Week 2 Worksheet for Math 61

1. (a) How many integers in $\{1, ..., 1000\}$ are divisible by 4?

   $1000/4 = 250$.
   The multiples of 4 in $\{1, ..., 1000\}$ are $\{4 \cdot x | x \in \{1, ..., 250\}$
   So there are 250 multiples of 4 in $\{1, ..., 1000\}$.

   (b) How many integers in $\{1, ..., 1000\}$ are divisible by 6?

   $1000/6 = 166.\bar{6}$.
   The multiples of 4 in $\{1, ..., 1000\}$ are $\{6 \cdot x | x \in \{1, ..., 166\}$ $(6 \cdot 167 = 1002)$
   Then there are 166 multiples of 6 in $\{1, ..., 1000\}$

   (c) How many integers in $\{1, ..., 1000\}$ are divisible by both 4 and 6?

   An integer is divisible by both 4 and 6 if and only if it is a multiple of 12.
   $1000/12 = 83.\bar{3}$ so there are 83 multiples of both 4 and 6 in $\{1, ..., 1000\}$.

   (d) How many integers in $\{1, ..., 1000\}$ are divisible by 4 or 6?

   Let A be the set of integers in $\{1, ..., 1000\}$ that are divisible by 4 and let B be the set of integers that are divisible by 6.

   We want to calculate $|A \cup B|$. By the principle of inclusion-exclusion we have $|A \cup B| = |A| + |B| - |A \cap B|$

   Then $|A \cup B| = 250 + 166 - 83 = 333$.

   (e) How many integers in $\{1, ..., 1000\}$ are divisible by 4 or by 6 but not both?

   $333 - 83 = 250$

2. Let A and B be dice with the following tuples (a tuple is an ordered list of numbers) as the numbers on their sides. What are the odds of rolling a sum of 7?

   (a) $A = (1, 2, 3, 4, 5, 6)$
   $B = (1, 2, 3, 4, 5, 6)$

   There are 36 possible pairs each with equal probability. 6 of them add up to 7, namely $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$.

   Then there is a $6/36 = 1/6$ chance to roll a 7.

   (b) $A = (1, 1, 2, 5)$
   $B = (2, 6, 6, 6)$

   If we label the 1s and the 6s there are 16 possible parings.
   In half of the 16 rolls A rolls 1 and in $3/4$ of those rolls B rolls a 6.
   In $1/4$ of the 16 rolls A rolls a 5 and in $1/4$ of those rolls B rolls a 2.
   Hence there are $16 \cdot \frac{1}{2} \cdot \frac{3}{4} + 16 \cdot \frac{1}{4} \cdot \frac{1}{4} = 7$ possible rolls that add to 7.
   Then there is a $7/16$ chance.

(c) $A = (0, 2, 4, 4, 6, 8)$
$B = (2, 3, 4, 5, 6, 7, 8, 9)$

There is a $4/(6 \cdot 8) = 1/12$ chance. Use a similar method to part (b).

3. For each part we consider functions $f : A \to B$.

(a) How many functions are there from a set of m elements to a set of n elements?

For each element in A we have n choices of where it goes. Then there are m^n possible functions.

(b) How many injections are there from a set of m elements to a set of n elements?

If $m > n$ then there are none by the pigeonhole principle.

There are n choices of where to send the first element, $n-1$ choices of where to send the second element, and so on.

Then there are $\prod_{i=0}^{m-1} n - i = \frac{n!}{(n-m)!} = m!\binom{n}{m}$ functions.

(c) How many bijections are there from a set of m elements to a set of m elements?

Any injection from a set of m elements to a set of m elements is also a bijection. Then from (b) take $m = n$ and we get $m!$.

(d) How many surjections are there from $\{1, ..., n\}$ to $\{1, 2, 3\}$?
Hint: Apply inclusion-exclusion

We will instead count the number of functions that are not surjections. Then the number of surjections will be the complement of this.

This is equal to the set of functions with codomain $A = \{1, 2\}, B = \{1, 3\},$ or $C = \{2, 3\}$.

By inclusion-exclusion, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

$|A| = |B| = |C| = 2^n$ by part (a).

$A \cap B$ is the set of functions with codomain $\{1\}$, $A \cap C$ is the set of functions with codomain 2 and $B \cap C$ is the set of functions with codomain 3. Then $|A \cap B| = |A \cap C| = |B \cap C| = 1$.

$A \cap B \cap C = \emptyset$.

Hence the number of non-surjective functions is $3 \cdot 2^n - 3$.

Then the number of surjections is $3^n - (3 \cdot 2^n - 3) = 3^n - 3 \cdot 2^n + 3$.

