SUPPLEMENT TO DENSE PAIRS OF O-MINIMAL STRUCTURES BY LOU VAN DEN DRIES

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Introduction

If in doubt, all notations and assumptions are with respect to [7]. Generally speaking, exercises are things which I know are true, or I strongly suspect are true, and questions are things which I do not know the answer to. Please share with me any comments, additional exercises or questions which you think are relevant or interesting.

1. Week 1

Exercise 1.1. Let T be the theory of dense linear orders without endpoints (DLO) in the language $L = \{<\}$, and let T' be a theory expanding T in some language $L' \supseteq L$ with the property that every model $\boldsymbol{M} \models T'$, every L'-definable $X \subseteq M^n$ is L-definable. Show that T' does not have definable Skolem functions.

Exercise 1.2 (cf. pg. 61 [7])**.** Show there is an L^2-theory T^2 whose models are exactly the elementary pairs.

Exercise 1.3 (cf. pg. 62 [7])**.** Show there is an L^2-theory T^d whose models are exactly the dense (elementary) pairs.

Exercise 1.4 (cf. pg. 62-63 [7])**.** Let $V \subseteq \mathbb{R}$ be a finite dimension $\mathbb{R}^a$-vector space. Show that V is $\mathbb{R}^a$-small in the dense pair $(\mathbb{R}, \mathbb{R}^a)$, where T is the theory of ordered divisible abelian groups.

Solution. Let $\{\beta_1, \ldots, \beta_n\} \subseteq \mathbb{R}$ be such that $V = \operatorname{span}_{\mathbb{R}^a}\{\beta_1, \ldots, \beta_n\}$. Then the $\mathcal{B}$-definable function $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i\beta_i : \mathbb{R}^n \to \mathbb{R}$ has the property that $V = f((\mathbb{R}^a)^n)$. Thus V is $\mathbb{R}^a$-small. □

Exercise 1.5 (cf. pg. 63 [7])**.** Give examples of o-minimal structures with the property that not every open interval has the same (infinite) cardinality.

Solution. The dense linear order $\mathbb{R} + \mathbb{Q}$ has this property. The ordered divisible abelian group $\mathbb{R} \oplus \mathbb{Q}$ with the lexicographic order also has this property. □

Question 1.6. Suppose T is a complete o-minimal theory that extends the theory of ordered abelian groups, and T has the property that for every $\mathcal{A} \models T$, and every $a < b \in A$, $|A| = |(a,b)|$. Then does T interpret RCF? Can one define in $\mathcal{A} = (A; +, <, \ldots) \models T$ a function $\times : A^2 \to A$ which makes $(A; +, \times, <)$ a model of RCF?

Exercise 1.7 (cf. Theorem 3(3) [7])**.** Suppose $f : A^n \to A$ is definable in $(\mathcal{B}, \mathcal{A})$ and $f_1, \ldots, f_k : A^n \to A$ are definable in $\mathcal{A}$ are such that for each $x \in A^n$ there is $i \in \{1, \ldots, k\}$ with $f(x) = f_i(x)$. Furthermore assume that for each $i \in \{1, \ldots, k\}$ the set $D_i := \{x \in A^n : f(x) = f_i(x)\} \subseteq A^n$ is definable in $\mathcal{A}$. Then f is definable in $\mathcal{A}$.

Solution. The following defines the graph of f in $\mathcal{A}$:

$$f(x) = y \ :\Longleftrightarrow \ \bigvee_{i=1}^{k} \big(D_i(x) \wedge f_i(x) = y\big).$$ □

Exercise 1.8 (cf. Theorem 3(3) [7])**.** Give an example of a dense pair $(\mathcal{B}, \mathcal{A})$ and a function $f : A^n \to A$ definable in $(\mathcal{B}, \mathcal{A})$ which is not definable in $\mathcal{A}$.

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Question 1.9 (cf. pg. 63 [7])**.** Given a dense pair $(\mathcal{B}, \mathcal{A})$, can you always find $(\mathcal{B}', \mathcal{A}') \equiv (\mathcal{B}, \mathcal{A})$ such that $|A| < |B|$?

Exercise 1.10 (cf. Theorem 4 in [7])**.** Given a dense pair $(\mathcal{B}, \mathcal{A})$, a set $X \subseteq \mathcal{B}$ definable in $(\mathcal{B}, \mathcal{A})$, and $b_0 < b_1$ from $B \cup \{\pm\infty\}$, we say that that X is **co-$\mathcal{A}$-small on** (b_1, b_2) if $(b_0, b_1) \setminus X$ is $\mathcal{A}$-small.

Show that if $S \subseteq B$ is definable in $(\mathcal{B}, \mathcal{A})$, then there is a partition

$$-\infty = b_0 < b_1 < \cdots < b_k < b_{k+1} = +\infty$$

of B such that for each $i = 0, \ldots, k$ either

(1) $S \cap (b_i, b_{i+1})$, or
(2) $(b_i, b_{i+1}) \subseteq S$, or
(3) $S \cap (b_i, b_{i+1})$ and $(b_i, b_{i+1}) \setminus S$ are dense in (b_i, b_{i+1}), and $S \cap (b_i, b_{i+1})$ is $\mathcal{A}$-small, or
(4) $S \cap (b_i, b_{i+1})$ and $(b_i, b_{i+1}) \setminus S$ are dense in (b_i, b_{i+1}), and $S \cap (b_i, b_{i+1})$ is co-$\mathcal{A}$-small on (b_i, b_{i+1}).

Prove this as a consequence of Theorems 1-5 in the introduction, not from scratch.

Exercise 1.11 (cf. pg. 64 [7])**.** Here T is an arbitrary complete theory with infinite models.

(1) Show that T has a universal axiomatization (UA) iff for every $\mathbf{M} \models T$, and every $A \subseteq M$, $\langle A \rangle \models T$.
(2) Show that if T has QE and UA, then for every $\mathbf{M} \models T$, and every $A \subseteq M$,

$$\langle A \rangle = \mathrm{dcl}(A) = \mathrm{acl}(A) \preccurlyeq M.$$

(3) Does the converse of (2) hold?

Exercise 1.12. Suppose $\mathbf{M}$ is a one-sorted structure, $A \subseteq M$, and there exists an A-definable total ordering on M. Then $\mathrm{dcl}(A) = \mathrm{acl}(A)$ inside $\mathbf{M}$. In particular, if there is a $\emptyset$-definable total ordering on $\mathbf{M}$ (as is the case for every structure considered in [7]), then for every $A \subseteq M$, $\mathrm{dcl}(A) = \mathrm{acl}(A)$.

Exercise 1.13 (cf. pg. 64 [7])**.** Let $\mathcal{B} \models T$, and $S \subseteq B^n$ be definable and nonempty. Show that for every $a \in B^n$, $d(a, S)$ exists, and the function $x \mapsto d(x, S) : B^n \to B^{\geq 0}$ is definable.

Exercise 1.14 (cf. pg. 64 [7])**.** Given $\mathcal{B} \models T$, say that $\mathcal{B}$ is **downward-pairable** if there is $\mathcal{A} \preccurlyeq \mathcal{B}$ such that $(\mathcal{B}, \mathcal{A}) \models T^d$. Now assume T has QE and UA, let $\mathbb{M} \models T$ be highly saturated, and define $\mathcal{P} := \langle 0 \rangle \preccurlyeq \mathbb{M}$.

