115 Week 3 Notes

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1 Warm-up

Show that if $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f$ is injective.
Proof:
Recall that we say a function $f : A \to B$ is injective if for all $x, y \in A, (f(x) = f(y) \Rightarrow x = y)$.
Suppose that $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$ and as g is injective, $f(x) = f(y)$. Then as f is injective, $x = y$.
Hence $g \circ f$ is injective.

Show that if $f : A \to B$ and $g : B \to C$ are both surjective, then $g \circ f$ is surjective.
Proof:
Recall that we say a function $f : A \to B$ is surjective if $\forall b \in B, \exists a \in A$ such that $f(a) = b$.
Then for all $c \in C$, there exists $b \in B$ such that $g(b) = c$ and as f is surjective there exists $a \in A$ such that $f(a) = b$.
Then $g(f(a)) = g(b) = c$ so $g \circ f$ is surjective.

Show that if $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f$ is bijective.
Proof:
Apply the previous 2 lemmas.

Show that if $f_1, \dots, f_n$ are injective, then so is $f_1 \circ \cdots \circ f_n$ is injective (assuming that the composition makes sense.)
Proof:
We proceed by induction. When $n = 1$ this is trivial.
Now assume the claim for all $m < n$.
Then $g = f_1 \circ \cdots \circ f_{n-1}$ is injective. By what was proven above, $g \circ f_n$ is injective. Hence $g \circ f_n = f_1 \circ \cdots \circ f_n$ is injective.

2 Linear transformations

Let $T : V \to W$ be a linear transformation. Show that $T(\vec{0}) = \vec{0}$.
Proof:
$T(\vec{0}) = T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0})$.
Then $\vec{0} = T(\vec{0})$.

Let $T : V \to W$ be a linear transformation and let $U \subset W$ be a subspace. Show that $T^{-1}(U) = \{v \in V : T(v) \in U\}$ is a subspace of V.
Proof:
Suppose that $v_1, v_2 \in T^{-1}(U)$. Then $T(v_1), T(v_2) \in U$, so $T(v_1) + T(v_2) = T(v_1 + v_2) \in U$ so $v_1 + v_2 \in T^{-1}(U)$.
Furthermore for $a \in \mathbb{R}, T(av_1) = aT(v_1) \in U$, so $av_1 \in T^{-1}(U)$.
Finally, $T(\vec{0}) = \vec{0}$ so $\vec{0} \in T^{-1}(U)$, hence $T^{-1}(U)$ is a subspace.

Under the same conditions as the previous problem, show that show that $\operatorname{Im} T = \{T(v) : v \in V\}$ is a subspace of W.
Proof:
Let $T(v_1), T(v_2) \in \operatorname{Im} T$. Then $T(v_1) + T(v_2) = T(v_1 + v_2) \in \operatorname{Im} T$. Similarly, let $a \in \mathbb{R}$. Then $aT(v_1) = T(av_1) \in \operatorname{Im} T$.
Finally, $\vec{0} = T(\vec{0}) \in \operatorname{Im} T$. Hence $\operatorname{Im} T$ is a subspace of W.