# Euler's method for the approximate solution of ODE's

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### Euler's method for a single ordinary differential equation.

If one is given the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t) \qquad y(t_0) = y_0 \tag{1}$$

and asked to approximate the solution over a time interval $[t_0, T]$, then a common way to do this is to create a *difference* equation whose solution $y_k$ approximates $y(t_k)$ at some finite set of times $t_k \subseteq [t_0, T]$ for $k = 0 \ldots N$ (see Figure 1).

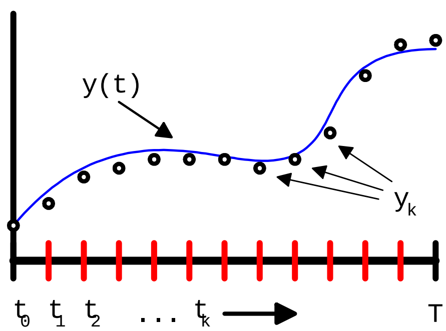

Figure 1

One of the simplest numerical methods based on this idea is Euler's method. In Euler's method the difference equation

$$y_{k+1} = y_k + \Delta t\, f(y_k, t_k) \qquad y_0 = y(0) \tag{2}$$

is used to create the approximate solution values. Here $\Delta t = \dfrac{T - t_0}{N}$, where N is the number of timesteps to be taken.

Euler's method can be "derived" or "understood" in a number of different ways. One way to understand Euler's method is to recognize it as a repeated application of a first order Taylor series approximation to the solution. Specifically, if we know $y(t)$ at time $t = t_k$ and we want to approximate it at time $t = t_{k+1}$, then we can use a Taylors series expansion with $\Delta t = t_{k+1} - t_k$;

$$y(t_{k+1}) = y(t_k) + \Delta t \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_{t_k} + \frac{\Delta t^2}{2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right)_{t_k} + \cdots$$

Assuming that the timestep $\Delta t$ is small enough, we neglect the higher order terms and use

$$y(t_{k+1}) \approx y(t_k) + \Delta t \, \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_{t_k} \tag{3}$$

To use (3) requires knowing the derivative of the function at $t_k$, $\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_{t_k}$. However, $y(t)$ satisfies the differential equation (1) so we have an explicit formula for its derivative at time $t_k$ in terms of its value at time $t_k$, namely, $\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_{t_k} = f(y(t_k), t_k)$. Substituting this relation into (3) gives us

$$y(t_{k+1}) \approx y(t_k) + \Delta t \, f(\, y(t_k), \, t_k). \tag{4}$$

Hence we see that Euler's method (2) is the repeated application of a first order Taylor series approximation (4). Now, one might worry that the errors in the repeated use of (4) accumulate in a catastrophic manner. In courses in numerical analysis, one proves that the errors tend to zero as the number of timesteps (over a fixed time interval) increases. In fact, under suitable assumptions on the differential equation, the difference between the computed and exact solution obeys the relation

$$\max_k |\, y(t_k) - y_k \,| \leq C\Delta t \qquad k = 0...N$$

for some constant $C$. Thus, as the number of steps $N \rightarrow \infty$, $\Delta t \rightarrow 0$, and so the approximate solution values converge to the true solution values.

## Euler's method for systems.

Systems of ordinary differential equations have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2, \ldots, y_n.t) \\ f_2(y_1, y_2, \ldots, y_n, t) \\ \vdots \\ f_n(y_1, y_2, \ldots, y_n, t) \end{pmatrix} \qquad \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \\ \vdots \\ y_n(t_0) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}_0 \tag{5}$$

or in vector notation

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} = \vec{F}(\vec{y}, t) \qquad \vec{y}(t_0) = \vec{y}_0$$

The solution one seeks is a vector valued function $\vec{y}(t)$ whose time derivative satisfies (5). Euler's method extends directly to systems of differential equations. Formally, Euler's method is obtained by just letting $y_k \rightarrow \vec{y}_k$, $y_{k+1} \rightarrow \vec{y}_{k+1}$, and $f \rightarrow \vec{F}$ in (2);

$$\vec{y}_{k+1} = \vec{y}_k + \Delta t \; \vec{F}(\vec{y}_k, t_k) \qquad \vec{y}_0 = \vec{y}(t_0) \tag{6}$$

Here $\Delta t = \dfrac{T - t_0}{N}$, where N is the number of timesteps to be taken.