(1) Show that $\mathcal{P}$ is not downward-pairable.
(2) Give an example of $\mathcal{B} \preccurlyeq \mathbb{M}$ such that $\mathcal{B} \neq \mathcal{P}$, and $\mathcal{B}$ is not downward-pairable. How "large" can you make $\mathcal{B}$?
(3) Given an $\emptyset$-indiscernible sequence $(a_i)_{i \in I}$ in $\mathbb{M}$, is $\mathcal{P}\langle (a_i)_{i \in I} \rangle$ downward-pairable or non-downward-pairable? Are there examples of both? What if $(a_i)_{i \in I}$ is a Morley sequence?

Solution. (1) Suppose $\mathcal{A} \preccurlyeq \mathcal{P}$. Since $0 \in \mathcal{A}$, $\mathcal{P} = \langle 0 \rangle \subseteq \mathcal{A}$ and so $\mathcal{A} = \mathcal{P}$. Thus $\mathcal{P}$ is not downward-pairable.

(2) Take $a \in \mathbb{M}$ such that $a > \mathcal{P}$. Define $\mathcal{B} := \mathcal{P}\langle a \rangle$. Note that $\mathrm{rk}(\mathcal{B}|\mathcal{P}) = 1$. Thus if $\mathcal{A} \preccurlyeq \mathcal{B}$, either $\mathcal{A} = \mathcal{P}$ or $\mathcal{A} = \mathcal{B}$. We claim that $(\mathcal{B}, \mathcal{P})$ is not a dense pair. Indeed, inside $\mathcal{B}$, $(a, a+1) > \mathcal{P}$, and thus $(a, a+1) \cap \mathcal{P} = \emptyset$. ☐

Exercise 1.15. Given $\mathcal{A} \models T$, way that $\mathcal{A}$ is **upward-pairable** if there is $\mathcal{B} \succcurlyeq \mathcal{A}$ such that $(\mathcal{B}, \mathcal{A}) \models T^d$.

(1) Give an example of T and $\mathcal{A} \models T$ such that $\mathcal{A}$ is not upward-pairable.
(2) How general of an example is there?

Solution. (1) Let T be an theory which has a model whose underlying set is $\mathbb{R}$, and let $\mathcal{A}$ be that model. We claim the $\mathcal{A}$ is not upward-pairable. Assume towards a contradiction there is $\mathcal{B} \succ \mathcal{A}$ is such that $(\mathcal{B}, \mathcal{A})$ is a dense pair. Then the underlying ordered group of $\mathcal{B}$ is not archimedean. If there is an element $b \in B$ such that $b > A$, then $(b, b+1) \cap A = \emptyset$. If there is an element $b \in B$ such that $0 < b$, and $b < a$ for every $a \in A$ such that $a > 0$, then $(b/2, b) \cap = \emptyset$. ☐

2. Pregeometries

Some of the material in this section is adapted from [6, Appendix C].

Definition 2.1. Given a set X and a function $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$, we say that cl is a **closure operator (on X)** if it satisfies:

- (Extension) for every $A \subseteq X$, $A \subseteq \mathrm{cl}(A)$,
- (Increasing) for every $A \subseteq B \subseteq X$, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$, and
- (Idempotent) for every $A \subseteq X$, $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$.

We say that a closure operator cl on X has **finite character** if it satisfies:

- (Finite Character) for every $A \subseteq X$, and $a \in \mathrm{cl}(A)$, there is some finite $A_0 \subseteq A$ such that $a \in \mathrm{cl}(A_0)$.

We say that a closure operator cl on X has **exchange** if it satisfies:

- (Exchange) for every $A \subseteq X$, and $a, b \in X$, if $a \notin \mathrm{cl}(A)$ and $b \notin \mathrm{cl}(A)$, then $a \in \mathrm{cl}(A \cup \{b\})$ iff $b \in \mathrm{cl}(A \cup \{a\})$.

We say the pair (X, cl) is a **pregeometry** if cl is a closure operator on X that satisfies both finite character and exchange.

By convention, if we refer to a pair (X, cl) as a *closure operator*, we mean that X is a set, and $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator on X.

Exercise 2.2. Give an example of a set X and a closure operator cl on X which does not satisfy finite character.

Solution. Let $X = \mathbb{R}$ and cl be the operation of topological closure with respect to the usual euclidean or order topology on $\mathbb{R}$. Then $0 \in \mathrm{cl}(A)$ where $A = \{1/n : n \geq 1\}$, but $0 \notin \mathrm{cl}(A_0)$ for every finite $A_0 \subseteq A$. □

Most closure operators that arise in model theory have finite character:

Exercise 2.3. Let $\boldsymbol{M}$ be a first-order structure. Note that for every $A \subseteq M$,

$$A \subseteq \langle A \rangle \subseteq \mathrm{dcl}(A) \subseteq \mathrm{acl}(A) \subseteq M.$$

(1) Show that the following operations are all closure operators of finite character on M:
  (a) $A \mapsto A$,
  (b) $A \mapsto \langle A \rangle$,
  (c) $A \mapsto \mathrm{dcl}(A)$,
  (d) $A \mapsto \mathrm{acl}(A)$,
  (e) $A \mapsto M$.
(2) Give examples of specific $\boldsymbol{M}$ where various combinations of the above closure operators are the same or different.

Definition 2.4. Let (X, cl) be a closure operator. A subset A of X is called

(1) **independent** if for every $a \in A$, $a \notin \mathrm{cl}(A \setminus \{a\})$,
(2) a **generating set** if $X = \mathrm{cl}(A)$, and
(3) a **basis** if A is an independent generating set.

Exercise 2.5. Suppose (X, cl) is a closure operator. Then $\emptyset$ is independent and X is a generating set.

Exercise 2.6. Suppose (X, cl) is a closure operator with finite character. Suppose $(A_i)_{i \in I}$ is an increasing union of independent subsets of X. Then $\bigcup_{i \in I} A_i$ is an independent set. Give an example of a closure operator without finite character where this fails (bonus points if the example has Exchange).

Lemma 2.7 (Basis Existence Lemma)**.** Let (X, cl) be a pregeometry, and suppose E is a generating set. Suppose B_0 is an independent subset of E. Then B_0 can be extended to a basis $B \subseteq E$. In particular, X has a basis.

Proof. Suppose B is an independent set. We claim that if $x \in X \setminus \mathrm{cl}(B)$, then $B \cup \{x\}$ is also independent. Clearly $x \notin \mathrm{cl}((B \cup \{x\}) \setminus \{x\})$ by assumption. Suppose $b \in B$. Then $b \notin \mathrm{cl}(B \setminus \{b\})$ since B is independent, so $b \notin \mathrm{cl}((B \setminus \{b\}) \cup \{x\})$ by Exchange (otherwise $x \in \mathrm{cl}(B)$, a contradiction).

