Week 10 Notes

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This week we will be reviewing every named theorem in the class in preparation for your final exam.

1 Vector Spaces

Subspace Theorem (Lecture 3)
Let V be a vector space over F, $\emptyset \neq W \subset V$ a subset. Then the following are equivalent:

1. W is a subspace of V.
2. W is closed under addition and scalar multiplication from V.
3. $\forall w_1, w_2 \in W, \forall \alpha \in F, \alpha w_1 + w_2 \in W$.

Toss-in Theorem (Lecture 5)
Let V be a vector space over F and $\emptyset \neq S \subset V$ a linearly independent subset. Suppose that $v \in (V - \operatorname{Span} S)$. Then $S \cup \{v\}$ is linearly independent.

Coordinate Theorem (Lecture 6)
Let V be a finite dimensional vector space over F with basis $\mathcal{B} = \{v_1, ..., v_n\}$ and $v \in V$. Then $\exists! \alpha_1, ..., \alpha_n \in F$ such that $v = \alpha_1 v_1 + ... + \alpha_n v_n$.

Important Exercise (Lecture 6)
Let V be a vector space over F, $v_1, ..., v_n \in V$. Then $\operatorname{Span}\{v_1, ..., v_n\} = \operatorname{Span}\{v_2, ..., v_n\}$ if and only if $v_1 \in \operatorname{Span}\{v_2, ..., v_n\}$.

Toss Out Theorem (Lecture 6)
Let V be a vector space over F. If V can be spanned by finitely many vectors then V is a finite dimensional vector space over F. More precisely, if $V = \operatorname{Span}\{v_1, ..., v_n\}$ then a subset of $\{v_1, ..., v_n\}$ is a basis for V.

(Steinitz) Replacement Theorem (Lecture 7)
Let V be a finte dimensional vector space over F and $\{v_1, ..., v_n\}$ be a basis for V. Suppose that $v \in V$ satisfies $V = a_1v_1 + ... + a_nv_n$ for $a_1, ..., a_n \in F, a_i \neq 0$. Then $\{v_1, ..., v_{i-1}, v, v_{i+1}, ..., v_n\}$ is also a basis for V.

Main Theorem (Lecture 7)
Suppose that V is a vector space over F with $V = \mathrm{Span}\{v_1, ..., v_n\}$. Then any linearly independent subset of V has at most n elements.

Extension Theorem (Lecture 8)
Let V be a finite dimensional vector space over F, $W \subset V$ a subspace. Then every linearly independent subset S in W is finite and part of a basis for W which is a finite dimensional vector space.

(Note from Andrew: A simpler an equivalent way to say this is as follows: Let V be a finite dimensional vector space over F and S a linearly independent subset of V. Then there exists a basis of V containing S. This also holds true over infinite dimensional vector spaces if you assume the axiom of choice, but you are not responsible for knowing the proof of the infinite dimensional case.)

Counting Theorem (Lecture 8)
Let V be a finite dimensional vector space over F, $W_1, W_2 \subset V$ subspaces. Suppose that both W_1 and W_2 are finite dimensinoal vector spaces over F. Then

1. $W_1 \cap W_2$ is a finite dimensional vector space over F.
2. $W_1 + W_2$ is a finite dimensional vector space over F.
3. $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$.

2 Linear Transformations

Dimension Theorem (Lecture 9)
Let $T : V \to W$ be linear with V a finite dimensional vector space over F. Then

1. $\mathrm{Im}\, T$ and $\ker T$ are finite dimensional vector spaces over F.
2. $\dim V = \dim \ker T + \dim \mathrm{Im}\, T$

(Note from Andrew: You will usually see this called the rank-nullity theorem.)

Monomorphism Theorem (Lecture 10)
Let $T : V \to W$ be linear. Then the following are equivalent:

1. T is $1-1$, so a monomorphism.
2. T takes linearly independent sets in V to linearly independent sets in W.
3. $\ker T = 0 := \{0_V\}$.
4. $\dim \ker T = 0$

Isomorphism Theorem (Lecture 10)
Suppose $T : V \to W$ is linear with $\dim V = \dim W < \infty$, i.e. V, W are finite dimensional vector spaces over F of the same dimension. Then the following are equivalent:

1. T is an isomorphism.
2. T is a monomorphism
3. T is an epimorphism
4. If $\mathcal{B} = \{v_1, ..., v_n\}$ is a basis for V, then $\{Tv_1, ..., Tv_n\}$ is a basis for W i.e. T takes bases of V to bases of W.
5. There exists a basis $\mathcal{B}$ of V that maps to a basis of W.

