Dynamical methods in fractal geometry

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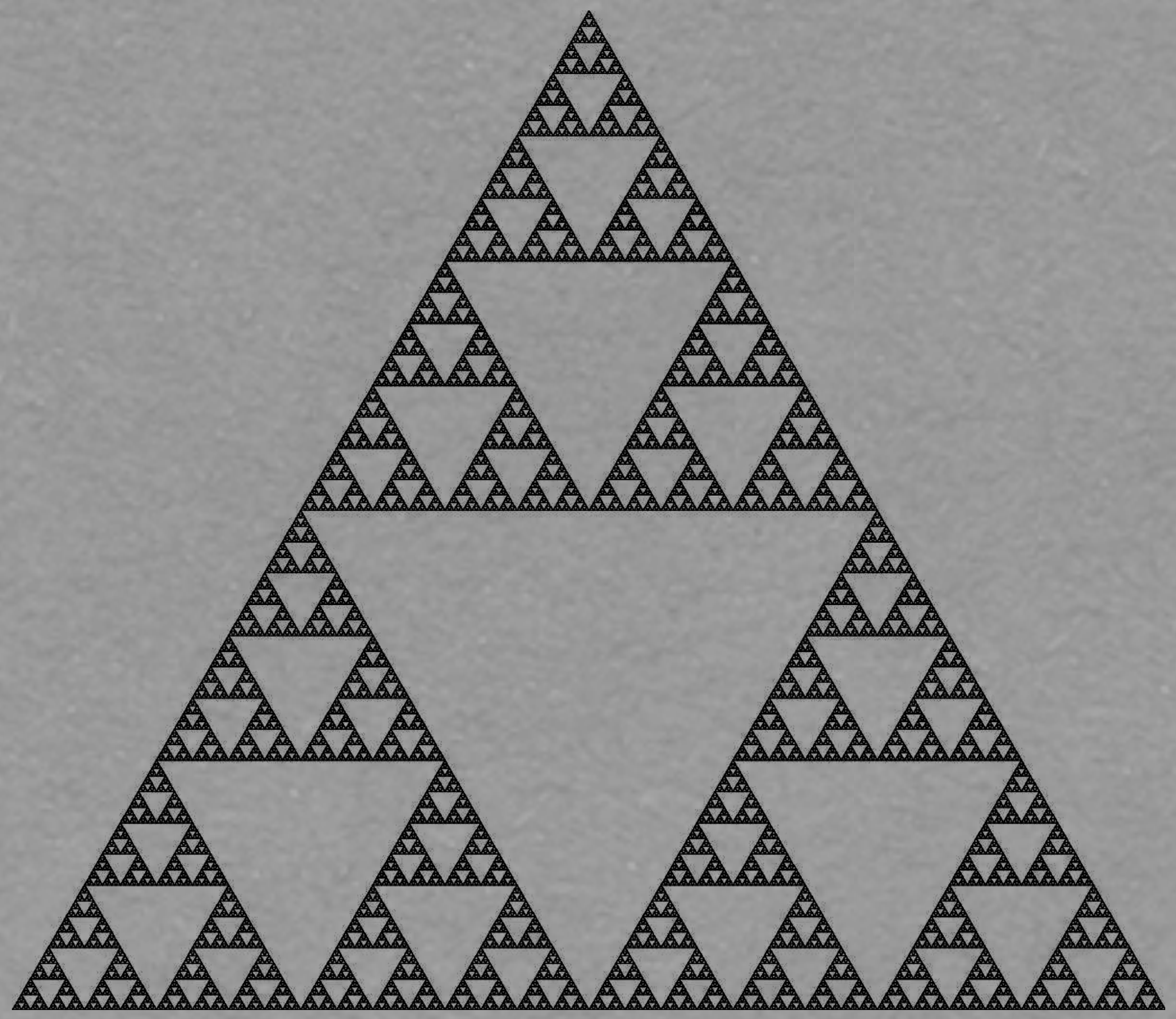
Fractals

- "Fractal" = some shape that's "made up of scaled copies of itself"