By Finite Character, and an exercise above, we may apply Zorn's Lemma to obtain a maximal independent subset B of E. The above claim shows that $E \subseteq \mathrm{cl}(B)$, and thus $X = \mathrm{cl}(B)$. □

Exercise 2.8. Give an example of a closure operator where the Basis Existence Lemma fails. Can you find an example that has Finite Character? An example that has Exchange?

Lemma 2.9 (Invariance of Basis Cardinality)**.** Let (X, cl) be a pregeometry, and let B_0 and B_1 be two bases of X. Then $|B_0| = |B_1|$.

Proof. Let A be independent and B a generating subset of X. It suffices to show that $|A| \leq |B|$. Assume first that A is infinite. By Basis Existence we extend A to a basis A'. Using Finite Character, choose for every $b \in B$ a finite subset A_b of A' with $b \in \mathrm{cl}(A_b)$. Since $\bigcup_{b \in B} A_b \subseteq A'$ is a generating set (its cl contains B, hence its $\mathrm{cl}^2 = \mathrm{cl}$ is all of X), we necessarily have $\bigcup_{b \in B} A_b = A'$. This implies that B is infinite (if it is finite, then the union is finite), and $|A| \leq |A'| \leq |B|$:

$$A' = \bigcup_{b \in B} A_b \leq_{\mathrm{card}} \underbrace{\bigcup_{b \in B} A_b \times \{b\}}_{\subseteq A \times B} \leq_{\mathrm{card}} \omega \times B =_{\mathrm{card}} B.$$

Now assume that A is finite. Then $|A| \leq |B|$ follows from the following claim: Given any $a \in A \setminus B$, there is some $b \in B \setminus A$ such that $A' = \{b\} \cup A \setminus \{a\}$ is independent (A' has the same cardinality as A, applying the claim enough times yields an independent subset of B with the same cardinality of A). To prove the claim, suppose $a \notin A \setminus B$. Since $a \in \mathrm{cl}(B)$, then $B \not\subseteq \mathrm{cl}(A \setminus \{a\})$. Choose $b \in B \setminus \mathrm{cl}(A \setminus \{a\})$. It follows from Exchange that $A' = \{b\} \cup A \setminus \{a\}$ is independent. ☐

Definition 2.10. Given a pregeometry (X, cl), the **dimension** $\dim(X)$ of (X, cl) is the cardinality of a basis. Note: In the setting of [7], the dimension of the relevant pregeometries is referred to as *rank*, as *dimension* already has a different meaning for o-minimal structures.

Exercise 2.11. Give an example of a closure operator (X, cl) which has two bases B_0 and B_1 such that $|B_0| \neq |B_1|$.

3. Week 2

Recall the *Monotonicity Theorem* [8, Chapter 3, (1.2)]:

Theorem 3.1. In an o-minimal structure $\mathcal{R}$, if $f : (a, b) \to \mathcal{R}$ is an A-definable function on the interval (a, b), then there are $a = a_0 < a_1 < \cdots < a_k = b$ such that on each subinterval (a_i, a_{i+1}) the function f is continuous, and (depending on i) either constant, strictly increasing, or strictly decreasing. Furthermore, the a_i can be taken to be A-definable.

Exercise 3.2 (cf. pg. 64 [7])**.** Given an o-minimal structure $\mathcal{A}$, show that the operation $C \mapsto \mathrm{dcl}(C)$ satisfies Exchange, and thus (A, dcl) is a pregeometry.

Solution. This argument is basically [4, Remark after Theorem 4.2]. Suppose $a, b \in A$ and $B \subseteq A$ are such that $a, b \notin \mathrm{dcl}(B)$, and suppose $a \in \mathrm{dcl}(Bb)$. Then there is a B-definable function $f : A \to A$ such that $f(b) = a$. Let $-\infty = a_0 < a_1 < \cdots < a_k = +\infty$ be as in the Monotonicity Theorem above applied to f; in particular, arrange that the a_i's are B-definable. Then $b \neq a_i$ for each i since $b \notin \mathrm{dcl}(B)$, thus b is in (a_i, a_{i+1}) for one of the i's. If $f \restriction (a_i, a_{i+1})$ is constant, then $f(b)$ is B-definable, so $f \restriction (a_i, a_{i+1})$ is strictly monotone. Let g be the B-definable partial inverse to f on (a_i, a_{i+1}). Then $g(a) = b$ which witnesses $b \in \mathrm{dcl}(B, a)$. ☐

Exercise 3.3 (cf. pg. 65 [7])**.** The point of this exercise is to figure out how sharp the "$\exists^\infty\exists^\infty$" hypothesis in the Peterzil-Starchenko Theorem is. Let $\mathcal{A}$ be an o-minimal expansion of an ordered vector space over an ordered field F.

(1) Given $\lambda_0, \ldots, \lambda_n$, show that there is $g : A^{p+1} \to A$ definable in $\mathcal{A}$ and tuples $a_0, \ldots, a_n \in A^p$ such that $g(a_i, x) = \lambda_i x$ for all x and $i = 0, \ldots, n$.
(2) Given $x_0, \ldots, x_n$, can you find a function $g : A^{p+1} \to A$ definable in $\mathcal{A}$ such that for infinitely many scalars $\lambda \in F$ there is a tuple $a_\lambda \in A^p$ such that $g(a_\lambda, x_i) = \lambda x_i$ for $i = 0, \ldots, n$. Prove or give counterexample.

Exercise 3.4 (cf. Corollary 1.3 [7])**.** Given an arbitrary complete o-minimal theory, generalize the definitions of *elementary pair*, *dense pair* and *$\mathcal{A}$-small* to this setting in the obvious way.

(1) Can you find a complete o-minimal theory and a dense pair $(\mathcal{B}, \mathcal{A})$ in this theory such that Corollary 1.3 of [7] fails? I.e., B is $\mathcal{A}$-small? What about an example with T expanding the theory of ordered divisible abelian groups?
(2) Suppose T extends RCF, and $(\mathcal{B}, \mathcal{A}) \models T^2$ with $B \neq A$. Show that B is not $\mathcal{A}$-small.

Exercise 3.5 (cf. Corollary 1.3 [7])**.** Assume T has QE and UA, working in $\mathbb{M}$ highly saturated, define $\mathcal{P} := \langle 0 \rangle$, and define $\operatorname{rk}(\mathcal{B}) := \operatorname{rk}(\mathcal{B}|\mathcal{P})$.

(1) Suppose for cardinals $\kappa \leq \lambda$ there is a model $\mathcal{B}$ with $\operatorname{rk}(\mathcal{B}) = \lambda$. Find $\mathcal{A} \preccurlyeq \mathcal{B}$ such that $\operatorname{rk}(A) = \kappa$.
(2) Suppose for cardinals $\kappa \leq \lambda$ there is a model $\mathcal{B}$ with $\operatorname{rk}(\mathcal{B}) = \lambda$. Find $\mathcal{A} \preccurlyeq \mathcal{B}$ such that $\operatorname{rk}(B|A) = \kappa$.
(3) Give an example of a theory T' with QE, UA, and at least one constant symbol in the language, and a model $\mathcal{B} \models T'$ such that $\mathcal{B} \neq \langle \emptyset \rangle$, however there is no $\mathcal{A} \prec \mathcal{B}$ such that there exists $b \in B \setminus A$ with $\mathcal{B} = \mathcal{A}\langle b \rangle$.

