Interestingly then, we see that moving to dual spaces reverses the direction of a linear map. We call this property being <u>contravariant.</u>

We will talk more about dual spaces in the coming weeks, but for the last thing that we will talk about today, let us investigate what happens to a quotient after taking a dual.

Let V be a finite dimensional vector space and let $W \subset V$ be a subspace.
Let $\beta = \{v_1, ..., v_m\}$ be a basis for W and extend β to a basis β' for V.

It was mentioned in discussion when we discussed quotients that we could try and think of V/W as being $\text{Span}\{v_{m+1}, ..., v_n\}$, but the issue was that this subspace on our choice of basis, whereas the definition of V/W does not depend on our choice of basis.

While it didn't work out perfectly back then, with the power of dual space we can make this idea work out beautifully.
Let's consider instead the subspace U of V^* spanned by $\{v_{m+1}^*, ..., v_n^*\}$. It seems like this still depends on our choice of basis. However, if you look closely, we can find a characterization of U that does not depend on our choice of basis.

Indeed, observe that for all $T \in U$ and $w \in W$, $T(w) = 0$. Furthermore, for $T \notin U$, there exists some $1 \leq i \leq m$ such that the component of v_i^* of T is non-zero. Then $T(v_i)$ is non-zero so there is a vector in U such that T is non-zero on it.

This should inspire a definition. Given a subspace $W \subset V$, define the <u>annihilator</u> of W, denoted W^0 by $W^0 := \{T \in V^* : \forall w \in W, T(w) = 0\}$.

We claim that $W^0 \simeq (V/W)^*$.
Indeed, define $\varphi : W^0 \to (V/W)^*$ by $\varphi(T)(v + W) = T(v)$.
This is well defined as if $v + W = v' + W$ then $v - v' \in W$. Then $T(v - v') = 0$, so $T(v) = T(v')$.
Furthermore, we claim that φ is an isomorphism.
Certainly, for $\alpha \in F, S, T \in W^0$ we have

$$\begin{aligned}\varphi(\alpha S + T)(v + W) &= (\alpha S + T)(v) + W \\ &= \alpha S(v) + T(v) + W \\ &= \alpha(S(v) + W) + (T(v) + W) \\ &= \alpha\varphi(S)(v) + \varphi(T)(v) \\ &= (\alpha\varphi(S) + \varphi(T))(v)\end{aligned}$$

so φ is a linear map.

Now suppose that $\varphi(T) = 0$ and let $v \in V$. Then, $0 = \varphi(T)(v + W) = T(v)$.
Hence $T = 0$ to $\ker\varphi = \{0\}$ and φ is injective.

Now let $T \in (V/W)^*$. Define $S : V \to 0$ by $S(v) = T(v+W)$. We claim that S is linear and that $\varphi(S) = T$.
Indeed, observe that $S = T \circ \pi$, and a composition of linear maps is linear. Furthermore, $\varphi(S)(v + W) = S(v) = T(v + W)$.
Hence $\varphi(S) = T$.

Hence φ is also surjective so is an isomorphism.

We will discuss more properties of dual spaces and annihilators in the coming weeks.