SUPPLEMENT TO T-CONVEXITY AND TAME EXTENSIONS BY LOU VAN DEN DRIES AND ADAM H. LEWENBERG

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Introduction

If in doubt, all notations and assumptions are with respect to [4]. Please share with me any comments, additional exercises or questions which you think are relevant or interesting.

Given an ordered set $S = (S, \leq)$ and $A \subseteq S$, we let

$$\operatorname{conv}(A) := \{x \in S : a \leq x \leq b \text{ for some } a, b \in A\}.$$

1. Day 1

Exercise 1.1. Let $K = (K; 0, 1, +, \cdot, <)$ be an ordered field, and suppose $V \subseteq K$ is a convex subring. Then the field of fractions of V inside K is K, and V is a valuation ring of K. In particular, all convex subrings considered in [4] are valuation rings.

Proof. Suppose $x \notin V$, and assume without loss of generality that $x > 0$. Then $x > 1$ and so $0 < x^{-1} < 1$ which implies that $x^{-1} \in V$. This shows that K is the field of fractions of V, and it also shows that V is a valuation ring of K. □

Exercise 1.2. Give an example of T, $\mathcal{R} \models T$, and $V \subseteq R$ such that V is a convex subring of $\mathcal{R}$, but it is not a T-convex subring of $\mathcal{R}$.

Proof. Let $\mathcal{R}$ be an elementary extension of $\bar{\mathbb{R}} = (\mathbb{R}; 0, 1, +, -, \cdot, <)$ with an element $t \in \mathcal{R}$ such that $t > \mathbb{R}$. Let $T = \operatorname{Th}(\mathcal{R}, t)$ in the language of ordered rings augmented with an additional constant symbol for t. Now define $V := \operatorname{conv}(\mathbb{Z}) \subseteq \mathcal{R}$. Then V is a convex subring of $(\mathcal{R}, t)$, but it is not T-convex since $t \notin V$, and the constant function $x \mapsto t : R \to R$ is 0-definable and continuous in $(\mathcal{R}, t)$.

Here is another example. Let $T = T_{\exp} = \operatorname{Th}(\mathbb{R}; 0, 1, +, -, \cdot, <, \exp)$. Let $\mathcal{R}$ be an elementary extension of $(\mathbb{R}; 0, 1, +, -, \cdot, <, \exp)$ with an element $t \in R$ such that $t > \mathbb{R}$. Define $V = \operatorname{conv}(\mathbb{Z}[t])$. Then $t \in V$, but $\exp(t) \notin V$. □

Exercise 1.3. Show that T is polynomially bounded iff for every $\mathcal{R} \models T$ and every convex subring $V \subseteq \mathcal{R}$, if $\mathcal{P} \subseteq V$, then V is T-convex.

Exercise 1.4. Let $(\mathcal{R}, V) \models T_{\text{convex}}$. Show that V is not a finite union of intervals and points. This shows that the best we can hope for is that T_{convex} is weakly o-minimal (which it is by [4, Proposition 3.16]).

Proof. By definition of T_{convex}, there is $t \in R$ such that $t > V$, so V has an upper bound. However V does not have a least upper bound since $t > V$ implies $t - 1 > V$ since V is a subring of R. Every nonempty finite union of intervals and points which is bounded above has a least upper bound. Thus V is not a finite union of intervals and points. □

Exercise 1.5. Suppose $\mathcal{R}' \preccurlyeq_{\text{tame}} \mathcal{R} \models T$. Given $r \in \mathcal{R} \cap \operatorname{conv}(R')$, show there is a unique $r' \in \mathcal{R}'$ such that $|r - r'| < \varepsilon$ for all positive $\varepsilon \in \mathcal{R}'$.

Proof. Assume that $r \in \mathcal{R} \cap \operatorname{conv}(R')$ and take $r_0, r_1 \in \mathcal{R}'$ such that $|r - r_i| < \varepsilon$ for all positive $\varepsilon \in \mathcal{R}'$ and $i = 0, 1$. By the triangle inequality, $|r_0 - r_1| < \varepsilon$ for all positive $\varepsilon \in \mathcal{R}'$. However, $r_0 - r_1 \in \mathcal{R}'$. Thus $|r_0 - r_1| = 0$ and so $r_0 = r_1$. □

Exercise 1.6. Give an example of $\mathcal{R}' \preccurlyeq \mathcal{R} \models T$ such that $\mathcal{R}'$ is not tame in $\mathcal{R}$.