Solution. In the situation of (1) and (2), we may take a basis H of $\mathcal{B}$ over $\mathcal{A}$, i.e., $H \subseteq B$ is $\mathcal{A}$-independent and $\mathcal{B} = \mathcal{A}\langle H \rangle$. Let $H_0 \subseteq H$ be such that $|H_0| = \kappa$. For (1), $\mathcal{A} := \langle H_0 \rangle$ works. For (2), $\mathcal{A} := \langle H \setminus H_0 \rangle$ works. □

4. Some independence relations

In this section we discuss some independence relations which are implicitly being used in [7]. For concreteness, we restrict our discussion to the setting of [7] and assume that T has QE and UA, although everything can be done in a more general pregeometry. *Let $\mathcal{D}$ be an arbitrary model of T and let A, B, C range over subsets of D.*

The first independence relation is rather coarse, but is automatically forced in arbitrary extensions of pairs:

Definition 4.1. We define $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$ which is read as B **is** (acl**-**) **disjoint from** C **over** A if:

$$B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C \;:\Longleftrightarrow\; \operatorname{acl}(BA) \cap \operatorname{acl}(CA) = \operatorname{acl}(A) \iff \langle BA \rangle \cap \langle CA \rangle = \langle A \rangle$$

Remark 4.2. If A, B, C are the underlying sets of models $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that $A \subseteq B$ and $A \subseteq C$, then

$$B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C \iff B \cap C = A.$$

In particular, if $(\mathcal{B}, \mathcal{A}) \subseteq (\mathcal{D}, \mathcal{C}) \models T^2$ is an extension of elementary pairs, then automatically $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$.

The next exercise is essentially [1, Prop 1.5]:

Exercise 4.3. The relation $\mathop{\smile\!\!\!\!\!\!|}^{0}$ satisfies the following axioms:

(1) (Full existence) Assume $\mathcal{D}$ is sufficiently saturated relative to A, B, and C. Then there is $B' \equiv_A B$ such that $B' \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$.
(2) (Invariance) Suppose σ is an automorphism of $\mathcal{D}$. If $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$, then $\sigma B \mathop{\smile\!\!\!\!\!\!|}^{0}_{\sigma A} \sigma C$.
(3) (Monotonicity) If $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$, $B' \subseteq B$ and $C' \subseteq C$, then $B' \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C'$.
(4) (Normality) If $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$, then $BA \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$.
(5) (Finite character) If $B_0 \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$ for every finite $B_0 \subseteq B$, then $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$.
(6) (Transitivity) Suppose $E \subseteq A \subseteq B$. If $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$ and $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{E} C$, then $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{E} C$.
(7) (Extension) Assume $\mathcal{D}$ is sufficiently saturated relative to A, B, C and $\hat{C} \supseteq C$. If $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$, then there is $B' \equiv_{CA} B$ such that $B' \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} \hat{C}$.
(8) (Local character) For every B, there is a cardinal $\kappa(B)$ such that for every C there is $A \subseteq C$ of cardinality $|A| < \kappa(B)$ such that $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$.
(9) (Anti-reflexivity) For every $b \in D$, $b \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} b$ implies $b \in \operatorname{dcl}(A)$.
(10) (Symmetry) $B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C$ iff $C \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} B$.

Solution. (Invariance) Suppose σ is an automorphism of $\mathcal{D}$. Note that

$$B \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} C \Rightarrow \langle BA \rangle \cap \langle CA \rangle = \langle A \rangle \Rightarrow \sigma(\langle BA \rangle \cap \langle CA \rangle) = \sigma(\langle A \rangle) \Rightarrow \sigma(\langle BA \rangle) \cap \sigma(\langle CA \rangle) = \langle \sigma(A) \rangle$$

$$\Rightarrow \langle\sigma(B)\sigma(A)\rangle \cap \langle\sigma(C)\sigma(A)\rangle = \langle\sigma(A)\rangle \Rightarrow \sigma(B) \mathop{\perp\!\!\!\!\!\smile}^{0}_{\sigma(A)} \sigma(C).$$

(Monotonicity) Suppose $B \mathop{\perp\!\!\!\!\!\smile}^{0}_{A} C$, $B' \subseteq B$ and $C' \subseteq C$. Then

$$\langle A\rangle \subseteq \langle B'A\rangle \cap \langle C'A\rangle \subseteq \langle BA\rangle \cap \langle CA\rangle = \langle A\rangle$$

so $\langle B'A\rangle \cap \langle C'A\rangle = \langle A\rangle$. We conclude that $B' \mathop{\perp\!\!\!\!\!\smile}^{0}_{A} C'$.

(Normality) Suppose $B \mathop{\perp\!\!\!\!\!\smile}^{0}_{A} C$. Then $\langle A\rangle = \langle BA\rangle \cap \langle CA\rangle = \langle (BA)A\rangle \cap \langle CA\rangle$, and so $BA \mathop{\perp\!\!\!\!\!\smile}^{0}_{A} C$.

(Finite Character) Assume that $B \not\mathop{\perp\!\!\!\!\!\smile}^{0}_{A} C$. Then there is $b \in \langle BA\rangle \cap \langle CA\rangle \setminus \langle A\rangle$. Let $B_0 \subseteq B$ be finite such that $b \in \langle B_0 A\rangle$. Then $b \in \langle B_0 A\rangle \cap \langle CA\rangle \setminus \langle A\rangle$. Thus $B_0 \not\mathop{\perp\!\!\!\!\!\smile}^{0}_{A} C$.

(Transitivity) By assumption, $B \mathop{\perp\!\!\!\!\!\smile}^{0}_{A} C$ which means $\langle B\rangle \cap \langle CA\rangle = \langle A\rangle$, and $A \mathop{\perp\!\!\!\!\!\smile}^{0}_{E} C$ which means $\langle A\rangle \cap \langle CE\rangle = \langle E\rangle$. Note that

$$\langle B\rangle \cap \langle CE\rangle = (\langle B\rangle \cap \langle CA\rangle) \cap \langle CE\rangle = \langle A\rangle \cap \langle CE\rangle = \langle E\rangle.$$

(Local character) Given sets B and C, construct $A_n \subseteq C$ and $D_n \subseteq D$ as follows: $A_0 = D_0 = \emptyset$. $D_{n+1} := \langle BA_n\rangle \cap \langle C\rangle$. For every $d \in D_{n+1}$, let $\bar{c}_d \in C$ be a finite tuple such that $d \in \langle \bar{c}_d\rangle$. Let A_{n+1} be the union over all tuples $\bar{c}_d$. Set $A := \bigcup_n A_n$. It is easy to see that $A \subseteq C$ and $|A| \leq |T| + |B|$. Furthermore, if $d \in \langle BA\rangle \cap \langle CA\rangle$, then $d \in \langle BA_n\rangle \cap \langle CA\rangle \subseteq D_{n+1}$ for some n, and so $d \in \langle A_{n+1}\rangle \subseteq \langle A\rangle$.