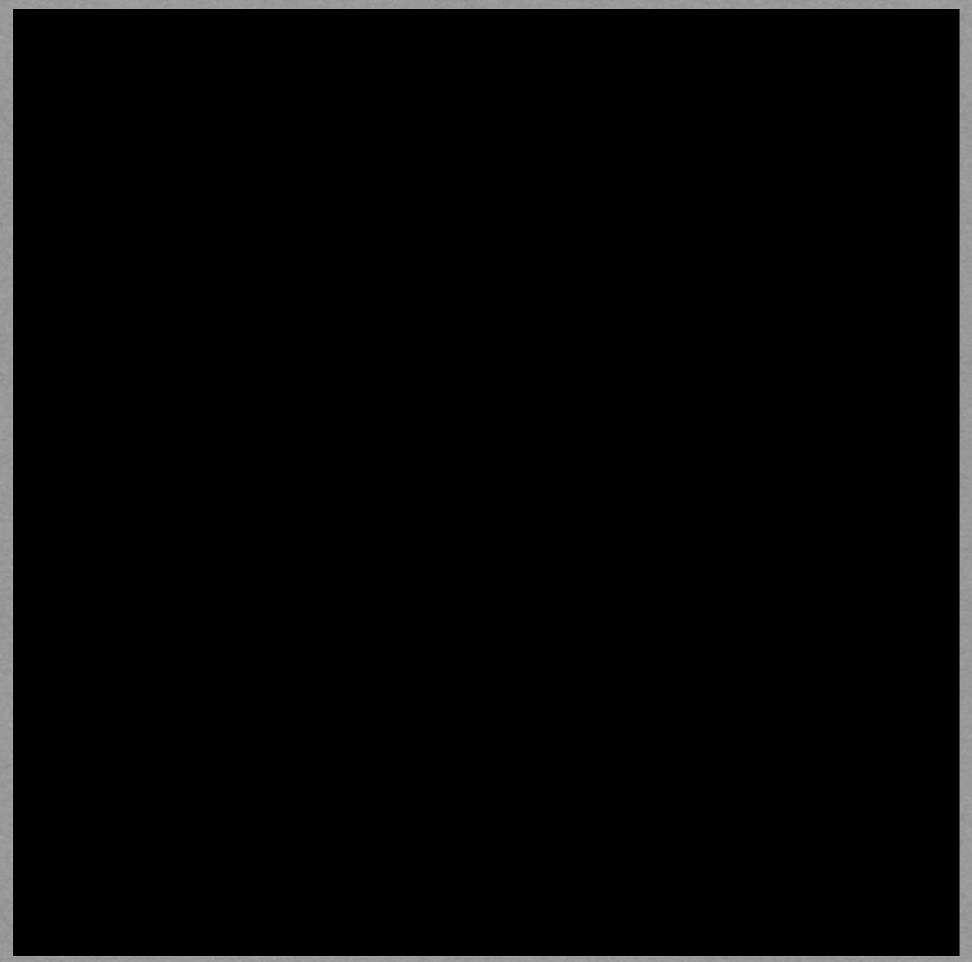


Iterated function systems

- A natural way of defining a large class of fractal objects
  - Let $\phi_1, \ldots, \phi_n$ be **contraction mappings** in $\mathbb{R}^d$
  - **Theorem:** there exists a **unique** compact set K such that $K = \bigcup_{1 \le i \le n} \phi_i(K)$
  - K is called the **attractor** of the **iterated function system (IFS)**
  - "Made up of scaled copies of itself"

