Problem [1] background information

The initial value problem for an nth order, linear, homogeneous, constant coefficient ODE consists of determining a function $y(t)$ such that the nth order differential equation

$$a_n\, y^n + a_{n-1}\, y^{n-1} + a_{n-2}\, y^{n-2} + \cdots + a_3\, y''' + a_2\, y'' + a_1\, y' + a_0\, y = 0 \tag{1}$$

is satisfied and $y(t)$ and it's first $n-1$ derivatives satisfy the initial data

$$\begin{pmatrix} y(0) \\ y'(0) \\ y''(0) \\ \vdots \\ y^{n-1}(0) \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{pmatrix} \tag{2}$$

where $a_0,\, a_1, \ldots a_n$ and $b_0,\, b_1, \ldots b_{n-1}$ are real constants

One associates with the ODE (1) a polynomial of degree n whose coefficients are obtained from the ODE:

$$a_n\, r^n + a_{n-1}\, r^{n-1} + a_{n-2}\, r^{n-2} + \cdots + a_3\, r^3 + a_2\, r^2 + a_1\, r + a_0 \;= 0 \tag{3}$$

If the n roots, $r_0, \ldots r_{n-1}$ of this polynomial are real and distinct, then a solution to the initial value problem is a function consisting of a sum of the n exponentials:

$$y(t) = c_0 e^{r_0 t} + c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_{n-1}\, e^{r_{n-1} t} \tag{4}$$

where the coefficients $c_0,\, c_1, \ldots c_{n-1}$ are determined by solving the following linear system of equations

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ r_0 & r_1 & \ldots & r_{n-1} \\ r_0^2 & r_1^2 & \ldots & r_{n-1}^2 \\ \vdots & \vdots & \ldots & \vdots \\ r_0^{n-1} & r_1^{n-1} & \ldots & r_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{pmatrix} \tag{5}$$

that is obtained by requiring that the solution of the form (4) satisfy the specified initial conditions (2).

Thus, a procedure for constructing the solution to the initial value problem consists of first finding the roots of the polynomial (3) and then, if the roots are real and distinct, constructing a solution of the form (4) where the coefficients are determined by solving the linear system of equations (5).