(Anti-reflexivity) Suppose $b \mathop{\perp\!\!\!\!\!\smile}^{0}_{A} b$. Then $\langle bA\rangle \cap \langle bA\rangle = \langle bA\rangle = \langle A\rangle$. Thus $b \in \langle A\rangle$.

(Symmetry) This is immediate from the definition. □

Exercise 4.4. Show that $\mathop{\perp\!\!\!\!\!\smile}^{0}$ satisfies the following, or find a counterexample:

$$\text{(Base monotonicity) Suppose } E \subseteq A \subseteq C. \text{ If } B \mathop{\perp\!\!\!\!\!\smile}^{0}_{E} C, \text{ then } B \mathop{\perp\!\!\!\!\!\smile}^{0}_{A} C.$$

The next independence relation $\mathop{\smile\!\!\!\!\!\vert}$ is in general finer than $\mathop{\perp\!\!\!\!\!\smile}^{0}$. It plays an important role in distinguishing which extensions of pairs are admissible, and serves as an obstruction to full model completeness.

Definition 4.5. We define $B \mathop{\smile\!\!\!\!\!\vert}_{A} C$ which is read as B **is free from** C **over** A if:

$$B \mathop{\smile\!\!\!\!\!\vert}_{A} C \;:\Longleftrightarrow\; \text{for every finite } Y \subseteq B,\ \mathrm{rk}(\langle YA\rangle|\langle A\rangle) = \mathrm{rk}(\langle YAC\rangle|\langle AC\rangle)$$

Exercise 4.6. If $B \mathop{\smile\!\!\!\!\!\vert}_{A} C$, then $B \mathop{\perp\!\!\!\!\!\smile}^{0}_{A} C$.

Solution. Suppose $B \mathop{\smile\!\!\!\!\!\vert}_{A} C$ and assume towards a contradiction that $B \not\mathop{\perp\!\!\!\!\!\smile}^{0}_{A} C$. Take $b \in \langle BA\rangle \cap \langle CA\rangle \setminus \langle A\rangle$. Then $b \in \langle BA\rangle \setminus \langle A\rangle$ and by Exchange there is a finite $B_0 \subseteq B$ such that B_0 is independent over A and $b \in \langle B_0 A\rangle \setminus \langle A\rangle$. Say $|B_0| = n$. Then by Exchange there are $b_2, \ldots, b_n \in \langle B_0 A\rangle$ such that $bb_2 \cdots b_n$ is independent over A and $\langle bb_2 \cdots b_n A\rangle = \langle B_0 A\rangle$. Thus

$$n = \mathrm{rk}(\langle B_0 A\rangle|\langle A\rangle) = \mathrm{rk}(\langle bb_2 \cdots b_n A\rangle|\langle A\rangle)$$

However, since $b \in \langle AC\rangle$, $bb_2 \cdots b_n$ is not independent over AC. Thus

$$n > \mathrm{rk}(\langle bb_2 \cdots b_n AC\rangle|\langle AC\rangle) = \mathrm{rk}(\langle B_0 AC\rangle|\langle AC\rangle).$$

We conclude that $B \not\mathop{\smile\!\!\!\!\!\vert}_{A} C$, a contradiction. □

Remark 4.7. If A, B, C are the underlying sets of models $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that $A \subseteq B$ and $A \subseteq C$, then

$$B \mathop{\smile\!\!\!\!\!\vert}_{A} C \iff \text{for every finite } Y \subseteq B,\ \mathrm{rk}(\mathcal{A}\langle Y\rangle|\mathcal{A}) = \mathrm{rk}(\mathcal{C}\langle Y\rangle|\mathcal{C}).$$

Exercise 4.8. The relation $\mathop{\smile\!\!\!\!\!\vert}$ has the following properties:

1. (Invariance) Suppose σ is an automorphism of $\mathcal{D}$. If $B \mathop{\smile\!\!\!\!\!\vert}_{A} C$, then $\sigma B \mathop{\smile\!\!\!\!\!\vert}_{\sigma A} \sigma C$.
2. (Finite character) If $B_0 \mathop{\smile\!\!\!\!\!\vert}_{A} C$ for every finite $B_0 \subseteq B$, then $B \mathop{\smile\!\!\!\!\!\vert}_{A} C$.
3. (Monotonicity) If $B \mathop{\smile\!\!\!\!\!\vert}_{A} C$, $B' \subseteq B$ and $C' \subseteq C$, then $B' \mathop{\smile\!\!\!\!\!\vert}_{A} C'$.
4. (Symmetry) If $B \mathop{\smile\!\!\!\!\!\vert}_{A} C$, then $C \mathop{\smile\!\!\!\!\!\vert}_{A} B$.
5. (Base monotonicity) Suppose $E \subseteq A \subseteq C$. If $B \mathop{\smile\!\!\!\!\!\vert}_{E} C$, then $B \mathop{\smile\!\!\!\!\!\vert}_{A} C$.
6. (Normality) $B \mathop{\smile\!\!\!\!\!\vert}_{A} C$ implies $BA \mathop{\smile\!\!\!\!\!\vert}_{A} C$.

Solution. (Invariance) This is immediate from the definition using the fact that all notions involved are invariant.

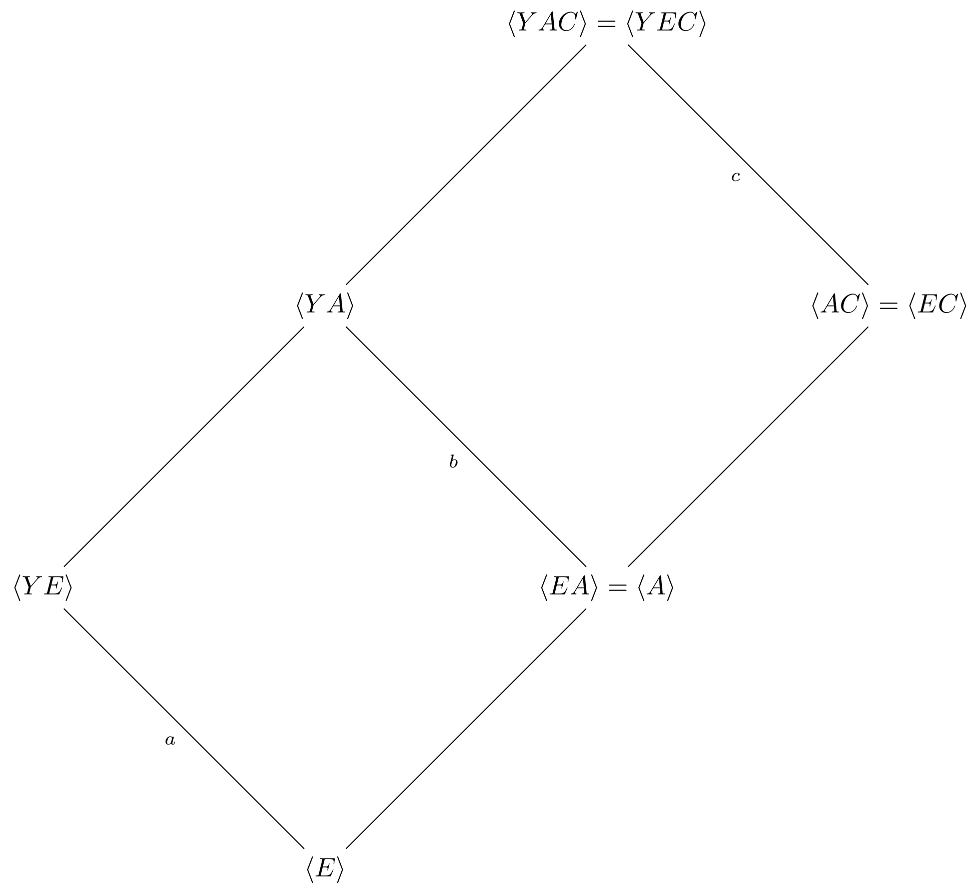
(Finite character) Suppose $B_0 \mathop{\smile\!\!\!\!\!\!\mid}_A C$ for every finite $B_0 \subseteq B$. Let Y be a finite subset of B. Then $Y \mathop{\smile\!\!\!\!\!\!\mid}_A C$ by assumption, and Y is a finite subset of itself, so $\mathrm{rk}(\langle YA\rangle | \langle A\rangle) = \mathrm{rk}(\langle YAC\rangle | \langle AC\rangle)$ by definition of $Y \mathop{\smile\!\!\!\!\!\!\mid}_A C$. We conclude that $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$.