Iterated function systems

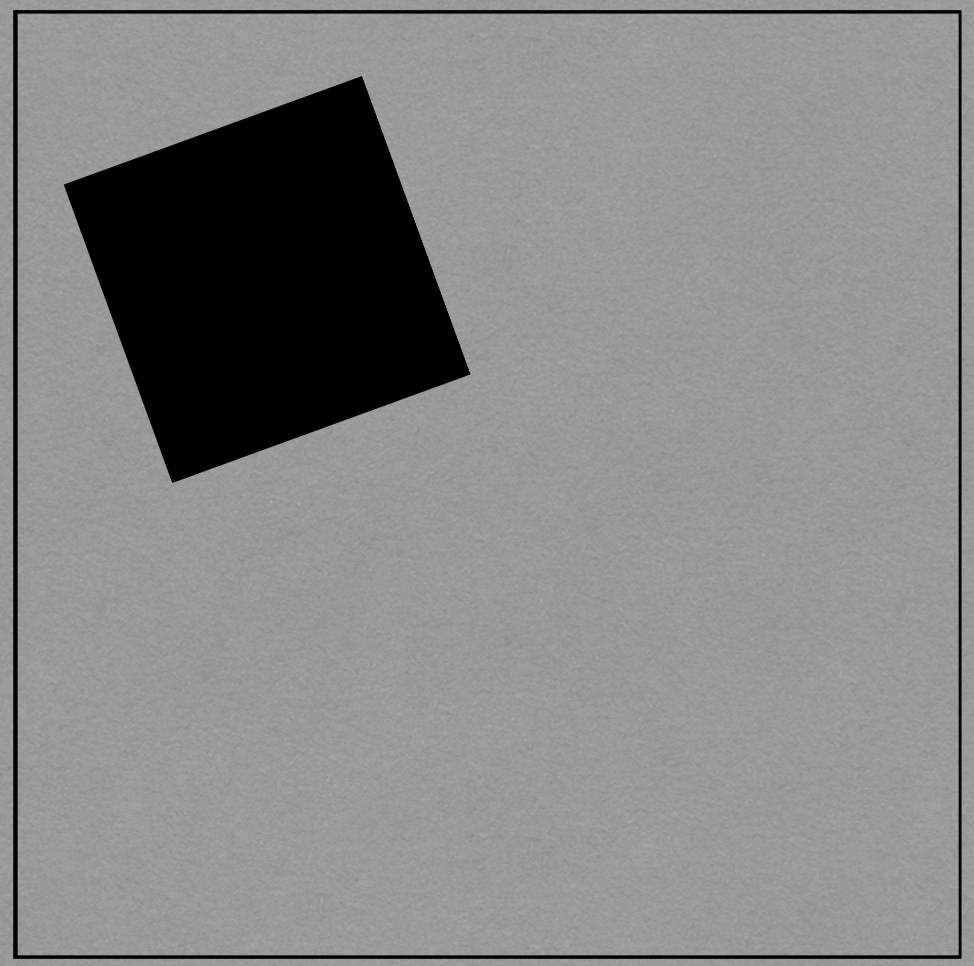
- Example:



ϕ_1

Iterated function systems

- Example:



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Iterated function systems

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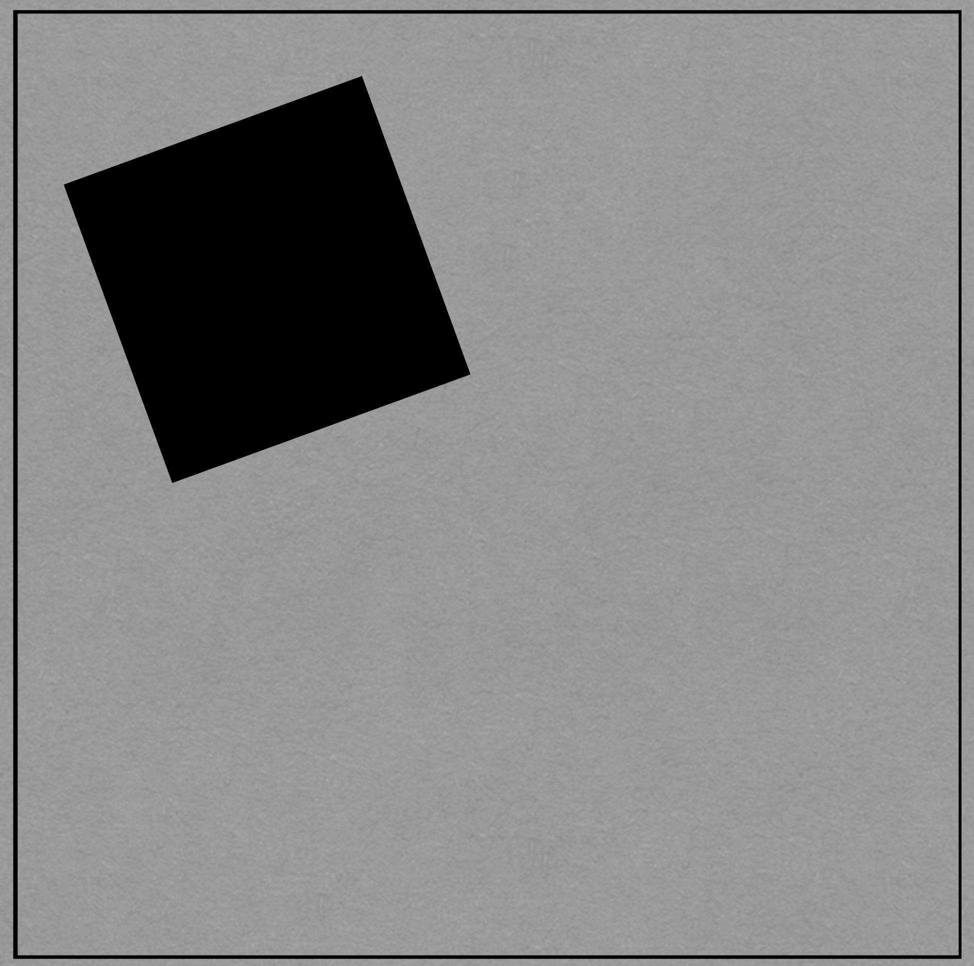

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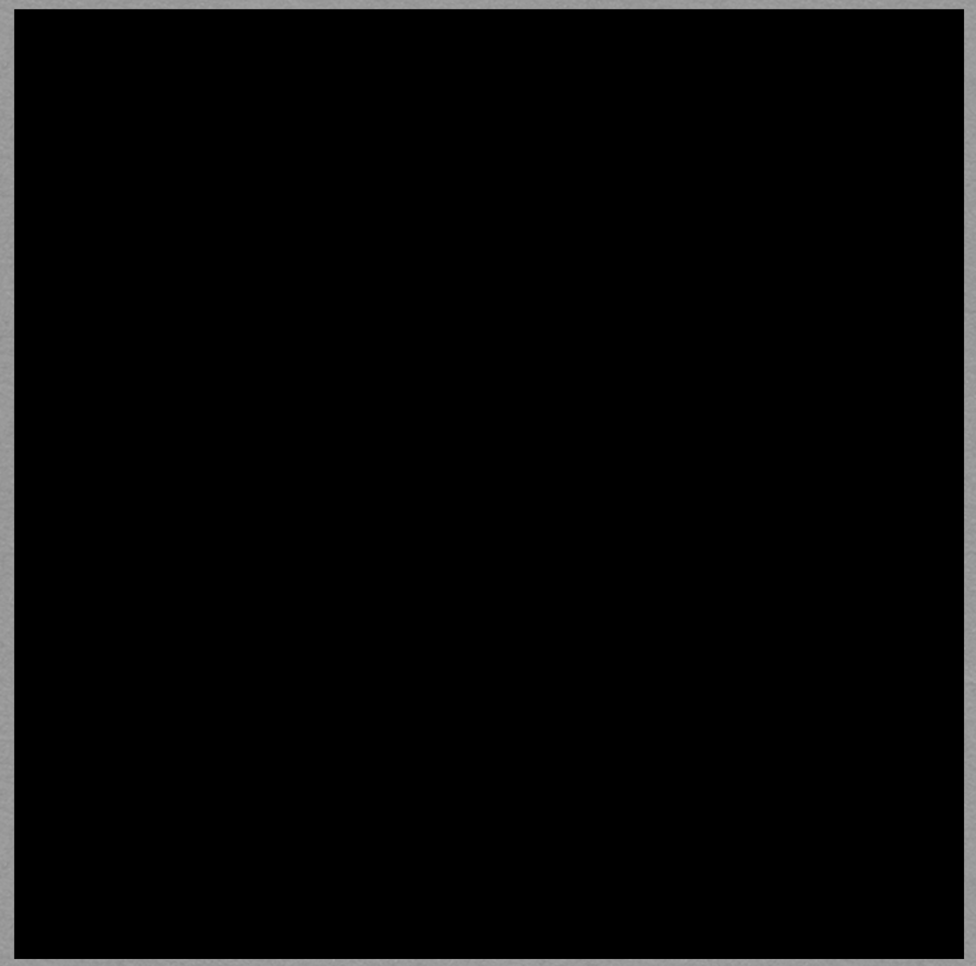