Just as in the single ODE case, one can "understand" Euler's method for systems as being the repeated application of a Taylor series approximation. Specifically, given $\vec{y}(t_k)$ one can think of approximating $\vec{y}(t_{k+1})$ using a Taylor Series approximation for each component,

$$
\begin{aligned}
y_1(t_{k+1}) = & \quad y_1(t_k) + \Delta t \left(\frac{\mathrm{d}y_1}{\mathrm{d}t}\right)_{t_k} + \frac{\Delta t^2}{2}\left(\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2}\right)_{t_k} + \_ \cdots \\
y_2(t_{k+1}) = & \quad y_2(t_k) + \Delta t \left(\frac{\mathrm{d}y_2}{\mathrm{d}t}\right)_{t_k} + \frac{\Delta t^2}{2}\left(\frac{\mathrm{d}^2 y_2}{\mathrm{d}t^2}\right)_{t_k} + \_ \cdots \\
\vdots \qquad & \qquad\qquad\qquad \vdots \\
y_n(t_{k+1}) = & \quad y_n(t_k) + \Delta t \left(\frac{\mathrm{d}y_n}{\mathrm{d}t}\right)_{t_k} + \frac{\Delta t^2}{2}\left(\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2}\right)_{t_k} + \_ \cdots
\end{aligned}
\tag{7}
$$

If we assume a sufficiently small timestep, then we use just the first two terms of each approximation

$$
\begin{aligned}
y_1(t_{k+1}) \approx & \quad y_1(t_k) + \Delta t \left(\frac{\mathrm{d}y_1}{\mathrm{d}t}\right)_{t_{k1}} \\
y_2(t_{k+1}) \approx & \quad y_2(t_k) + \Delta t \left(\frac{\mathrm{d}y_2}{\mathrm{d}t}\right)_{t_k} \\
\vdots \qquad & \qquad\qquad \vdots \\
y_n(t_{k+1}) \approx & \quad y_n(t_k) + \Delta t \left(\frac{\mathrm{d}y_n}{\mathrm{d}t}\right)_{t_k}
\end{aligned}
$$

or in vector notation,

$$
\vec{y}(t_{k+1}) \approx \vec{y}(t_k) + \Delta t \left(\frac{\mathrm{d}\vec{y}}{\mathrm{d}t}\right)_{t_k} \tag{8}
$$

As before, we need the derivatives of each component to evaluate this approximation. The differential equations provide us with a means of obtaining the derivatives, namely, $\left(\frac{\mathrm{d}\vec{y}}{\mathrm{d}t}\right)_{t_k} = \vec{F}(\vec{y}(t_k), t_k)$. Thus, using this expressions in (8) we obtain

$$
\vec{y}(t_{k+1}) \approx \vec{y}(t_k) + \Delta t \, \vec{F}(\vec{y}(t_k), t_k)
$$

So Euler's method for a system is just repeated application of a Taylor series approximation to each component. As with single ordinary differential equations, one finds that under suitable assumptions on the differential equation the numerical solutions $\vec{y}_k$ converge to $\vec{y}(t_k)$, in fact,

$$
\max_k || \, \vec{y}(t_k) - \vec{y}_k \, || \leq C\Delta t \qquad k = 0 \ldots N
$$

for some constant $C$. Thus, as the number of steps $N \to \infty$, $\Delta t \to 0$, and the approximate solution vector converges to the true solution vector at each $t_k$.