115 Week 3 Notes

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1 Problem 7 from October 14 Homework

Prove or give a counterexample: Suppose that V is a vector space and $u, v, w \in V$. If u, v are linearly independent, v, w are linearly independent, and u, w are linearly independent, then u, v, w are linearly independent.

Counter-example:

$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2 Problem 4 from October 16 Homework

Show that if V is a vector space, $\vec{v} \in V$ and $a \in \mathbb{R}$ satisfy $a\vec{v} = 0$, then $a = 0$ or $\vec{v} = \vec{0}$.

Proof:
If $a = 0$ then we are done.
If $a \neq 0$, then we multiply on both sides $\frac{1}{a}$ to get $\frac{1}{a} \cdot a\vec{v} = \frac{1}{a}\vec{0}$ so $\vec{v} = \vec{0}$.

3 Problem 5 from October 19 Homework

Suppose that V is a vector space with dimension n. Show that $u_1, \ldots, u_n \in V$ are linearly independent if and only if $\operatorname{span}\{u_1, \ldots, u_n\} = V$.

Suppose that $u_1, \ldots, u_n \in V$ are linearly independent and let $v \in V$. Recall from class that if you have $n + 1$ vectors in V with $\dim V = n$, then they are linearly dependent.

Hence $u_1, \ldots, u_n, v$ are linearly dependent so there exists $a_1, \ldots, a_n, b \in \mathbb{R}$ such that $bv +$

$\sum_{i=1}^{n} a_i u_i = \vec{0}$ and at least one of $a_1, ..., a_n, b$ is non-zero.
If $b = 0$, then $\sum_{i=1}^{n} a_i u_i = \vec{0}$ and as $u_1, ..., u_n$ are linearly independent, this implies that $a_i = 0$ for all i.
Then $bv = \vec{0}$ and as $b \neq 0$, $v = \vec{0}$. $\vec{0} \in \text{span}\{u_1, ..., u_n\}$.

Otherwise if $b \neq 0$, then $bv = -\sum_{i=0}^{n} a_i u_i$ and multiplying by $\frac{1}{b}$ on both sides we get $v = \sum_{i=0}^{n} \frac{-a_i}{b} u_i$.
Hence $v \in \text{span}\{u_1, \ldots, u_n\}$.

Now suppose that $\text{span}\{u_1, \ldots u_n\} = V$ and suppose by way of contradiction that $u_1, ..., u_n$ are linearly dependent.
Then by lemma 2, for some $i_1, ..., i_k$, $\text{span}\{u_{i_1}, ..., u_{i_k}\} = \text{span}\{u_1, \ldots u_n\}$ and $u_{i_1}, ..., u_{i_k}$ are linearly independent with $k < n$.
But then $\{u_{i_1}, ..., u_{i_k}\}$ is a smaller set of linearly independent vectors that span V (that is to say, a basis), but $\dim V = n > k$ a contradiction.