ϕ_2

Iterated function systems

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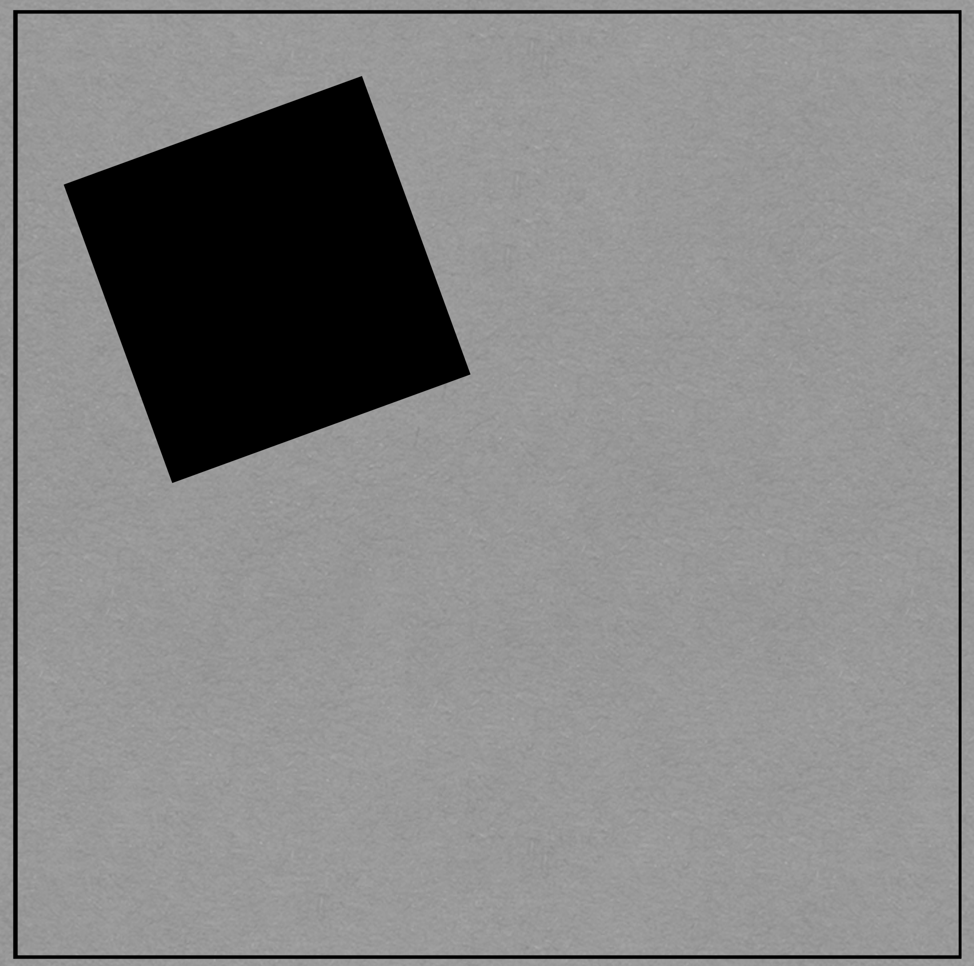