4. What are the odds of having a royal flush in a hand of 6 cards?

There are 4 possible royal flushes, 1 for each suit. For each royal flush, there are $52 - 5 = 47$ other cards that can be paired with it so there are $4 \cdot 47$ possible hands containing a royal flush.

There are $\binom{52}{6}$ possible hands of 6 cards, so there is a $\frac{4 \cdot 47}{\binom{52}{6}}$ chance of having a royal flush.

We can in fact reduce this fraction further to:
$\frac{4 \cdot 47 \cdot 6! \cdot 46!}{52!} = \frac{4 \cdot 47 \cdot 6!}{47 \cdot 48 \cdot 49 \cdot 50 \cdot 51 \cdot 52} = \frac{6!}{12 \cdot 49 \cdot 50 \cdot 51 \cdot 52} = \frac{1}{1 \cdot 49 \cdot 10 \cdot 17 \cdot 13}$ but you should probably leave this unsimplified.

5. How many ways can you arrange the letters in the word MISSISSIPPI?

   $\frac{11!}{1!4!4!2!}$

6. In a 5×6 grid, how many paths are there from the upper left corner to the bottom right corner that only involve moves down or to the right?

   $\frac{11!}{5!6!}$

7. Let $X = \{1, ..., n\}$. Let $f : X \to X$ be a bijection. Define $f^{(k)} = \underbrace{f \circ f \circ ... \circ f}_{k \text{ times}}$.

   (a) Fix $x \in X$. Show that there exists $k \in \mathbb{Z}^+$ such that $f^{(k)}(x) = x$.

   Consider the sequence $x = f^{(0)}(x), f(x), f^{(2)}(x), ..., f^{(n)}(x)$. There are only n possible elements of the sequence, but the sequence has length $n+1$.

   Then there must some k, ℓ with $k < \ell$ such that $f^{(k)}(x) = f^{(\ell)}(x)$.

   As f is a bijection, so is $f^{(k)}$. Moreover, $(f^{(k)})^{-1} = (f^{-1})^{(k)}$. Then $x = (f^{-1})^{(k)}(f^{(k)}(x)) = (f^{-1})^{(k)}(f^{(\ell)}(x)) = f^{(\ell-k)}(x)$.

   As $\ell - k > 0$ we are done.

   (b) Show that there exists $k \in \mathbb{Z}^+$ such that $\forall x \in X, f^{(k)}(x) = x$.

   We will do this problem two different ways: First by induction and then by the pigeonhole principle.

   Way 1:

   I claim by induction on n that such a k exists.

   When $n = 1$, there is only one function $f : X \to X$, namely $f(1) = 1$. Then we can take $k = 1$ and be done.

   Now suppose by induction that for all bijections $g : \{1, ..., n-1\} \to \{1, ..., n-1\}$ there exists a k such that $\forall x \in \{1, ..., n-1\}, g^{(k)}(x) = x$.

   By part (a) there exists some ℓ such that $f^(\ell)(n) = n$.

   Then the restriction of the domain of $f^{(\ell)}$ to $\{1, ..., n-1\}$ is a bijection to $\{1, ..., n-1\}$.

   Then there exists k such that $\forall x \in \{1, ..., n-1\}$, $(f^{(\ell)})^{(k)}(x) = 1$.

   Furthermore, $(f^{(\ell)})^{(k)}(n) = \underbrace{f^\ell \circ f^\ell \circ ... \circ f^\ell}_{k \text{ times}}(n) = n$.

   Finally, $(f^{(\ell)})^{(k)} = f^{(\ell k)}$.

   Then we are done.

Way 2:

For an n-tuple $\vec{x} = (x_1, ..., x_n)$ define $F(x) = (f(x_1), ..., f(x_n))$.

Observe that F has an inverse, $F^{-1}(x) = (f^{-1}(x_1), ..., f^{-1}(x_n))$.

Let $\vec{\vec{v}}_0 = (1, 2, ..., n)$.

Define $\vec{v}_i = F(v_{i-1})$.

$\vec{v}_i \subset \{1, ..., n\}^n$ and $|\{1, ..., n\}^n| = n^n$.

Then for some k, ℓ such that $1 \le k < \ell \le n^n + 1, v_k = v_\ell$.

Then $v_0 = (F^{-1})^{(k)}(v_k) = (F^{-1})^{(k)}(v_\ell) = v_{\ell-k}$.

Then $F^{(\ell-k)}(v_0) = v_0$.

Hence $\forall x \in \{1, ..., n\}, f^{(\ell-k)}(x) = x$.

As $\ell - k > 0$ we are done.