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Proof. Let $T = \mathrm{RCF}$, let $\mathcal{R}' = (\mathbb{Q}^{\mathrm{rc}}; 0, 1, +, -, \cdot, <)$ and let $\mathcal{R} = (\mathbb{R}; 0, 1, +, -, \cdot, <)$. Then $\mathcal{R}' \preccurlyeq \mathcal{R}$, but $\mathcal{R}'$ is not tame in $\mathcal{R}$. Indeed, $\pi \in R \setminus R'$ does not have a best approximation in $\mathbb{Q}^{\mathrm{rc}}$. □

Exercise 1.7. If T has QE and is universally axiomatizable, then T_{convex} and T_{tame} have QE by [4, Theorems 3.10 and 5.9]. Show that in this situation neither T_{convex} nor T_{tame} are universally axiomatizable.

Exercise 1.8. Let $\gamma : \mathcal{R} \to \mathcal{R}$ be a continuous R-definable function. Define the 2-cells $C_1 := (-\infty, \gamma), C_2 := (\gamma, +\infty)$. Note that $\mathcal{R}^2 = C_1 \cup \Gamma(\gamma) \cup C_2$ is a cell decomposition of $\mathcal{R}^2$. Suppose $f : \mathcal{R}^2 \to \mathcal{R}$ is an R-definable function such that its restriction to each of C_1, $\Gamma(\gamma)$, and C_2 separately is continuous, and for each $i \in \{1, 2\}$, the function $f|C_i$ is either independent of the second variable, strictly increasing in the second variable, or strictly decreasing in the second variable. Then f is continuous on all of $\mathcal{R}^2$. This is the key argument in [4, Lemma 1.5].

Exercise 1.9. Let T be an arbitrary complete theory. T has definable skolem functions iff for every $\boldsymbol{M} \models T$ and for every $A \subseteq M$, if A is definably closed, then $A \preccurlyeq \boldsymbol{M}$, i.e., A is the underlying set of an elementary substructure of $\boldsymbol{M}$.

2. Day 2

Exercise 2.1. Prove Definable Choice [4, (1.7)].

Exercise 2.2. Show that Curve Selection [4, (1.8)] holds for $a \in R^m$ iff $a \in \mathrm{cl}(A \setminus \{a\})$.

Exercise 2.3. Show how to prove [4, Lemma 1.10] in the other cases for C.

Exercise 2.4. Show how [4, Lemma 1.10] can fail when the hypothesis "bounded" is removed.

Exercise 2.5. [3, pg. 96] If $f : X \to R^n$ is an injective continuous definable map on a closed bounded set $X \subseteq R^m$, then f is a homeomorphism from X onto $f(X)$.

Exercise 2.6. [3, pg. 96] Let $f : X \to R^n$ be a definable continuous map on a closed bounded set $X \subseteq R^m$ and let $Y = f(X)$. Then we have:

1. A definable set $S \subseteq Y$ is closed iff $f^{-1}(S)$ is closed;
2. A definable set $g : Y \to R^p$ is continuous iff $g \circ f : X \to R^p$ is continuous.

Exercise 2.7. Show that $(\mathbb{R}, <)$ is tame in every linearly ordered extension.

3. Day 3

Exercise 3.1. Assume $\mathcal{R}' \models T$. Show that there is a proper elementary extension $\mathcal{R} \succcurlyeq \mathcal{R}'$ such that $\mathcal{R}'$ is maximal among elementary substructures of $\mathcal{R}$ contained in $V := \mathrm{conv}(\mathcal{R}')$. In such a situation we have $\mathcal{R}' \preccurlyeq_{\mathrm{tame}} \mathcal{R}$.