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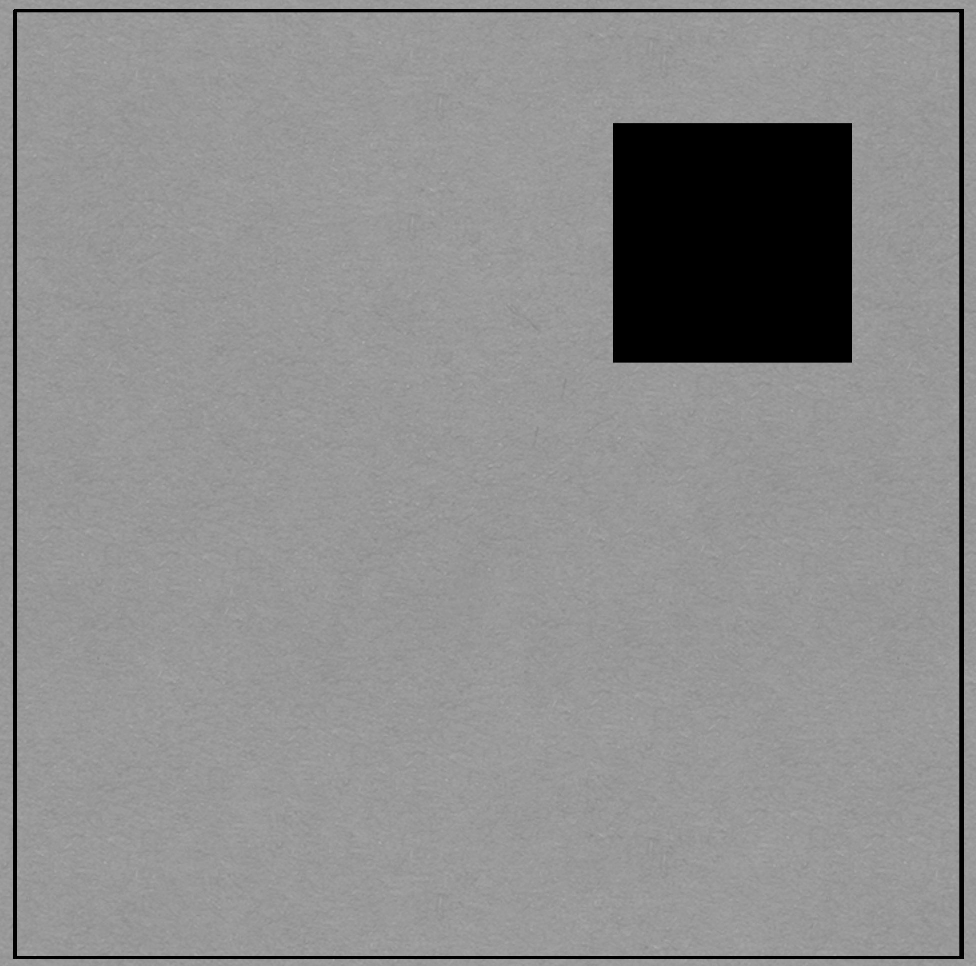

ϕ_2

Iterated function systems

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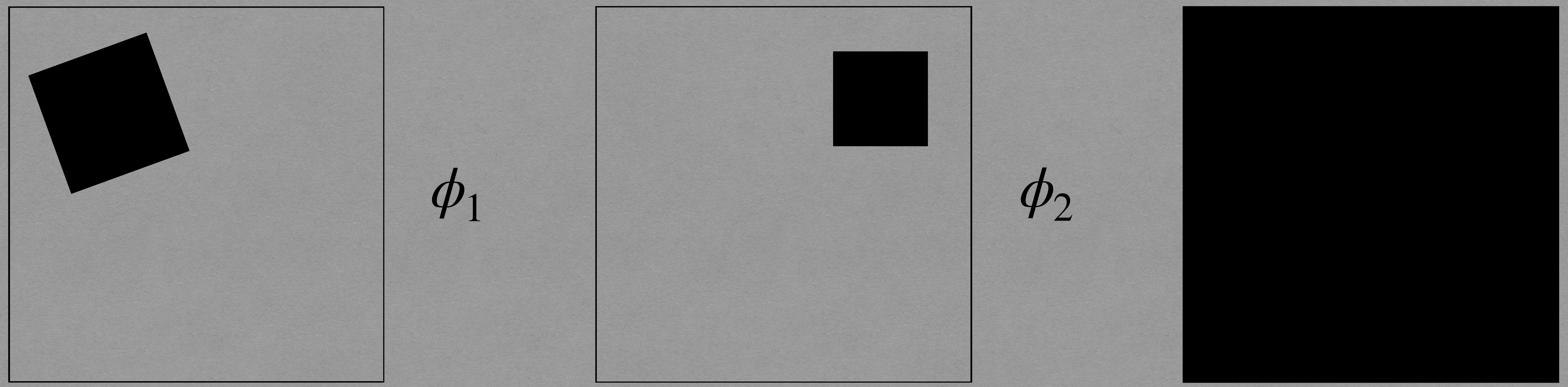

