Dynamical methods in fractal geometry

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Fractals

• "Fractal" = some shape that's "made up of scaled copies of itself"
Iterated function systems

• A natural way of defining a large class of fractal objects
  ▶ Let $\phi_1, \ldots, \phi_n$ be contraction mappings in $\mathbb{R}^d$
  ▶ **Theorem:** there exists a unique compact set $K$ such that $K = \bigcup_{1 \leq i \leq n} \phi_i(K)$
  ▶ $K$ is called the **attractor** of the **iterated function system (IFS)**
  ▶ "Made up of scaled copies of itself"
Iterated function systems

• Example:
Iterated function systems

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• One method for constructing the attractor:
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Example

- Sierpinski triangle
  - $\phi_1(x, y) = \frac{1}{2}(x, y)$
  - $\phi_2(x, y) = \frac{1}{2}(x, y) + (1/2, 0)$
  - $\phi_3(x, y) = \frac{1}{2}(x, y) + (1/4, 1/2)$
Example

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Example

• Sierpiński triangle

\[
\begin{align*}
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• Remarks/definitions:
  ▶ If the union in $K = \bigcup_{1 \leq i \leq n} \phi_i(K)$ is disjoint, the IFS satisfies the strong separation condition (SSC)
  ▶ If all of the $\phi_i$ are affine, the attractor $K$ is a self-affine set. If all of the $\phi_i$ are similarity maps (scaling + isometry), $K$ is a self-similar set.
• Without these conditions, very hard to analyze
A natural quantity associated to a fractal set is its **Hausdorff dimension**

- Nice properties:
  - \( \dim_H(\cdot) = 0, \ \dim_H(-) = 1, \ \dim_H(\square) = 2, \) etc.
  - \( \dim_H \left( \bigcup_{i \in \mathbb{N}} A_i \right) = \sup_{i \in \mathbb{N}} \dim_H(A_i) \)
  - \( \dim_H(A \times B) = \dim_H(A) + \dim_H(B) \) (under some conditions on \( A \) and \( B \))
  - \( \dim_H(f(A)) = \dim_H(A) \) if \( f \) is bi-Lipschitz
Slices of fractals

- **Theorem (Marstrand, 1950s):** Let $A \subseteq \mathbb{R}^2$. Then for a.e. line $L$, $\dim(A \cap L) \leq \max(0, \dim(A) - 1)$.

  - More generally, if $A \subseteq \mathbb{R}^d$, then $\dim(A \cap W) \leq \max(0, \dim(A) - \text{codim}(W))$ for a.e. affine subspace $W$.

- Marstrand's theorem holds for any set (doesn't even have to be measurable!). If $A$ has nice fractal structure maybe Marstrand's theorem is true for **every** line $L$.

  - Conjectured in various forms by Furstenberg
Obvious obstructions

\[ \dim(A \cap L) = 1 \]

\[ \dim(A) - 1 = \frac{\log(3)}{\log(2)} - 1 \approx 0.58 \]

\[ \dim(A \cap L) = \frac{\log(2)}{\log(3)} \approx 0.63 \]

\[ \dim(A) - 1 = \frac{\log(4)}{\log(3)} - 1 \approx 0.26 \]
• From now, make the following assumptions:
  ‣ Each $\phi_i$ is a **similarity**
  ‣ The attractor $K$ satisfies the **SSC**
  ‣ Each $\phi_i$ has the **same rotation part** $\theta \notin 2\pi\mathbb{Q}$

• For example:
Attractor system

- The attractor $K$ can be turned into a dynamical system

- Define $S : K \rightarrow K$ to be the local inverse to the $\phi_i$, i.e. $S(z) = \phi_i^{-1}(z)$ for $z \in \phi_i(K)$ (well-defined by SSC)
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An observation

• Suppose there exists $L$ with $\dim(K \cap L) > \dim(K) - 1$.
  
  ▪ $L$ points in the direction of $e^{2\pi it_0}$
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An observation

- Suppose there exists $L$ with $\dim(K \cap L) > \dim(K) - 1$.
  - $L$ points in the direction of $e^{2\pi it_0}$
  - $S(K \cap L)$ is a union of several slices, each in the direction $e^{2\pi i(t_0 - \theta)}$, and at least one of them also has dimension $> \dim(K) - 1$
An observation

• Iterate this procedure: for every $n \geq 0$ there is a line $L_n$ in the direction $e^{2\pi i(t_0 - n\theta)}$ such that $\dim(K \cap L_n) > \dim(K) - 1$
  
  ▶ One bad slice $\longrightarrow$ a lot of bad slices whose slopes are dense in $S^1$

• Furstenberg (1960s) was able to push this observation much further with two key insights:
  
  ▶ Introduce randomness into the picture
  
  ▶ Work with dimensions of measures rather than dimensions of sets
Dimension of measures

• Say a probability measure $\mu$ on $\mathbb{R}^d$ is $\alpha$-regular if $\mu(B_r(x)) \lesssim r^\alpha$ as $r \downarrow 0$ for all $x$.

