Week 4 Tuesday Notes

Andrew Sack

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1 Set Cardinality

The following will be stated without proof. If you are interested in a proof, I recommend reading Wikipedia or a textbook on introductory set theory. You can also come to office hours for more details on at least some of these.

The cardinality of a set S is the "number of elements" in S denoted $|S|$. For a finite set this already makes perfect sense, but it turns out there's a way to extend the notion to infinite sets as well under the standard axioms of set theory.

Let A and B be sets.

We say that $|A| \leq |B|$ if there exists an injection $f : A \to B$.
We says that $|A| \geq |B|$ if there exists a surjection $f : B \to A$.
Assuming the axiom of choice, this behaves like you expect it to. That is, if $|A| \leq |B|$ then $|B| \geq |A|$.
We say that $|A| = |B|$ if there exists a bijection $f : A \to B$. Immediately we see that if $|A| = |B|$ then $|B| = |A|$.
It turns out that if $|A| \leq |B|$ and $|B| \leq |A|$ then $|B| = |A|$. This is known as the Cantor-Schroeder Bernstein theorem.

All of this together tells us that comparing the "sizes" of sets is possible. You should verify that all of the above tracks with your beliefs about finite sets.

There is some strange behavior when we discuss infinite sets however.

For example, $|\mathbb{Z}| = |\mathbb{Z}^+| = |\mathbb{Q}|$.

Let A and B be infinite sets with $|A| \leq |B|$. We have the following facts:

- $|A \cup B| = |B|$
- $|A \times B| = |B|$

- $|A^n| = |A|$

There are more rules, you can check the Wikipedia article on so-called cardinal numbers.

We call a set S countable if $|S| = |\mathbb{Z}|$ or if S is finite. As I mentioned before, $|\mathbb{Z}| = |\mathbb{Z}^+| = |\mathbb{Q}|$. One interesting fact that is true: A countable union of countable sets is countable. More formally, if $\mathcal{I}$ is an indexing set and S_i is a countable set for each $i \in I$, then $\bigcup\limits_{i \in \mathcal{I}} S_i$ is countable.

One particularly interesting fact is that there are multiple "sizes" of infinite sets under this definition of size. We can in fact prove that there is no largest cardinality.
In particular, we show that for any set S , the power set of S denoted $\mathcal{P}(S)$ has strictly greater cardinality than S. Specifically, we show that there is no surjection from S to $\mathcal{P}(S)$. This is the only proof given in this section of the notes.

Let $f : S \to \mathcal{P}(S)$ be a function and define $T = \{s \in S : s \notin f(s)\}$.
We claim that $T \notin \operatorname{Im} f$. Suppose that $x \in S$ and $f(x) = T$.
Then if $x \in T$, then $x \notin f(x) = T$ a contradiction. If $x \notin T$ then $x \notin f(x) = T$ so $x \in T$ a contradiction.
Hence $T \notin \operatorname{Im} f$, so f is not a surjection.

Fact: $|\mathcal{P}(\mathbb{Z})| = |\mathbb{R}|$.

With this background, we can take what I consider to be the most obvious approach to the following exercise from your book.

2 Exercise 1

Show that $\mathbb{R}$ is an infinite dimensional vector space over $\mathbb{Q}$.

2.1 Proof 1

We show that any finite dimensional vector space over $\mathbb{Q}$ is countable. Let V be finite dimensional vector space over $\mathbb{Q}$.

Let $\beta = \{v_1, ..., v_n\}$ be a basis for V. Define a surjection $T : \mathbb{Q}^n \to V$ by $T \begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix} = \sum\limits_{i=1}^{n} a_i v_i$.

Hence $|V| \leq |\mathbb{Q}^n| = |\mathbb{Q}|$.

However $\mathbb{R}$ is uncountable and hence not finite dimensional over $\mathbb{Q}$.

As a challenge, try showing that $\mathbb{R}$ has uncountable dimension over $\mathbb{Q}$. We say that the dimension of a vector space is countable if it has a countable basis.

2.2 Proof 2

Here is a very clever proof I found on the internet of this fact that does not use anything fancy about set theory.

We explicitly give an infinite linearly independent set of $\mathbb{R}$ over $\mathbb{Q}$.

Let $p_1, p_2, ...$ be an infinite list of distinct primes.

We claim that $\log(p_1), \log(p_2), \ldots$ is an independent set.

Indeed, suppose that $\sum_{i=1}^{n} \frac{a_i}{b_i} \log(p_i) = 0$ where $a_i, b_i \in \mathbb{Z}, b_i \neq 0$.

Multiply by $\prod_{i=1}^{n} b_i$ to clear denominators. Define $c_i = \frac{a_i}{b_i} \prod_{i=1}^{n} b_i$ and observe that $c_i \in \mathbb{Z}$ and that $c_i = 0$ if and only if $a_i = 0$.

Then we have that $\sum_{i=1}^{n} c_i \log(p_i) = 0$.

Then $\log(\prod_{i=1}^{n} p_i^{c_i}) = 0$ so $\prod_{i=1}^{n} p_i^{c_i} = 1$.

Without loss of generality, let $c_1, ..., c_m \geq 0$ and $c_{m+1}, ..., c_n < 0$.

Then $\prod_{i=1}^{m} p_i^{c_i} = \prod_{i=m+1}^{n} p_i^{c_i}$.

By the fundamental theorem of arithmetic, every positive integer has unique prime factorization. As the primes on the left hand side of the equation are all distinct from the primes on the right hand side of the equation, $c_i = 0$ for each i and hence so is $\frac{a_i}{b_i}$.

Hence $\{\log(p_1), \log(p_2), ...\}$ is a linearly independent set over $\mathbb{Q}$.

3 Exercise 2

If there is time remaining in class, we will look at the following application of the rank-nullity theorem (called the dimension theorem in class.)

Recall from Homework 1 that if A and B are finite sets of the same size, then $f : A \to B$ is injective if and only if f is surjective.

We have a similar version of vector space.

Lemma: If V and W are finite dimensional vector spaces of the same dimension, then a linear map $T : V \to W$ is injective if and only if T is surjective.

Proof:

$\Rightarrow$
Recall from homework that T is injective if and only if $\ker T = \{0\}$. Hence $\dim V = \dim \ker T + \dim \operatorname{Im} T$. As $\dim \ker T = 0$, we have that $\dim V = \dim \operatorname{Im} T$. Hence $\dim \operatorname{Im} T = \dim W$ and as shown last Thursday and on this week's homework, this implies that $\operatorname{Im} T = W$ so T is surjective.

$\Leftarrow$
If T is surjective then $\operatorname{Im} T = W$ so $\dim \operatorname{Im} T = \dim W$. Hence $\dim V = \dim \ker T + \dim V$ so $\dim \ker T = 0$.

Hence $\ker T = \{0\}$ so T is injective.

As an exercise, give examples of this failing if we do not assume that V and W are finite dimensional.