(Monotonicity) Suppose $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$, $B' \subseteq B$ and $C' \subseteq C$. Let Y be an arbitrary finite subset of B'. Since $Y \subseteq B$ and $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$, it follows that $\mathrm{rk}(\langle YA\rangle | \langle A\rangle) = \mathrm{rk}(\langle YAC\rangle | \langle AC\rangle)$. Thus $B' \mathop{\smile\!\!\!\!\!\!\mid}_A C$.

We will now show $B \mathop{\smile\!\!\!\!\!\!\mid}_A C'$. Take a finite subset Y of B such that $n = \mathrm{rk}(\langle YA\rangle | \langle A\rangle)$. Let $y_1, \ldots, y_n \in \langle YA\rangle$ be a basis of $\langle YA\rangle$ over $\langle A\rangle$. Then $y_1, \ldots, y_n$ is a basis of $\langle YAC\rangle$ over $\langle A\rangle$ since $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$. Thus $y_1, \ldots, y_n$ is a basis of $\langle YAC'\rangle$ over $\langle AC'\rangle$. We conclude that $\mathrm{rk}(\langle YAC'\rangle | \langle AC'\rangle) = n$.

(Symmetry) Suppose towards a contradiction that $C \not\mathop{\smile\!\!\!\!\!\!\mid}_A C$. Then there is a finite $Y \subseteq C$ such that $n = \mathrm{rk}(\langle YA\rangle | \langle A\rangle) > \mathrm{rk}(\langle YAB\rangle | \langle AB\rangle) = m$. Then there is some finite $B_0 \subseteq B$ such that $\mathrm{rk}(\langle YAB_0\rangle | \langle AB_0\rangle) = m$. By Monotonicity and $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$ however, $p := \mathrm{rk}(\langle AB_0\rangle | \langle A\rangle) = \mathrm{rk}(\langle YAB_0\rangle | \langle YA\rangle)$. By additivity of rank, $\mathrm{rk}(\langle YAB_0\rangle | \langle A\rangle) = n + p = m + p$, which forces $n = m$, a contradiction.

(Base monotonicity) Let Y be a finite subset of B. Define $b = \mathrm{rk}(\langle YA\rangle | \langle A\rangle)$ and $c = \mathrm{rk}(\langle YAC\rangle | \langle AC\rangle)$. It suffices to show that $b = c$. Define $a = \mathrm{rk}(\langle YE\rangle | \langle E\rangle)$ and note that by the assumption that $B \mathop{\smile\!\!\!\!\!\!\mid}_E C$, we have $a = \mathrm{rk}(\mathrm{rk}\langle YEC\rangle | \langle EC\rangle) = c$. Since $a \geq b \geq c$, we must necessarily have $b = c$.



(Normality) Suppose $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$. Let Y be a finite subset of BA. Partition $Y = Y_0 \cup Y_1$ such that $Y_0 \in B \setminus A$ and $Y_1 \subseteq A$. Then $\mathrm{rk}(\langle YA\rangle | \langle A\rangle) = \mathrm{rk}(\langle Y_0 A\rangle | \langle A\rangle) = \mathrm{rk}(\langle Y_0 AC\rangle | \langle AC\rangle) = \mathrm{rk}(\langle YAC\rangle | \langle AC\rangle)$. □

Exercise 4.9. If $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$, then $\langle B\rangle \mathop{\smile\!\!\!\!\!\!\mid}_A C$.

Solution. Suppose $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$. By Symmetry, $C \mathop{\smile\!\!\!\!\!\!\mid}_A B$. By definition of $\mathop{\smile\!\!\!\!\!\!\mid}$, $C \mathop{\smile\!\!\!\!\!\!\mid}_A \langle B\rangle$. By Symmetry again, $\langle B\rangle \mathop{\smile\!\!\!\!\!\!\mid}_A C$. □

Exercise 4.10 (cf. pg. 66 [7])**.** Suppose A, B, C are the underlying sets of models of $\mathcal{D}$ such that $A \subseteq B$, $A \subseteq C$, and $\mathcal{B} \perp_{\mathcal{A}} \mathcal{C}$. If $Z \subseteq C$, then $\mathcal{B}\langle Z\rangle \perp_{\mathcal{A}\langle Z\rangle} \mathcal{C}$.

Solution. Suppose $\mathcal{B} \perp_{\mathcal{A}} \mathcal{C}$ and $Z \subseteq C$. Then $\mathcal{A} \subseteq \mathcal{A}\langle Z\rangle \subseteq \mathcal{C}$ and by Base Monotonicity, $\mathcal{B} \perp_{\mathcal{A}\langle Z\rangle} \mathcal{C}$. By Normality, $\mathcal{B}\mathcal{A}\langle Z\rangle \perp_{\mathcal{A}\langle Z\rangle} \mathcal{C}$. Finally, by the previous exercise, $\mathcal{B}\langle Z\rangle \perp_{\mathcal{A}\langle Z\rangle} \mathcal{C}$. □

Exercise 4.11 (cf. Case 2 of Theorem 2.5 [7])**.** (Transcendental extension) Suppose $b \in D$ is such that $b \notin \operatorname{dcl}(ABC)$. If $B \perp_A C$, then $Bb \perp_A C$, $B \perp_A Cb$, and $B \perp_{Ab} C$.

