Week 3 Thursday Notes

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1 Homework Problem 2

A good mental model to use for vector spaces is $\mathbb{R}^2$ and $\mathbb{R}^3$. In these vector spaces, the subspaces are $\{0\}$, lines through the origin, and planes through the origin.

Is the union of two lines through the origin a line or a plane? Not usually. For example in $\mathbb{R}^2$, the union of the subspace spanned by $\begin{pmatrix}1\\0\end{pmatrix}$ (the x-axis) and the subspace spanned by $\begin{pmatrix}0\\1\end{pmatrix}$ (the y-axis) is two crossing lines, certainly not a line or a plane.

Indeed, this fails the subspace test. $\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ which is not in either of the two subspaces.

However one easy condition that implies the union of two subspaces is a subspace is if one is contained by another. If $X \subseteq W$ then $X \cup W = W$, a subspace by assumption and similarly if $W \subseteq X$.

We should ask, is this every possibility? It turns out that the answer is yes! And by more or less the same logic as above. Suppose that X and W are subspaces, that $X \cup W$ is a subspace, and that $W \not\subseteq X$.
Then let $v \in W - X$ and $u \in X$. Then $v + u \in X \cup W$ as $X \cup W$ is a subspace.
But then either $v + u \in X$ or $v + u \in W$.
If $v + u \in X$ then $v + u - u = v \in X$. But this case cannot happen as $v \notin X$ by assumption.
Then $v + u \in W$. Hence $v + u - v = u \in W$ so $X \subset W$.

2 Exercise 2 for today

If $W \subset V$ is a subspace of V with V finite dimensional, then if $\dim W = \dim V, W = V$.

Proof:

First we will do a lemma. (This may have been done in class. If so we will skip the proof in discussion.)
Let V be a vector space over a field F and let $S \subset V$ be linearly independent. Then if $v \in V - \text{Span}(S)$ then $S \cup \{v\}$ is also linearly independent.
Proof:
Let $u_1, ..., u_n \in S$ and suppose that $a_1u_1 + \cdots + a_nu_n + bv = \vec{0}$ for $a_i, b \in F$.
If $b = 0$ then $\sum_{i=1}^{n} a_iu_i = 0$ so $a_i = 0$ for all i by the linear independence of S.
Otherwise if $b \neq 0$ then $bv = -\sum_{i=1}^{n} a_iu_i$ so $v = \sum_{i=1}^{n} -\frac{a_i}{b}u_i$.
But then $v \in \text{Span}(S)$ a contradiction.
Hence $S \cup \{v\}$ is linearly independent.

(Side question: Does this generalize to the following: Let S and T be disjoint linearly independent subsets. Then $S \cup T$ is also linearly independent? If yes say why. If no, give a counterexample.)

Let $\beta = \{v_1, ..., v_n\}$ be a basis for W. We claim that β is also a basis for V. In particular, we must show that $\text{Span}\,\beta = V$.
Indeed, suppose to the contrary. Then there exists $u \in V$ such that $u \notin \text{Span}\,\beta$.
Then by the lemma, $\beta \cup \{u\}$ is also linearly independent. But this contradicts a corollary from lecture that $n + 1$ vectors in an n-dimensional vector space are linearly independent.

3 Exercise 3 for today

Let V and W be vector spaces over F and let $\phi : V \to W$ be a linear transformation. Then if $v_1, ..., v_n \in V$ and $\{Av_1, ..., Av_n\}$ is linearly independent, then $\{v_1, ..., v_n\}$ is linearly independent as well.

Lemma: Let $T : V \to W$ be a linear transformation. Then $T(\vec{0}) = \vec{0}$.
Proof:
$T(\vec{0}) = T(0 \cdot \vec{0}) = 0 \cdot T(\vec{0}) = \vec{0}$.

Now for the exercise:

Suppose that $\sum_{i=1}^{n} a_i v_i = \vec{0}$.

$$\begin{aligned}\text{Then } \vec{0} &= A(\vec{0}) \\ &= A\left(\sum_{i=1}^{n} a_i v_i\right) \\ &= \sum_{i=1}^{n} a_i A(v_i)\end{aligned}$$

Then by the linear independence of $\{Av_1, ..., Av_n\}$, $a_i = 0$ for all i.

Hence $v_1, ..., v_n$ are linearly independent.