Week 4 Thursday Notes

Andrew Sack

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1 Isomorphism

Let's talk briefly about isomorphism in a general setting. In each of your math classes, you will generally study a few types of mathematical structures. An important question to ask is, "When are two mathematical structures the same?"

This is a context dependent question. For example, in high school geometry we have a number of ways to ask if two shapes in the plane are the same. We could say that they are the same if and only if they are the exact same set of points, but this isn't very useful. Depending on the context, it is useful to look at properties invariant under rigid transformations of a shape in which case we call being the same "similarity" or "congruence" if we further care about the specific size of the shape.

Similarly in vector spaces, it should feel intuitively correct that $\mathbb{R}^2$ and $\mathbb{C}$ behave the same as vector spaces over $\mathbb{R}$. After all, without being able to talk about multiplication by i, there is nothing that really distinguished $a + bi$ from the point (a, b).

As another example that some of you have seen in Math 61, we can talk about isomorphisms of trees. Any two trees with 3 vertices are isomorphic, but if we talk about rooted trees, that is, a tree where we pick a vertex and call it the root, this is no longer the case. If we pick a vertex of degree 1 as the root this is a different mathematical structure than if we pick the vertex of degree 2.

A little more formally, you can think of an isomorphism as a bijection between two structures that preserves all properties "viewable" within that type of structure, back and forth. Essentially, it provides a bijective mapping not just between the elements of the structure, but between the properties as well. This is why we require that the inverse of a isomorphism is also a type of function that our type of object is "able to see."

One important thing I was asked to point out about isomorphisms is that they should map "special points" of a structure to other special points. For example, in rooted trees, a rooted tree isomorphism should map the root of one tree to the root of another tree because we want to view the roots as essentially being "the same objects but viewed from a different perspective." For Dr. Elman, this is why it is important that in our definition of isomorphism of vector spaces requires that the zero vector maps to the zero vector, even though we ultimately show that this requirement isn't something we have to check extra.

One of the amazing facts that you saw in class yesterday is that any two vector spaces of the same dimension are isomorphic. This tells us that vector spaces are remarkably simple structure, able to be totally characterized by a number. In particular, for an n-dimensional vector space over a field F, it is isomorphic to F^n.

This lets us simplify a great deal of proofs. If we want to show that some vector space property is true over a general n-dimensional vector space, it will generally suffices to just show that it is true over F^n. Funnily enough, while at the start of the course we told you that you shouldn't think of vectors as having coordinates, we have now shown that more or less you can do so with respect to a basis, at least in the finite dimensional case. That being said, it does remain useful to know how to work with vectors viewing them abstractly.

2 Quotient Spaces

In the extra lecture yesterday, you talked a little bit about quotient spaces. I would like to provide a bit of a more intuitive understanding of what they mean, outside of the formality of the extra lecture.

Let V be a vector space and $W \subseteq V$ a subspace. Recall that we define V/W as $V/\sim$ where for $v_1, v_2 \in V, v_1 \sim v_2$ if $v_1 - v_2 \in W$. There are three standard ways to write an element of W/V. One way is to say $v + W \in V/W$, where $v + W$ is the equivalence class of v. Another way is to write $\tilde{v}$ where $\tilde{v}$ is the equivalence class of $v \in V$ and the final is to say $[v]$ where $[v]$ is the equivalence class of $v \in V$.

Personally I prefer the first way. This notation should also suggest to you that $v + W = \{v + w : w \in W\}$. This is indeed the equivalence class of v.

This defines a vector space with addition $(v + W) + (v' + W) = (v + v') + W$ and for $a \in F, a \cdot (v + W) = a \cdot v + W$.

All of this defines the formal business with defining a quotient space, but I am personally more interested in understanding what this actually means.

You should think of V/W as V but after "forgetting" in the information in W. In some sense, this is like forgetting the portion of each vector that "comes from W".

Let's look at an example: Consider

$$V = \mathbb{R}^3 \text{ and } W = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = 0 \right\}.$$

The “information” in W is the second and third component of every vector. The equivalence classes in V/W consist of all vectors whose first component is the same. We can then think of an element in V/W as “the set of first coordinates of a vector in $\mathbb{R}^3$”, which you should see is really just the same as $\mathbb{R}$.

Later on in the course we will see a way to make this intuition of “chopping off the last two coordinates” a little more rigorous and well defined. (At least in the case of vector spaces over $\mathbb{R}$ and $\mathbb{C}$.)