Solution. Suppose $B \perp_A C$ and $b \notin \operatorname{dcl}(ABC)$. We will first show $Bb \perp_A C$. Suppose Y is a finite subset of Bb such that $Y = Y_0 b$ where $Y_0 \subseteq B$. Then

$$\operatorname{rk}(\langle YA\rangle | \langle Y\rangle) = 1 + \operatorname{rk}(\langle Y_0 A\rangle | \langle A\rangle) = 1 + \operatorname{rk}(\langle Y_0 AC\rangle | \langle AC\rangle) = \operatorname{rk}(\langle YAC\rangle | \langle AC\rangle).$$

To get $B \perp_A Cb$, we apply Symmetry (twice) and the first result. To get $B \perp_{Ab} C$, suppose $B \perp_A Cb$, then apply Base Monotonicity to get $B \perp_{Ab} Cb$, then Monotonicity to get $B \perp_{Ab} C$. □

5. Week 3

Exercise 5.1. Give an example of T with models $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that $\mathcal{B} \perp^{0}_{\mathcal{A}} \mathcal{C}$ but $\mathcal{B} \not\perp_{\mathcal{A}} \mathcal{C}$.

Solution. This example comes from [5, pg. 183]. The theory is T_{RCF}, let $\mathcal{R} = (\mathbb{R}; 0, 1, +, \cdot, <)$, and let $\mathcal{A}$ be the submodel of $\mathcal{R}$ with underlying set $\mathbb{R}^a$. Let $s_1, s_2, t_1 \in \mathbb{R}$ be three real algebraically independent transcendental numbers. Define $t_2 := -s_1 t_1 - s_2$. Then both $\{t_1, t_2\}$ and $\{s_1, s_2\}$ are algebraically independent over $\mathbb{R}^a$. Define $B = \mathbb{R}^a(s_1, s_2)^{rc}$, $C = \mathbb{R}^a(t_1, t_2)^{rc}$, and $D = \mathbb{R}^a(s_1, s_2, t_1, t_2)^{rc} = \mathbb{R}^a(s_1, s_2, t_1)^{rc}$.

We will first show that $\mathcal{B} \perp^{0}_{\mathcal{A}} \mathcal{C}$. Suppose towards a contradiction that there is some $\xi \in (B \cap C) \setminus A$. Then there are irreducible polynomials $p(X) \in \mathbb{R}^a[s_1, s_2][X]$ and $q(X) \in \mathbb{R}^a[t_1, t_2][X]$ such that $p(\xi) = q(\xi) = 0$. As the rings $\mathbb{R}^a[s_1, s_2]$ and $\mathbb{R}^a[t_1, t_2]$ are UFDs, we may arrange that the polynomials p and q are primitive in these rings (i.e., the gcd of the coefficients of p in $\mathbb{R}^a[s_1, s_2]$ is 1, same for q in $\mathbb{R}^a[t_1, t_2]$). Now define $R_0 = \mathbb{R}^a[s_1, s_2, t_1, t_2] = \mathbb{R}^a[s_1, s_2, t_1] = \mathbb{R}^a[t_1, t_2, s_1]$. Let Q_0 be the field of fractions of R_0. Since p and q have a common root ξ in a finite extension of Q_0, they must have a common factor in $Q_0[X]$. The coefficients of $p(X)$ belong to R_0, and so if $p(X)$ is reducible in $Q_0[X]$ it must be reducible also in $R_0[X]$ by Gauss's Lemma. Since $p(X)$ is irreducible in $\mathbb{R}^a[s_1, s_2][X]$ and t_1 is transcendental over $\mathbb{R}^a[s_1, s_2]$, it follows that $p(X)$ is irreducible in $R_0[X]$. Similarly, $q(X)$ is irreducible in $R[X]$. Since $p(X)$ and $q(X)$ have a common factor, there is $d \in Q_0$ such that $p(X) = dq(X)$. We may write $d = d_1(s_1, s_2, t_2)/d_2(x_1, s_2, t_1)$ where $d_1(X), d_2 \in \mathbb{R}^a[X]$ and do not have a nontrivial common factor. Then we get $d_2(s_1, s_2, t_1)p(X) = d_1(s_1, s_2, t_1)q(X)$. It follows that $d_1(s_1, s_2, t_1)$ divides the coefficients of $p(X)$, but $p(X)$ is primitive in $M_0[s_1, s_2][X]$ and hence also in $R_0[X]$. Thus $d_1 \in \mathbb{R}^a$. Writing $-s_1 t_1 - t_2$ for s_2 in d_2, we obtain a polynomial of t_1, t_2, s_2 with coefficients in $\mathbb{R}^a$ and conclude in the same way that $d_2 \in \mathbb{R}^a$, hence $p(X) = dq(X)$ for some $d \in \mathbb{R}^a$. Now for $n \geq 0$, we'll compare the coefficients of X^n in $p(X)$ and $q(X)$. Denote this coefficient by $p_n(s_1, s_2)$, $q_n(t_1, t_2)$ for $p_n, q_n \in \mathbb{R}^a[X_1, X_2]$. Then $p_n(s_1, s_2) = dq_n(t_1, t_2)$, and so $p_n(s_1, s_2) = dq_n(t_1, -s_1 t_1 - s_2)$. Since s_1, s_2, t_1 are algebraically independent over $\mathbb{R}_0$, we may substitute 0 for s_1 and get $p_n(0, s_2) = dq_n(t_1, -s_2)$. It then follows that the LHS and RHS are both independent of t_1, and so $q_n(t_1, t_2)$ is also independent of t_1. Similarly $p_n(s_1, s_2)$ is independent of s_1. This implies that $p_n(s_1, s_2) = dq_n(t_1, t_2)$ really is an algebraic relation between s_2 and t_2, which are algebraically independent over $\mathbb{R}^a$, and so the coefficients of p, q must all be in $\mathbb{R}^a$ (to prevent a nontrivial algebraic relation from occurring). This implies that $\xi \in \mathbb{R}^a$, a contradiction.

It is clear that $\mathcal{B} \not\perp_{\mathcal{A}} \mathcal{C}$ since s_1, s_2 are algebraically independent over $\mathcal{A}$, but not algebraically independent over $\mathcal{C}$ since $s_1 t_1 + s_2 + t_2 = 0$. □

Exercise 5.2 (cf. Lemma 2.3 [7])**.** Give an example of T and an extension $(\mathcal{B}, \mathcal{A}) \subseteq (\mathcal{D}, \mathcal{C})$ of models of T^d such that $B \not\perp_A C$. By completeness of T^d, this shows that T^d is *not* model complete in general.

Solution. We'll use Exercise 5.1 with $T = T_{RCF}$, $\mathcal{D} = \mathcal{R}$, and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ occurring as in that exercise. By completeness of T^d, we have $(\mathcal{B}, \mathcal{A}) \equiv (\mathcal{D}, \mathcal{C})$, however by [7, Lemma 2.3], $(\mathcal{B}, \mathcal{A}) \subseteq (\mathcal{D}, \mathcal{C})$ is *not* an elementary extension of models of T^d. Thus T^d is *not* model complete. □

Exercise 5.3. Give an example of T such that T^d actually is model complete.

Solution. Let K be an ordered field, and let T_K be the (complete) theory of infinite ordered K-vector spaces. Then T_K^d has QE and hence is model complete, see [3, 5.8]. $\square$

Exercise 5.4. Read [2, Proposition B.5.4] and justify why the existence of a back-and-forth system as in the proof of [7, Theorem 2.5] between $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{D}, \mathcal{C})$ proves that T^d is complete.