Exercise 3.2. Assume T satisfies the assumptions of [4]. Let $\mathcal{R} \models T$, and let $f : C \to R$ be an R-definable function. Show that the following are equivalent:

1. f is continuous;
2. given every $\mathcal{R} \preccurlyeq_{\mathrm{tame}} \mathcal{R}'$, if $x \in C_{\mathcal{R}'}$ is $\mathcal{R}$-bounded, then $f_{\mathcal{R}'}(x)$ is $\mathcal{R}$-bounded and
$$\mathrm{st}_{\mathcal{R}}(f_{\mathcal{R}'}(x)) = f(\mathrm{st}_{\mathcal{R}}(x)).$$
3. there is $\mathcal{R} \preccurlyeq_{\mathrm{tame}} \mathcal{R}'$, such that $R \subsetneq R'$ and if $x \in C_{\mathcal{R}'}$ is $\mathcal{R}$-bounded, then $f_{\mathcal{R}'}(x)$ is $\mathcal{R}$-bounded and
$$\mathrm{st}_{\mathcal{R}}(f_{\mathcal{R}'}(x)) = f(\mathrm{st}_{\mathcal{R}}(x)).$$

In some sense this is a converse to [4, Lemma 1.13].

Exercise 3.3. Let $\mathcal{R} \models T$. Show that the underlying valued field of $\mathcal{R}$ with valuation ring V is henselian.

Exercise 3.4. (This is taken from [2, §2]) Let $f : \mathbb{R} \to \mathbb{R}$ be *any* function (not necessarily definable in some o-minimal expansion of $\mathbb{R}$). Show that

1. $Y := \{y \in \mathbb{R}^{>0} : \lim_{x \to +\infty}(f(xy) - f(x)) \in \mathbb{R}\}$ is a multiplicative subgroup of $(\mathbb{R}^{\times}, \cdot, 1)$.
2. $Z := \{z \in \mathbb{R} : \exists y \in \mathbb{R}^{>0}, \lim_{x \to +\infty}(f(xy) - f(x)) = z\}$ is an additive subgroup of $(\mathbb{R}, +, 0)$.

(3) The function $L(f)(y) = \lim_{x\to+\infty}(f(xy) - f(x)) : Y \to Z$ is a surjective homomorphism. The notation is to suggest that $L(f)$ is somehow the "logarithmic part" of f, but this should not be taken too seriously, as we could easily have $Y = \{1\}$ and $Z = \{0\}$.
(4) The sets Y, Z and the function $L(f)$ are $\emptyset$-definable in $(\mathbb{R}; 0, 1, +, -, \cdot, <, f)$ (again, no o-minimality is assumed).
(5) If $(\mathbb{R}; 0, 1, +, -, \cdot, <, f)$ is o-minimal and $\lim_{x\to+\infty}(f(2x) - f(x)) \in \mathbb{R}^\times$, then $Y = \mathbb{R}^{>0}$ and $L(f) = \log_a$ for some $a \in \mathbb{R}^{>0}$. (Recall that every subgroup of $(\mathbb{R}, +)$ is either cyclic or dense, and every endomorphism of $(\mathbb{R}, +)$ is either nowhere continuous or linear.) Conclude that log is definable, hence so is e^x.

Exercise 3.5. (Also taken from [2, §2]) Let $f : \mathbb{R} \to \mathbb{R}$ be ultimately nonzero. Show that:

(1) The sets

$$Y := \{y \in \mathbb{R}^{>0} : \lim_{x\to+\infty} f(xy)/f(x) \in \mathbb{R}\}$$
$$Z := \{z \in \mathbb{R}^{>0} : \exists y \in \mathbb{R}^{>0}, \lim_{x\to+\infty} f(xy)/f(x) = z\}$$

are multiplicative subgroups of $(\mathbb{R}^\times, \cdot, 1)$.
(2) The function $P(f)(y) = \lim_{x\to+\infty}(f(xy)/f(x)) : Y \to Z$ is a surjective homomorphism. The notation is to suggest that $P(f)$ is somehow the "power part" of f, but again, this should not be taken too seriously. We tend to write just Pf as convenient.
(3) The sets Y, Z and the function Pf are $\emptyset$-definable in $(\mathbb{R}; 0, 1, +, -, \cdot, <, f)$.
(4) If $(\mathbb{R}; 0, 1, +, -, \cdot, <, f)$ is o-minimal and $2 \in Y$, then $Y = \mathbb{R}^{>0}$ and Pf is a power function.
(5) If there exists $r \in \mathbb{R}$ such that $\lim_{x\to+\infty} f(x)/x^r \in \mathbb{R}^\times$, then $Y = \mathbb{R}^{>0}$ and $Pf = x^r$.
(6) Calculate Y, Z and Pf directly for the functions $\log x, x^r \log x, (\log x)^{\log x}$.

