Week 2 Worksheet for Math 61

1. Define the sequence a by $a_0 = 3$ and $a_n = 2 + a_{n-1}$.

   (a) Write out 5 terms of a starting with a_0.

   $3, 5, 7, 9, 11$

   (b) Calculate $\sum_{i=0}^{4} a_i$.

   $3 + 5 + 7 + 9 + 11 = 35$

   (c) Find a formula for a_i.

   $a_i = 3 + 2i$

   (d) Find a formula for $s_n = \sum_{i=0}^{n} a_i$ in terms of n.

   $$\begin{aligned} s_n &= \sum_{i=0}^{n} a_i \\ &= \sum_{i=0}^{n} 3 + 2i \\ &= \sum_{i=0}^{n} 3 + 2\sum_{i=0}^{n} i \\ &= 3(n+1) + 2(\frac{n(n+1)}{2}) \\ &= 3n + 3 + n^2 + n \\ &= n^2 + 4n + 3 \end{aligned}$$

   (e) Is a non-decreasing? Is a increasing?

   Yes. Yes.

   (f) Is s non-decreasing? Is s increasing?

   Yes. Yes.

2. For the following relations R, determine if R is reflexive, symmetric, antisymmetric, transitive, a partial order, and/or an equivalence relation.

   (a) $x, y \in \mathbb{R}$, $xRy \Leftrightarrow xy = 1$.

   - $2 \cdot 2 = 4 \neq 1$ so R is not reflexive.
   - If $xy = 1$ then $yx = 1$ so R is symmetric.
   - $2R\frac{1}{2}$ and $\frac{1}{2}R2$ but $2 \neq \frac{1}{2}$ so R is not antisymmetric.
   - $2R\frac{1}{2}$ and $\frac{1}{2}R2$ but $2 \cdot 2 \neq 1$ so R is not transitive.

- R is not reflexive so it is not a partial order.
- R is not reflexive so it is not an equivalence relation.

(b) $x, y \in \{\text{Rock, Paper, Scissors}\}, xRy \Leftrightarrow x$ beats y.

- Paper does not beat paper so R is not reflexive.
- Paper beats Rock, but Rock does not beat paper so R is not symmetric.
- As we never have the case that A beats B and also B beats A, R is antisymmetric.
- Paper beats Rock and Rock beats Scissors, but Paper does not beat Scissors, so R is not transitive.
- R is not reflexive so it is not a partial order.
- R is not reflexive so it is not an equivalence relation.

(c) $x, y \in \{\text{Rock, Paper, Scissors}\}, xRy \Leftrightarrow x$ beats or ties y.

- Everything ties itself so R is reflexive.
- Paper beats Rock, but Rock does not beat paper so R is not symmetric.
- We never have the case that A beats B and also B beats or ties A. Then aRb only if a ties b which only occurs if $a = b$. So R is antisymmetric.
- Paper beats Rock and Rock beats Scissors, but Paper does not beat Scissors, so R is not transitive.
- R is not transitive so it is not a partial order.
- R is not symmetric so it is not an equivalence relation.

(d) $x, y \in \mathbb{Z}, xRy \Leftrightarrow x - y$ is even.

- Recall that n is even if $n = 2k$ for some integer k.
- $x - x = 0 = 2 \cdot 0$ which is even, so R is reflexive.
- If $x - y = 2k$ then $y - x = -(x - y) = 2(-k)$ so R is symmetric.
- $2R0$ and $0R2$ but $0 \neq 2$ so R is not antisymmetric.
- If $x - y = 2k$ and $y - z = 2\ell$ then $(x - z) = (x - y) + (y - z) = 2(k + \ell)$ so R is transitive.
- As R is not antisymmetric R is not a partial order.
- As R is reflexive, symmetric, and transitive, R is an equivalence relation.

(e) $x, y \in \mathbb{Z}, xRy \Leftrightarrow x \mid y$. (Recall that $x \mid y$ iff $\exists z \in Z$ such that $y = xz$)

- $x = x\ cdot 1$ so $x|x$. Hence R is reflexive.
- $2|4$ but $4 \not| 2$ so R is not symmetric.
- $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$ so R is not antisymmetric.
- If $x = yn$ and $y = zm$ then $x = z(nm)$ so R is transitive.
- As R is not antisymmetric, R is not a partial order.
- As R is not symmetric, R is not an equivalence relation.