  ‣ Example: $d$-dimensional Lebesgue measure is $d$-regular
  ‣ Example: a point mass is $0$-regular (and not $\alpha$-regular for any larger $\alpha$)
Dimension of measures

- Strong relationship between regularity of measures and dimensions of sets they can "see"
  - If $\mu$ is $\alpha$-regular and $\dim(A) < \alpha$, then $\mu(A) = 0$.
  - "Frostman's lemma": If $\dim(A) \geq \alpha$, then there exists an $\alpha$-regular $\mu$ supported on $A$.

- Morally, there is a rough correspondence
  $\dim(\mu) := \sup\{\alpha : \mu \text{ is } \alpha\text{-regular}\} = \inf\{\dim(A) : \mu(A) > 0\}$
Magnification dynamics

- Idea: run the attractor system, but keep track of more data
- "Zooming in" dynamics on $K \times \mathbb{T} \times \text{Prob}(K)$:
  - $M : (z, t, \nu) \mapsto (S_z, t - \theta, S_*(\nu_z))$, where $\nu_z := \nu$ conditioned on the piece $\phi_i(K)$ that contains the point $z$
Furstenberg's construction

- Suppose there is a slice \( L \) in the direction \( e^{2\pi it_0} \) with \( \dim(K \cap L) =: \alpha > \dim(K) - 1 \)
  - Frostman's lemma \( \mapsto \) let \( \nu_0 \) be an \( \alpha \)-regular measure supported on \( K \cap L \)

- Let \( \bar{\mu}_0 = \nu_0 \times \delta_{t_0} \times \delta_{\nu_0} \in \text{Prob}(K \times \mathbb{T} \times \text{Prob}(K)) \) ("introduce randomness")

- Let \( \bar{\mu} \) be a weak-* limit point of the sequence \( \bar{\mu}_n := \frac{1}{n} \sum_{j=0}^{n-1} M^j_\ast \bar{\mu}_0 \) (Krylov-Bogolyubov machine)
  - \( \bar{\mu} \) is an \( M \)-invariant measure on \( K \times \mathbb{T} \times \text{Prob}(K) \) supported on \( \{(z, t, \nu) : \nu \text{ is } \alpha \text{-dimensional and } \nu(L_{z,t}) = 1\} \)

The line through the point \( z \) in the direction \( e^{2\pi it} \)
Furstenberg's construction

- The projection of $\mu$ onto the $\mathbb{T}$ coordinate is invariant for the irrational circle rotation $t \mapsto t - \theta$
  - Must be Lebesgue measure
- Conclusion:
  - **Theorem (Furstenberg):** Suppose there exists a line $L$ with $\dim(K \cap L) > \alpha$. Then for Lebesgue-a.e. $t \in \mathbb{T}$ there exists some line $L_t$ in the direction $e^{2\pi it}$ such that $\dim(K \cap L_t) > \alpha$ also.
• The conclusion of Furstenberg's theorem should translate to an upper bound on $\alpha$ in terms of $\dim(K)$ (think Kakeya problem)
  ▶ Turned out to be hard to get the "correct" value

• **Theorem (2019):** Let $\{\phi_1, \ldots, \phi_n\}$ be a similarity IFS in $\mathbb{R}^2$ satisfying the **SSC** and such that each $\phi_i$ has **common rotation part** $\theta \notin 2\pi\mathbb{Q}$. Let $K$ be the attractor. Then $\dim(K \cap L) \leq \max(0, \dim(K) - 1)$ for **every** line $L$.
  ▶ Independent & simultaneous proofs by Pablo Shmerkin and Meng Wu (appeared in back-to-back Annals issues)
Generalizations/extensions

• For what other sets $K$ does Marstrand's theorem hold for every line/affine subspace?

• $K = A \times B$, where $A, B \subseteq \mathbb{T}$ are $(\times 2), (\times 3)$-invariant respectively (not including lines parallel to axes)
  ‣ Also proven by Shmerkin/Wu

• $K =$ attractor of IFS with different irrational rotation parts
  ‣ ????

• $K =$ attractor of IFS in higher dimensions
  ‣ ????
Generalizations/extensions

• Projections:
  - **Theorem (Marstrand):** Let $A \subseteq \mathbb{R}^2$. Then for a.e. line $L$, 
    $$\dim(\Pi_L A) = \max(1, \dim(A)),$$
    where $\Pi_L$ is orthogonal projection onto the line $L$.

• Can also apply magnification dynamics to this problem

• Hochman & Shmerkin (2012) used similar ideas to prove the above holds for **every** line $L$, for many nice fractals $A$. 