Existence of Linear Transformation Theorem (Universal Property of Vector Spaces) (Lecture 10)
Let V be a finite dimensional vector space over F, $\mathcal{B} = \{v_1, ..., v_n\}$ a basis for V and W an arbtirary vector space over F. Let $w_1, ..., w_n \in W$ not necessarily distinct. Then

$$\exists! T : V \to W \text{ linear, such that } Tv_i = w_i \forall i.$$

Classification of Finite Dimensional Vector Spaces Theorem (Lecture 11)
Let V and W be finite dimensional vector spaces over F. Then $V \simeq W$ if and only if $\dim V = \dim W$.

(Note from Andrew: $V \simeq W$ means “there exists an isomorphism $T : V \to W$”)

3 Matrices

Matrix Theory Theorem (Lecture 12)
Let V, W be finite dimensional vector spaces over F, $\dim V = n, \dim W = m$ and $\mathcal{B}$ and $\mathcal{C}$ be ordered bases for V and W respectively. Then the map

$$\varphi : L(V, W) \to F^{m \times n} \text{ by } T \mapsto [T]_{\mathcal{B},\mathcal{C}}$$

is an isomorphism. In particular, $\dim L(V, W) = mn$.

Change of Basis Theorem (Lecture 12)
Let V, W be finite dimensional vector spaces over F with ordered bases $\mathcal{B}, \mathcal{B}'$ for V and $\mathcal{C}, \mathcal{C}'$ for W. Let $T : V \to W$ be linear. Then:

$$[T]_{\mathcal{B},\mathcal{C}} = [1_W]_{\mathcal{C}',\mathcal{C}}[T]_{\mathcal{B}',\mathcal{C}'}[1_V]_{\mathcal{B},\mathcal{B}'}$$

$$\text{Also } [1_W]_{\mathcal{C}',\mathcal{C}} = [1_W]^{-1}_{\mathcal{C},\mathcal{C}'} \text{ and } [1_V]_{\mathcal{B}',\mathcal{B}} = [1_V]^{-1}_{\mathcal{B},\mathcal{B}'}$$

(Note from Andrew: This was written slightly differently in lecture.)

4 Inner Product Spaces

Cauchy-Schwarz Inequality (Lecture 16)
Let V be an inner producr space over F. Then $\forall v_1, v_2 \in V$,

$$|\langle v_1, v_2 \rangle| \leq ||v_1|| \cdot ||v_2||.$$

Minkowski Inequality (Special Case) (Lecture 16)
Let V be an inner producr space over F. Then $\forall v_1, v_2 \in V$,

$$||v_1 + v_2|| \leq ||v_1|| + ||v_2||.$$

(Note from Andrew: This special case is normally called the triangle inequality. You can look up the general case of the Minkowski inequality online. Fun fact: Minkowski was Albert Einstein's teacher and also worked a lot on relativity.)

Gram-Schmidt Theorem (Lecture 18)
Let V be an inner product space over F and $\emptyset \neq S_n = \{v_1, ..., v_n\} \subset V$ a linealry independnet set. Then there exists $y_1, ..., y_n \in V$ such that

(i) $y_1 = v_1$

(ii) $T_n = \{y_1, ..., y_n\}$ is an orthogonal set and linearly independent.

(iii) $\operatorname{Span} T_n = \operatorname{Span} S_n$.

(Note from Andrew: We can enhance property 3 as follows: for $1 \leq k \leq n, \operatorname{Span}\{v_1, .., v_k\} = \operatorname{Span}\{y_1, ..., y_k\}$.)

OR Theorem (Lecture 18)
Let V be a finite dimensional inner product space over F. Then V has an orthogonal basis. If $F = \mathbb{R}$ or $\mathbb{C}$ then V has an orthonormal basis.