(f) $x, y \in \mathbb{Z}^+, xRy \Leftrightarrow x \mid y$. (Recall that $x \mid y$ iff $\exists z \in Z$ such that $y = xz$)

- $x = x\ cdot 1$ so $x|x$. Hence R is reflexive.
- $2|4$ but $4 \not| 2$ so R is not symmetric.
- If $x = yn$ and $y = xm$ then $x = x^2nm$. As $x > 0$ we have $1 = nm$ so $n = m = \pm 1$.
  However as x and y are both greater than 0 we have $n = m = 1$ so $x = y$. Hence R is antisymmetric.
- If $x = yn$ and $y = zm$ then $x = z(nm)$ so R is transitive.
- As R is reflexive, antisymmetric, and transitive, R is a partial order.
- As R is not symmetric, R is not an equivalence relation.

3. For each of the following relations. verify that it is an equivalence relation and give one member from each of its equivalence classes. (There many be infinitely many equivalence classes.)

   (a) $x, y \in \mathbb{Z}, xRy \Leftrightarrow 3|x - y$.
   (Recall that $3|x-y$ means "3 divides $x-y$". That is, $\exists z \in \mathbb{Z}$ such that $x-y = 3z$).

   (b) $x, y \in \mathbb{Z}, xRy \Leftrightarrow x \mid y$. (Recall that $x \mid y$ iff $\exists z \in Z$ such that $y = xz$)

   - The proof that R is an equivalence relation is essentially the same as $2(d)$ so I will not repeat it.
   - $\{0, 1, 2\}$ is a set containing one element from each equivalence class. For any integer n, n dividing by 3 gives a remainder of $0, 1$, or 2 so any integer is equivalent to one of these under R. (If you want to see a proof of this fact about remainders, read "What is a Proof?" on my website.

   (c) $x, y \in \mathbb{R}, xRy \Leftrightarrow \cos(x) = \cos(y)$.

   - $\cos(x) = \cos(x)$ so R is reflexive.
   - If $\cos(x) = \cos(y)$ then $\cos(y) = \cos(x)$ so R is symmetric.
   - If $\cos(x) = \cos(y)$ and $\cos(y) = \cos(z)$ then $\cos(x) = \cos(z)$ because $=$ is transitive. Hence R is transitive.
   - $\cos(x)$ is injective when we restrict $x \in [0, \pi)$ and surjective onto $[-1, 1]$, the range of $\cos(x)$ on all of $\mathbb{R}$.
     Then exactly one member of each equivalence class is in $[0, \pi)$

   (d) $x, y \in \mathbb{R}, xRy \Leftrightarrow x - y \in \mathbb{Z}$.

   - $x - x = 0 \in \mathbb{Z}$ so R is reflexive.
   - If $x - y \in \mathbb{Z}$ then $y - x = -(x - y) \in \mathbb{Z}$ so R is symmetric.
   - If $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$ then $x - z = (x - y) + (y - z) \in \mathbb{Z}$ because the sum of two integers is an integer.
   - Exactly one member of each equivalence class is in $[0, 1)$.

   (e) $(a, b), (x, y) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}), (a, b)R(x, y) \Leftrightarrow ay = bx$.

   - $ab = ba$ so R is reflexive.
   - If $ay = bx$ then $xb = ya$ so R is symmetric.

- If $(a,b)R(x,y)$ and $(x,y)R(m,n)$ then $ay = bx$ and $xn = ym$. Then $x(b-n) = y(m-a)$
  Hence $ayxn = bxym$ so $an(xy) = bm(xy)$ so $an = bm$ so $(a,b)R(m,n)$.
- $\{(\mathrm{num}(q), \mathrm{den}(q)) : q \in \mathbb{Q}\}$ for some choice of numerator and denominator of q has exactly one element from each equivalence class.

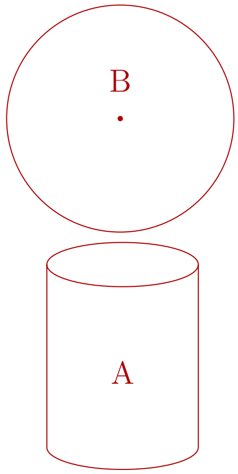
4. Describe the following sets geometrically (or draw a picture)

   (a) $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

   - One way to visualize this set is to tile 3-dimensional space with $1 \times 1 \times 1$ cubes and then look at only the corners of the cubes.

   (b) $S^1 \times (0,1)$ where $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ i.e. a circle.

   You can visualize this as a collection of circles of radii in $(0,1)$ not inclusive or you can view this as a cylinder with side length 1 and radius 1.



   (c) $([0,1] \times \{0\}) \cup (\{0\} \times [0,1]) \cup ([0,1] \times \{1\}) \cup (\{1\} \times [0,1])$

   - This is the border of a square.