4. Hardy fields

This section mostly follows [2, §3], but also borrows some things from [1, §9.1].

Let $\mathcal{G}$ be the ring of germs at $+\infty$ of real-valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$, $a \in \mathbb{R}$; the domain may vary and the ring operations are defined as usual. Given some property P of real-valued functions as above (for instance, P could be "continuous", or "differentiable"), we say that a germ $g \in \mathcal{G}$ has property P if it is the germ of some function with property P. For differentiable $g \in \mathcal{G}$, we let $g' \in \mathcal{G}$ denote the germ of the derivative of some differentiable representative of g.

Definition 4.1. A **Hardy field** is a subring K of $\mathcal{G}$ such that K is a field, all $g \in K$ are differentiable, and $g' \in K$ for all $g \in K$.

The following shows the relationship between Hardy fields and o-minimal expansions of the ordered field of real numbers $\bar{\mathbb{R}} = (\mathbb{R}; 0, 1, +, \cdot, -, <)$:

Proposition 4.2. If $\mathfrak{R}$ is an expansion of $\bar{\mathbb{R}}$, then the following are equivalent:

(1) $\mathfrak{R}$ is o-minimal.
(2) The germs of definable unary functions form a Hardy field.
(3) Every unary definable function is either ultimately zero or ultimately nonzero.

Proof. (1) $\Rightarrow$ (2) follows from the C^1-Monotonicity Theorem.

(2) $\Rightarrow$ (3) is immediate from the field structure and definition of a germ.

(3) $\Rightarrow$ (1) Let $A \subseteq \mathbb{R}$ be definable. We must show that A is a finite union of points and open intervals. It suffices to show that $\mathrm{bd}(A)$ of A is finite, which (by Bolzano-Weierstrass) means showing that $\mathrm{bd}(A)$ is bounded and discrete. Let f be the (definable) characteristic function of A. Then f is ultimately identically 1 or identically 0, so there is $b \in \mathbb{R}$ such that (b, ∞) is either contained in A or disjoint from A. Similarly, there is some $a \in \mathbb{R}$ such that $(-\infty, a)$ is either contained in A or disjoint from A. Hence $\mathrm{bd}(A)$ is bounded. Fix $x_0 \in \mathrm{bd}(A)$. By arguing as before with $\{1/(a - x_0) : a \in A\}$, there is $\epsilon > 0$ such that $(x_0, x_0 + \epsilon)$ is either contained in A or disjoint from A, and similarly for $(x_0 - \epsilon, x_0)$. Thus $\mathrm{bd}(A)$ is discrete. ☐

We now fix $\mathfrak{R}$ an o-minimal expansion of $\bar{\mathbb{R}}$ with field of exponents K and associated Hardy field $\mathcal{H}$.

Exercise 4.3. (1) If $f \in \mathcal{H}$, then $\lim_{x\to+\infty} f(x) \in \mathbb{R} \cup \{\pm\infty\}$.

(2) $\{(f,g) \in \mathcal{H}^\times \times \mathcal{H}^\times : \lim_{x\to+\infty} f(x)/g(x) \in \mathcal{R}^\times\}$ is an equivalence relation. Denote the natural quotient map by v. The image $v(\mathcal{H}^\times)$ is an ordered group by setting $v(f) + v(g) = v(fg)$ and $v(f) > 0$ iff $\lim_{x\to+\infty} f(x) = 0$. Denote the resulting absolute value on $v(\mathcal{H}^\times)$ by $|\cdot|$. (Be careful: This does *not* mean that $|v(\cdot)| = v(|\cdot|)$.)
(3) If $f, g \in \mathcal{H}^\times$ and $|f| \geq |g|$, then $v(f) \leq v(g)$ (note the reversal or the order!). The converse fails.
(4) If $f, g \in \mathcal{H}^\times$ and $f \neq -g$, then $v(f+g) \geq \min(v(f), v(g))$, with equality if $v(f) \neq v(g)$.
(5) If $f \in \mathcal{H}^\times$ and $r \in \mathbb{R}^\times$, then $v(rf) = v(f) = v(|f|)$.
(6) If $f \in \mathcal{H}^\times$ and $v(f) \neq 0$, then exactly one of $f, 1/f, -f$ or $-1/f$ is infinitely increasing (i.e., $\lim_{x\to+\infty} = +\infty$), and $|v(f)| = |v(-f)| = |v(1/f)| = |v(-1/f)|$.
(7) If $f \in \mathcal{H} \setminus \mathbb{R}$, then

