115AH Week 1 Tuesday Notes

January 4, 2021

1 Sets

1.1 Set Basics

A set is a "collection of elements".

Here are some examples you should already be familiar with:

- $\mathbb{R}$ = the set of real numbers.
- $\mathbb{C}$ = the set of complex numbers.
- $\mathbb{Z}$ = the set of integers.

We write sets in one of two ways:

(1) $A = \{$ (a list of all of the elements in A.$\}$
Ex: $\{a, b, c\}$
Ex: $\mathbb{Z}^+ = \{1, 2, 3, ...\}$
Ex: $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\}$.

(2) $A = \{x|x$ has the defining properties of A.$\}$
Ex: $\mathbb{Z} = \{x|x$ is an integer $\}$.
Ex: $\mathbb{Z}^+ = \{x|x > 0$ and $x \in \mathbb{Z}\}$.

If a is an element of A we write $a \in A$. The symbol $\in$ should be read as "is an element of" or "is in" so the statement $a \in A$ should read "a is in A".

Let A and B be two sets. If for all $a \in A$ we have that $a \in B$, we call A a subset of B and we write $A \subset B$ or $A \subseteq B$.

We say that $A = B$ if $a \in A$ if and only if $a \in B$. A common way to show the equality of two sets is to show that $A \subset B$ and $B \subset A$.

If $A \subset B$ but $A \neq B$ then we call A a "proper" subset of B. It is common to denote this as $A \subsetneq B$. Professor Elman also writes this as $A < B$. For example, $\mathbb{Z} \subsetneq \mathbb{R}$ and $\mathbb{R} \subsetneq \mathbb{C}$.

1.2 Operations on sets

Let A and B be sets.

We have the following definitions:

- $A \cup B := \{x : x \in A \text{ or } x \in B\}$.
  This is called the union of A and B.
- $A \cup B := \{x : x \in A \text{ and } x \in B\}$.
  This is called the intersection of A and B.
- $A \times B := \{(a, b) : a \in A \text{ or } b \in B\}$.
  This is called the Cartesian product of A and B.

Ex: For $a \in \mathbb{R}$, $(-\infty, a] \cup [a, \infty) = \mathbb{R}$
$(-\infty, a] \cap [a, \infty) = \{a\}$
$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

More generally if $A_1, \ldots, A_n$ are sets we write:

- $\bigcup_{i=1}^{n} A_i := A_1 \cup \cdots \cup A_n = \{x : x \in A_i \text{ for some } i, 1 \leqslant i \leqslant n\}$.
- $\bigcap_{i=1}^{n} A_i := A_1 \cap \cdots \cap A_n = \{x : x \in A_i \text{ for all } i, 1 \leqslant i \leqslant n\}$.
- $A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) : a_i \in A_i \text{ for all } i, 1 \leqslant i \leqslant n\}$.

Even more generally, let $\mathcal{I}$ be a set and for each $i \in \mathcal{I}$ let A_i be a set. (We call $\mathcal{I}$ an <u>indexing set</u>.)

We write

- $\bigcup_{\mathcal{I}} A_i = \bigcup_{i \in \mathcal{I}} A_i := \{x : x \in A_i \text{ for some } i \in \mathcal{I}\}$
- $\bigcap_{\mathcal{I}} A_i = \bigcap_{i \in \mathcal{I}} A_i := \{x : x \in A_i \text{ for all } i \in \mathcal{I}\}$

You may ask about "$\bigtimes_{i\in\mathcal{I}} A_i$" (or as I would write it $\prod_{i\in\mathcal{I}} A_i$) However this is a slightly more complicated story which you can ask me about during office hours or read about on Wikipedia.

If A and B are sets, we define a relation R on A and B to be a subset $R \subset A \times B$.
For example, $R \subset \mathbb{Z}\times\mathbb{Z}$ where $R = \{(x, \varepsilon)|x \in \mathbb{Z}$ and $\varepsilon = 0$ if x is even and $\varepsilon = 1$ if x is odd$\}$

2 Functions

2.1 Function Basics

Let A and B be sets. A function (or map) f from A to B is a "rule" that assigns to each element in A a **unique** element in B.
We write such an f by $f : A \to B$ by $a \mapsto f(a)$ where $f(a)$ is where a is sent to.
We call A the domain of f and we call B the codomain.
The image of f is the set $\operatorname{Im} f = \{b \in B|$ there exists an $a \in A$ such that $f(a) = b\}$.
We could also write this as $\{f(a)|a \in A\}$ commonly denoted as $f(A)$.

For example we could have $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^2$.

This function has domain $\mathbb{R}$, codomain $\mathbb{R}$ and image $\mathbb{R}^{\geqslant 0} := \{x \in \mathbb{R}|x \geqslant 0\}$.

More formally, a function f from A to B is a relation on A and B such that for all $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$. However we essentially never use this level of formality and will stick with the first notation.

2.2 Properties of functions

Let $f : A \to B$ be a function.

We say that f is surjective or onto if $\operatorname{Im} f = f(a) = B$.

Examples:

- $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto 2x$ is onto.
- $f : [0, 1] \to \mathbb{R}$ by $x \mapsto x^2$ is not onto.
- $f : \mathbb{Z} \to \{0, 1\}$ by $n \mapsto \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{is odd} \end{cases}$ is surjective.
- $f : \{2n|n \in \mathbb{Z}\} \to \mathbb{Z}$ by $2n \mapsto n$ is surjective.

We say that f is injective or one-to-one (or 1-1) if $f(x) = f(y)$ implies $x = y$.
Equivalently, for all $b \in \operatorname{Im} f$, there is exactly one element $a \in A$ satisfying $f(a) = b$.
Equivalently, for all $b \in B$, there is at most one element a such that $f(a) = b$.
Equivalently, there exists a function $g : \operatorname{Im} f \to A$ by $f(a) \mapsto a$.
If this is the case we write f^{-1} for g and say $f^{-1} : \operatorname{Im} f \to A$ by $f(a) \mapsto a$.

Examples:

- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$.
- If $A \subset B$, the inclusion map $\iota : A \to B$ by $a \mapsto a$ is injective.
  If $A \subsetneq B$ then this is not onto.

If a function is both a surjection and an injection we call it a bijection.

Example:
The identity map $1_A : A \to A$ by $a \mapsto a$ is a bijection. (We also write $\mathrm{id}_A : A \to A$ for the identity map.)

2.3 Composition

Let $f : A \to B$ and $g : B \to C$ be functions.

Then we define $g \circ f : A \to C$ by $a \mapsto g(f(a))$.
This is a function. Furthermore if $f : A \to B$ is a bijection then $f^{-1} : \operatorname{Im} f \to A$ by $f(a) \mapsto a$ is a function, and as f is onto, $\operatorname{Im} f = B$, so $f^{-1} : B \to A$ by $f(a) \mapsto a$.
In particular, $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

We also have that if $f : A \to B, g : B \to C, h : C \to D$ are maps, then $(h \circ g) \circ f = h \circ (g \circ f)$.