Solution. Let $(\mathcal{B}_0, \mathcal{A}_0)$ and $(\mathcal{D}_0, \mathcal{C}_0)$ be two arbitrary models of T^d. Let $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{D}, \mathcal{C})$ be κ-saturated elementary extensions of $(\mathcal{B}_0, \mathcal{A}_0)$ and $(\mathcal{D}_0, \mathcal{C}_0)$ respectively. By [2, Proposition B.5.4] and the proof of the existence of the back-and-forth system between $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{D}, \mathcal{C})$ in [7, Theorem 2.5], it follows that $(\mathcal{B}, \mathcal{A}) \equiv (\mathcal{D}, \mathcal{C})$. By the definition of elementary extension, it also follows that $(\mathcal{B}_0, \mathcal{A}_0) \equiv (\mathcal{B}, \mathcal{A})$ and $(\mathcal{D}_0, \mathcal{C}_0) \equiv (\mathcal{D}, \mathcal{C})$. By transitivity of $\equiv$, we get $(\mathcal{B}_0, \mathcal{A}_0) \equiv (\mathcal{D}_0, \mathcal{C}_0)$. We conclude that T^d is complete. $\square$

Exercise 5.5. In Case 2 of the proof of Theorem 2.5 in [7], justify why $\mathcal{B}'\langle b\rangle \mathop{\smile\hskip -0.9em ^| \ }_{\mathcal{A}'} \mathcal{A}$. Also justify why there is necessarily an element $d \in \mathcal{D} \setminus \mathcal{D}'\langle C\rangle$ such that the cuts realized by b in $\mathcal{B}'$ and by d in $\mathcal{D}'$ correspond via i.

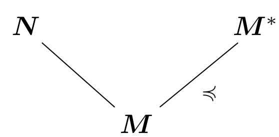
Solution. $\mathcal{B}'\langle b\rangle \mathop{\smile\hskip -0.9em ^| \ }_{\mathcal{A}'} \mathcal{A}$ follows from Exercise 4.11 above. By assumption, $(\mathcal{D}, \mathcal{C})$ is κ-saturated and so $\operatorname{rk}(\mathcal{D}|\mathcal{C}) \geq \kappa$ by [7, Lemma 1.5]. Furthermore, $|D'| < \kappa$ and so $\operatorname{rk}(\mathcal{D}|\mathcal{D}'\langle C\rangle) \geq \kappa > 0$. Thus $\mathcal{D} \neq \mathcal{D}'\langle C\rangle$ and so $(\mathcal{D}, \mathcal{D}'\langle C\rangle) \models T^d$. By [7, Lemma 2.4], $\mathcal{D} \setminus \mathcal{D}'\langle C\rangle$ is dense in $\mathcal{D}$. Thus κ-saturation and $|B'| < \kappa$ allows us to find such a d. $\square$

6. Week 4

Exercise 6.1. Read [2, B.9.2 and B.9.3], and use this to explain how we are obtaining the quantifier reduction result in Theorem 2.5 of [7].

Exercise 6.2. Suppose there is a back-and-forth system Γ between two structures $\boldsymbol{M}$ and $\boldsymbol{N}$, which share a common substructure B such that the identity $f : A \to A$ is in Γ. Let $a \in M^n$ and $b \in N^n$ be two tuples. Suppose there is $g \in \Gamma$ such that $g \supseteq f$, $a \in \operatorname{dom}(g)$, and $g(a) = b$. Then $\operatorname{tp}_{\boldsymbol{M}}(a/A) = \operatorname{tp}_{\boldsymbol{N}}(b/A)$.

Exercise 6.3. (An elementary extension test, [7, Cor 2.7]) Suppose $\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{M^*}$ are models of an L'-theory T' such that $\boldsymbol{M} \subseteq \boldsymbol{N}$ and $\boldsymbol{M} \preccurlyeq \boldsymbol{M}^*$, in picture form:



Suppose for every finite tuple $a \in M^n$, $\operatorname{tp}_{\boldsymbol{M}^*}(a) = \operatorname{tp}_{\boldsymbol{N}}(a)$. Then $\boldsymbol{M} \preccurlyeq \boldsymbol{N}$. (This basically says "If $\boldsymbol{N}$ views everything down in $\boldsymbol{M}$ as if $\boldsymbol{N}$ were an elementary extension, then $\boldsymbol{N}$ actually *is* an elementary extension.")

Exercise 6.4 (cf. Proof of Theorem 2 [7])**.** Let $(\mathcal{B}, \mathcal{A}) \models T^d$ and let $\phi(y)$ be an $L^2(B)$-formula, with $y = (y_1, \ldots, y_n)$. Suppose for any two elementary extensions $(\mathcal{B}_1, \mathcal{A}_1)$ and $(\mathcal{B}_2, \mathcal{A}_2)$ of $(\mathcal{B}, \mathcal{A})$ and any two n-tuples $a_1 \in (A_1)^n$ and $a_2 \in (A_2)^n$ that realize the same types over B (in $\mathcal{B}_1$ and $\mathcal{B}_2$) we have $(\mathcal{B}_1, \mathcal{A}_2) \models \phi(a_1)$ iff $(\mathcal{B}_2, \mathcal{A}_2) \models \phi(a_2)$. Show that there is an $L(B)$-formula $\psi(y)$ such that $(\mathcal{B}, \mathcal{A}) \models U(y) \to (\phi(y) \leftrightarrow \psi(y))$.

7. Week 5

Exercise 7.1 (cf. Proof of Corollary 3.4 [7])**.** (Two forms of transcendence) Let $(\mathcal{B}, \mathcal{A}) \prec (\mathcal{B}^*, \mathcal{A}^*) \models T^d$ and suppose $b^* \in B^*$. Show that $b^* \notin X^*$ for all $\mathcal{A}$-small sets $X \subseteq B$ iff $b^* \notin \mathcal{B}\langle A^*\rangle$.

Exercise 7.2 (cf. Proof of Corollary 3.4 [7])**.** Assume $(\mathcal{B}, \mathcal{A}) \prec (\mathcal{B}^*, \mathcal{A}^*) \models T^d$ is a sufficiently saturated elementary extension. Suppose that if $b^* \in \mathcal{B}^* \setminus \mathcal{B}\langle A^*\rangle$, then $F(b^*) \in \mathcal{B}\langle b^*\rangle$. Conclude that F agrees off some $\mathcal{A}$-small subset of B with a function $\hat{F} : B \to B$ that is definable in $\mathcal{B}$.

Exercise 7.3 (cf. Proof of Corollary 3.6 [7])**.** Here $(\mathcal{B}, \mathcal{A}) \models T^d$, $f : A^n \to A$ is definable in $(\mathcal{B}, \mathcal{A})$. Suppose that given an elementary extension $(\mathcal{B}^*, \mathcal{A}^*)$ of $(\mathcal{B}, \mathcal{A})$ and a point $a^* \in (A^*)^n$, we have $f(a^*) \in \mathcal{A}\langle a^*\rangle$. Conclude that there are functions $f_1, \ldots, f_k : A^n \to A$ definable in $\mathcal{A}$ such that for each $a \in A^n$ we have $f(a) = f_i(a)$ for some $i \in \{1, \ldots, k\}$.

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