OR Decomposition Theorem (Lecture 19)
Let V be an inner product space over F. (Not necessarily finite dimensional.) Let $S \subset V$ be a finite dimensional subspace and $v \in V$. Then

$$\exists! s \in S, s^{\perp} \in S^{\perp} \text{ such that } v = s + s^{\perp}.$$

In particular, $V = S + S^{\perp}, S \cap S^{\perp} = 0,$ so $V = S \perp S^{\perp}$.

Moreover, if $v = s + s^{\perp}, s \in S, s^{\perp} \in S^{\perp}$ then $||v||^2 = ||s||^2 + ||s^{\perp}||^2$.
If, in addition, V is a finite dimensional inner product space over F, then

$$\dim V = \dim S + \dim S^{\perp}.$$

Approximation Theorem (Lecture 20)
Let V be an inner product space over F, $S \subset V$ a finite dimensional subspace, and $v \in V$. Then v_S, the orthogonal projection of v onto S is closer to v than any other vector in S, i.e.

$$d(v, v_S) = ||v - v_S|| \leq ||v - r|| = d(v, r)$$

for all $r \in S$. Equivalently, $d(v, S) = d(v, v_S)$.
Moreover, if $r \in S$ then $||v - V_S| = ||v - r||$ if and only if $r = v_S$. We say that v_S gives the best approximation to v in S.

Hermitian Corollary (Lecture 21)
Let V be an inner product space over F, $T : V \to V$ linear. Suppose that T is Hermitian. Then any eigenvalue of T (if any) is real, i.e. lies in $F \cap \mathbb{R}$.

Key Lemma (for Hermitian Operators) (Lecture 21)
Let V be an inner product space over F, $T : V \to V$ Hermitian, $S \subset V$ a T-invariant subspace. Then

1. $S^{\perp}$ is T-invariant, i.e. $T(S^{\perp}) \subset S^{\perp}$.
2. $T|_{S^{\perp}} : S^{\perp} \to S^{\perp}$ is Hermitian.

Spectral Theorem (for Hermitian Operators) (Refined Version) (Lecture 22)
Let $F = \mathbb{R}$ or $\mathbb{C}$, V be a finite dimensional inner product space over F, $T : V \to V$ Hermitian. Then there exists an orthonormal basis $\mathcal{C}$ of eigenvectors for V of T and every eigenvalue of T is real. Moreover, if $\mathcal{B}$ is any ordered orthonormal basis for V then $[T]_{\mathcal{C}} = C[T]_{\mathcal{B}}C^*$ for some invertible matrix $C \in M_n(F)$, i.e. $C = [1_V]_{\mathcal{B},\mathcal{C}}$.

(Note from Andrew: C is called a unitary matrix. That is, on such that $C^{-1} = C^*$.)

New Key Lemma (Lecture 24)
Let V be a finite dimensional inner product space over F, $T : V \to V$ linear. Suppose

that V has an orthonormal basis and $W \subset V$ is a T-invariant subspace. Then $W^\perp \subset V$ is T^*-invariant. In particular, $T^*|_{W^\perp} : W^\perp \to W^\perp$ is linear.

Schur's Theorem (Lecture 24)
Let V be a finite dimesinonal inner product space over C, $T : V \to V$ linear. Then T is triangularizable. Moreover, there exists an ordered orthonormal basis $\mathcal{B}$ for V such that $[T]_\mathcal{B}$ is upper triangular.

(Note from Andrew: The matrix version of this says that any complex matrix A can be written as $A = QUQ*$ where Q is unitary and U is upper triangular. This is commonly known as a Schur Decomposition.)

Spectral Theorem (for Normal Operators) (Lecture 24)
Let V be a finite dimensional inner product space over C, $T : V \to V$ normal. Then there exists an ordered orthonormal basis $\mathcal{C}$ for V consisting of eigenvectors of T. In particular, T is diagonalizable. Moreover, if $\mathcal{B}$ is an ordered orthonormal basis, then

$$[T]_\mathcal{C} = [1_V]_{\mathcal{B},\mathcal{C}}[T]_\mathcal{B}[1_V]^*_{\mathcal{B},\mathcal{C}}.$$