115 Week 1 Notes

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We make heavy use of the following (related) rules of logic and sets, so I put them here to jog your memory:

Version in Logic	Version for Sets
a and b	$A \cap B$
a or b	$A \cup B$
not a	A^c
(a and b) and $c \Leftrightarrow a$ and (b and c)	$(A \cap B) \cap C = A \cap (B \cap C)$
(a or b) or $c \Leftrightarrow a$ or (b or c)	$(A \cup B) \cup C = A \cup (B \cup C)$
a and (b or c) $\Leftrightarrow$ (a and b) or (a and c)	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
a or (b and c) $\Leftrightarrow$ (a or b) and (a or c)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
not(a and b) $\Leftrightarrow$ (not a) or (not b)	$(A \cap B)^c = A^c \cup B^c$
not(a or b) $\Leftrightarrow$ (not a) and (not b)	$(A \cup B)^c = A^c \cap B^c$

Here we use the notation A^c is the complement of A, that is, given some universal set U that A is a subset of, $A^c = \{x \in U : x \notin U\}$.

We also make use of the following rules for quantifiers (we call $\forall$ and $\exists$ "quantifiers"):
Given a proposition $P(x)$, we have:
$\not(\forall x, P(x)) \Leftrightarrow \exists x, \text{not } P(x)$
$\not(\forall x, P(x)) \Leftrightarrow \exists x, \text{not } P(x)$

Given a proposition of two variables $P(x, y)$ we have the following rules:
$\not(\forall x \forall y, P(x, y)) \Leftrightarrow \exists x \exists y, \text{not } P(x, y)$
$\not(\forall x \exists y, P(x, y)) \Leftrightarrow \exists x \forall y, \text{not } P(x, y)$
$\not(\exists x \exists y, P(x, y)) \Leftrightarrow \forall x \forall y, \text{not } P(x, y)$
$\not(\exists x \forall y, P(x, y)) \Leftrightarrow \forall x \exists y, \text{not } P(x, y)$

Generically, we move from left to right, exchanging $\forall$ and $\exists$, and negating the final proposition.

I encourage you to think through each of these rules and convince yourself that they are true.

It is true that $\forall x \forall y, P(x,y) \Leftrightarrow \forall y \forall x, P(x,y)$ and similarly $\exists x \exists y, P(x,y) \Leftrightarrow \exists y \exists x, P(x,y)$. Because of this, it is common to just write $\forall x, y, P(x,y)$ and $\exists x, y, P(x,y)$.

HOWEVER, it is NOT true that $\forall x \exists y P(x,y) \Leftrightarrow \exists y \forall x P(x,y)$.
As an example demonstrating this, consider the statement: For all integers m, there exists an integer n such that $n = 2m$. This is obviously true as we can just take $n = 2m$.
However the statement: There exists an integer n such that for all integers m, $n = 2m$. This would be making the claim that for a single fixed integer n, every integer m is equal to $m/2$, which is obviously false.

Here are some write-ups of the homework problems that we did in class:

1 Number 6 from 10/5

This asks us which statement we would have to prove to DISPROVE the following: "There exists an integer n such that $P(n,m)$ is true for all integers m."
We can rewrite that statement using symbols as follows: $\exists n \forall m, P(n,m)$.
In order to disprove the statement, we must prove its negation. Using the rules from above we have that the negation is not $(\exists n \forall m, P(n,m))$ which is equivalent to $\forall n \exists m, \text{not } P(n,m)$.
Translating this back to words, we get option B.

2 Number 5 from 10/7

We only did the first set equality from number 5, but the second part is very similar.

Claim: $B \setminus (A_1 \cup ... \cup A_n) = (B \setminus A_1) \cap ... \cap (B \setminus A_n)$
Proof:

We start by showing that $B \setminus (A_1 \cup ... \cup A_n) \subseteq (B \setminus A_1) \cap ... \cap (B \setminus A_n)$
Let $x \in B \setminus (A_1 \cup ... \cup A_n)$. Then $x \in B$ and $x \notin A_1 \cup \cdots \cup A_n$.
Then $x \notin A_i$ for any i.
Then $x \in B \setminus A_i$ for all i, so $x \in (B \setminus A_1) \cap \cdots \cap (B \setminus A_n)$.
Hence $B \setminus (A_1 \cup ... \cup A_n) \subseteq (B \setminus A_1) \cap ... \cap (B \setminus A_n)$.

Now we show that $B \setminus (A_1 \cup ... \cup A_n) \supseteq (B \setminus A_1) \cap ... \cap (B \setminus A_n)$
Let $x \in (B \setminus A_1) \cap ... \cap (B \setminus A_n)$
Then $x \in B \setminus A_i$ for all i.
Then $(x \in B \text{ and } x \notin A_1)$ and $\dots$ and $(x \in B \text{ and } x \notin A_n)$.
Using the rules from the start of the PDF, this is equivalent to $(x \in B)$ and $(x \notin A_1 \text{ and } \dots \text{ and } x \notin A_n)$.
Again using the rules from the start of the pdf, this is equivalent to

$$(x \in B) \text{ and not } (x \in A_1 \text{ or } \dots \text{ or } x \in A_n)$$

Then $x \in B$ and $x \notin (A_1 \cup \cdots \cup A_n)$.
Hence $x \in B \setminus (A_1 \cup ... \cup A_n)$.

3 Number 3 from 10/7

We also showed half of number 3, in particular we showed that $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$.

Proof:

Let $x \in (A \cup B) \setminus (A \cap B)$.
Then $x \in A \cup B$ and $x \notin A \cap B$.
Then ($x \in A$ or $x \in B$) and not ($x \in A$ and $x \in B$).
Applying the rules from the start of the PDF, this is equivalent to

$$(x \in A \text{ or } x \in B) \text{ and } (x \notin A \text{ or } x \notin B)$$

Again applying the rules from the start of the PDF, this is equivalent to

$$((x \in A \text{ or } x \in B) \text{ and } x \notin A) \ \text{ OR } \ ((x \in A \text{ or } x \in B) \text{ and } x \notin B)$$

This is equivalent to

$$((x \in A \text{ and } x \notin A) \text{ or } (x \in B \text{ and } x \notin A)) \ \text{ OR } \ ((x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin B))$$

Observe that ($x \in A$ and $x \notin A$) and ($x \in B$ and $x \notin B$) are always false. For a false statement p and any statement q, $q \Leftrightarrow p$ or q.

Then we have that

$$(x \in B \text{ and } x \notin A) \text{ OR } (x \in A \text{ and } x \notin B)$$

Hence $x \in (B \setminus A) \cup (A \setminus B)$ so $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$.

While we didn't get to the other direction in class, if you notice, each of the steps that we took in this proof is reversible. This tells you how to show that $(A \cup B) \setminus (A \cap B) \supseteq (A \setminus B) \cup (B \setminus A)$.

When each of the steps are equivalences, we have actually shown that $x \in (A \cup B) \setminus (A \cap B) \Leftrightarrow x \in (A \setminus B) \cup (B \setminus A)$, so the two sets must be equal. This approach is similar to one of the proofs from Tuesday's class.