$$v(f'/f) = v((1/f)'/(1/f)) = v((-f)'/(-f)) = v((-1/f)'/(-1/f)).$$

Lemma 4.4 (HC)**.** Let $a, b \in \mathcal{H}^\times$ be such that $0 < |v(a)| \leq |v(b)|$. Then $v(a'/a) \geq v(b'/b)$.

Proof. Without altering $|v(a)|$, $|v(b)|$, $v(a'/a)$ or $v(b'/b)$, we replace a by $\pm a$ or $\pm 1/a$, and b by $\pm b$ or $\pm 1/b$, to reduce to the case that a and b are infinitely increasing

If $va = vb$, then $va' = vb'$ by l'Hôpital's Rule, so $v(a'/a) = v(a') - v(a) = v(b') - v(b) = v(b'/b)$.

Suppose $v(b) < v(a)$. Then b/a is infinitely increasing, so all of a, a', b' and $(b/a)'$ are positive, yielding $b'/b > a'/a > 0$ by the quotient rule. Hence $v(a'/a) \geq v(b'/b)$. □

For $f, g \in \mathcal{G}$, we write $f \sim g$ if g is ultimately nonzero and $\lim_{x\to+\infty} f(x)/g(x) = 1$. If $f, g \in \mathcal{H}^\times$, then $vf = vg$ iff $f \sim cg$ for some $c \in \mathbb{R}^\times$.

Lemma 4.5 (AC3)**.** Let $a, b \in \mathcal{H}^\times$ be such that $v(a) \geq 0$ and $v(b) \neq 0$. Then $v(a') > v(b'/b)$.

Proof. We may assume that $v(a) = 0$ by replacing a with $a+1$ if necessary. By l'Hôpital,

$$\frac{ab}{b} \sim \frac{ab' + a'b}{b'},$$

and so

$$a = \frac{ab}{b} \sim \frac{ab' + a'b}{b'} = a + a'\frac{b}{b'}.$$

Then $1 \sim 1 + (a'/a)b(b'/b)$, yielding $v(a'/a) > v(b'/b)$. Finish by observing that $v(a'/a) = v(a') - v(a)$, and $v(a) = 0$ by assumption. □

Lemma 4.6 (Partial asymptotic integration)**.** If $f \in \mathcal{H}^\times$ and $v(f) < v(1/x)$, then there exists $g \in \mathcal{H}^\times$ such that $g' \sim f$.

Proof. Note that $(xf)' \neq 0$.

Suppose $v(f) \geq v((xf)')$, that is, $v(f/(xf)') \geq 0$. By (HC), $v((f/(xf)')') > v(1/x)$, that is, $v(x(f/(xf)')') > 0$. Put $g_1 = xf^2/(xf)'$. Then $g_1'/f = 1 + x(f/(xf)')'$, so $g_1' \sim f$.

Suppose $v(f) < v((xf)')$, equivalently, $f'/f \sim -1/x$. Then $g_1' \neq 0$. Put $g_2 = fg_1/g_1'$ and then $g_2' \sim f$ follows, using

$$\frac{g_2'}{f} - 1 = \frac{f'/f}{g_1'/g_1} - \frac{g_1''/g_1'}{g_1'/g_1}.$$ □

Note:

(1) $\mathcal{H}$ is "closed under composition": if $f, g \in \mathcal{H}^\times$ and f is infinitely increasing, then $g \circ f$ lies in $\mathcal{H}^\times$ as well. The sign of $v(g \circ f)$ is the same as that of $v(g)$.
(2) Not all Hardy fields are closed under composition: $\mathbb{R}(x, e^x)$ is Hardy field that does not contain $e^x \circ x^2$.
(3) $\mathcal{H}$ is "closed under compositional inverse": if $f \in \mathcal{H}$ and $v(f) < 0$, then f^{-1} of the ultimately-defined compositional inverse of f also belongs to $\mathcal{H}$.

