# 115AH Week 2 Tuesday Notes

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## 1 Basis Basics

Recall the following:

Let $V$ be a vector space over a field $F$.

A set $S \subset V$ is said to be linearly independent if for any finite subset $\{v_1, ..., v_n\} \subseteq S$ and scalars $a_1, ..., a_n \in F$, $(\sum_{i=1}^{n} a_i v_i = 0) \Rightarrow (a_i = 0 \text{ for all } 1 \leqslant i \leqslant n)$.
A set is called linearly dependent if is not linearly independent.

Let $S \subset V$. Define $\operatorname{Span} S = \{\sum_{i=1}^{n} a_i v_i | a_i \in F, v_i \in S, n \in \mathbb{N}\}$. That is, $\operatorname{Span} S$ is the set of all finite linear combinations of vectors in $S$.

A set $S \subset V$ is called a basis for $V$ if $S$ is linearly independent and $\operatorname{Span} S = V$.

Here are some easy exercises (we may do some of these in discussion). Prove the following:

(a) Let $S \subset V$. Then $\operatorname{Span} S$ is a subspace.

(b) A subset $S \subset V$ is a subspace if and only if $\operatorname{Span} S = S$.

(c) $\operatorname{Span}(\operatorname{Span} S) = \operatorname{Span} S$ (When an applying an operation twice is the same as applying it once, we call this being "idempotent".)

(d) For two subspaces $U, W \subset V$ define $U + W = \{u + w : u \in U, w \in W\}$. Show that $U + W$ is a subspace.

(e) Let $S_1, S_2 \subset V$ be subsets. Then $\operatorname{Span} S_1 + \operatorname{Span} S_2 = \operatorname{Span}(S_1 \cup S_2)$.

(f) Let $u, v \in V$. If $\{u, v\}$ is linearly dependent, then either $u = a \cdot v$ or $v = a \cdot u$ for some $a \in F$.

(g) Let $u, v \in V$. If $\{u, v\}$ is linearly independent, then so is $\{u, u+v\}$ and so is $\{u+v, u-v\}$. (Give two generalizations of this.)

Feel free to ask me about any of these in office hours if you are stuck.

Useful facts:

- Every vector space has a basis.
- All bases for a vector space have the same cardinality. We call the cardinality of a basis the <u>dimension</u> of the vector space.
- Every vector can be written <u>uniquely</u> (up to ordering) as a linear combination of vectors in a given basis.
- Let $W \subset V$ be a subspace. Then $\dim W \leqslant \dim V$.

You have or will see proofs of these in class for the finite dimensional cases. If you would like to see proof for the infinite dimensional cases, see me in office hours.

Useful Facts about <u>finite dimensional</u> vector spaces:

- If $\dim W = \dim V$ then $W = V$.
- If $\dim V = n, S \subset V$ and $|S| = n$, then $\operatorname{Span} S = V$ if and only if $S$ is linearly independent.
- If $\dim V = \dim W$, then a <u>linear</u> function $f : V \to W$ is injective if and only if it is surjective if and only if it is bijective. (We will have the tools to prove this soon, but not yet.)

These will also eventually be proven or will be good exercises.

As a (maybe more tricky) exercise, give examples of when each of these fail in the infinite dimensional case.