Proposition 4.7 (Growth dichotomy)**.** Either $\mathfrak{R}$ is exponential or $v(\mathcal{H}^\times) = K \cdot v(x)$.

Proof. There are two cases to consider:

Case 1: There exists $f \in \mathcal{H}^\times$ such that $v(f'/f) \neq v(1/x)$ and $v(f) \neq 0$.

We show that $\mathfrak{R}$ is exponential. By replacing f with $-f$ if necessary, we arrange $f > 0$. By further replacing f with $1/f$ if necessary, we arrange f to be infinitely increasing. By replacing f with f^{-1} if necessary, we suppose that $v(f'/f) < v(1/x)$. By partial asymptotic integration, there is $g \in \mathcal{H}^\times$ such that $g' \sim f'/f$. Put $h = g \circ f^{-1}$; then $h' \sim 1/x$. By MVT, we have

$$h \circ (2x) - h = \frac{x}{\xi} \cdot \xi h' \circ \xi$$

for some $\xi \in \mathcal{H}$ such that $x < \xi < 2x$. (Why?) Note that $v(x/\xi) = 0 = v(\xi h' \circ \xi)$, the latter by substituting ξ into xh'. Thus

$$v(h \circ (2x) - h) = v(x/\xi) + v(\xi h \circ \xi) = v(x/\xi) + v(xh') = 0.$$

By exercise on $L(f)$, $\mathfrak{R}$ is exponential.

Case 2: For all $f \in \mathcal{H}^\times$, if $v(f'/f) \neq v(1/x)$, then $v(f) = 0$.

We show that $v(\mathcal{H}^\times)$, if $v(f'/f) \neq v(1/x)$, then $v(f) = 0$.

We show first that $Pf = x^r$ for some $r \in K$. This is immediate if $v(f) = 0$ (for then $Pf = 1 = x^0$), so assume that $v(f) \neq 0$. Put $g = (f \circ (2x))/f \in \mathcal{H}^\times$. Observe that

$$x\frac{g'}{g} = 2x\frac{f' \circ (2x)}{f \circ (2x)} - \frac{xf'}{f}.$$

Since $v(f) \neq 0$, we have $v(xf'/f) = 0$, and so $v(g'/g) > v(1/x)$. By the case assumption, $v(g) = 0$, that is, $f \circ (2x) \sim cf$ for some $c \in \mathbb{R}^\times$. Now apply Exercise on $P(f)$.

To finish the proof, we now let $f \in \mathcal{H}^\times$ and show that $v(f) = v(Pf)$. Since $P((Pf)/f) = 1 \restriction \mathbb{R}^{>0}$ (why?), we are reduced to showing that if $Pf = 1 \restriction \mathbb{R}^{>0}$, then $v(f) = 0$. By case assumption, it suffices to show that $v(xf'/f) \neq 0$. By MVT,

$$\frac{f \circ (2x)}{f} - 1 = \frac{xf' \circ \xi}{f} = \frac{x}{\xi} \cdot \frac{\xi f' \circ \xi}{f \circ \xi} \cdot \frac{f \circ \xi}{f}$$

where $\xi \in \mathcal{H}^\times$ and $x < \xi < 2x$. It suffices now to show that $v((\xi f' \circ \xi)/f \circ \xi) \neq 0$ (for then $v(xf'/f) \neq 0$ as well). Since $Pf(2) = 1$, we have

$$0 = v(\frac{f \circ (2x)}{f} - 1) = v(x/\xi) + v(\frac{\xi f' \circ \xi}{f \circ \xi}) + v(\frac{f \circ \xi}{f}).$$

Since $v(\xi) = v(x)$, it suffices now to show that $f \circ \xi \sim f$, which follows easily from monotonicity – either $f \leq f \circ \xi \leq f \circ (2x)$ or $f \geq f \circ \xi \geq f \circ (2x)$ – and that $Pf(2) = 1$ (that is, $f \circ (2x) \sim f$). □

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