ϕ_1 ϕ_2 ϕ_3

Iterated function systems

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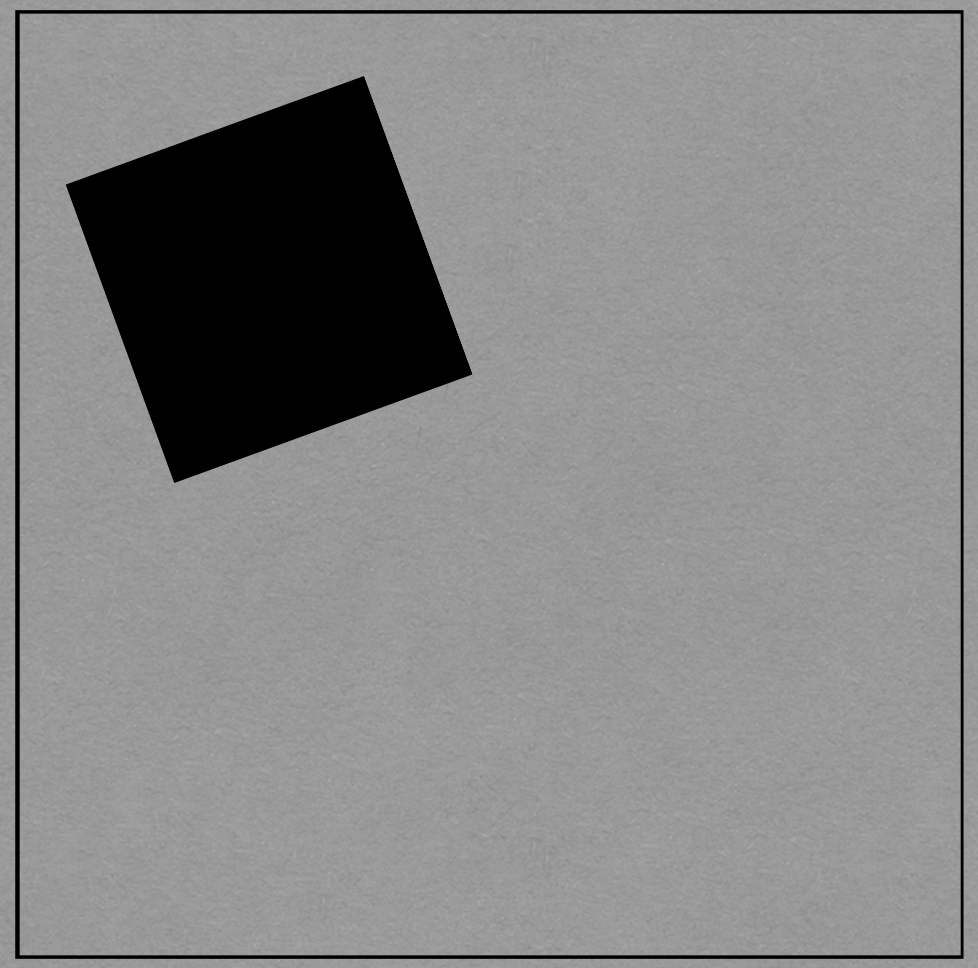
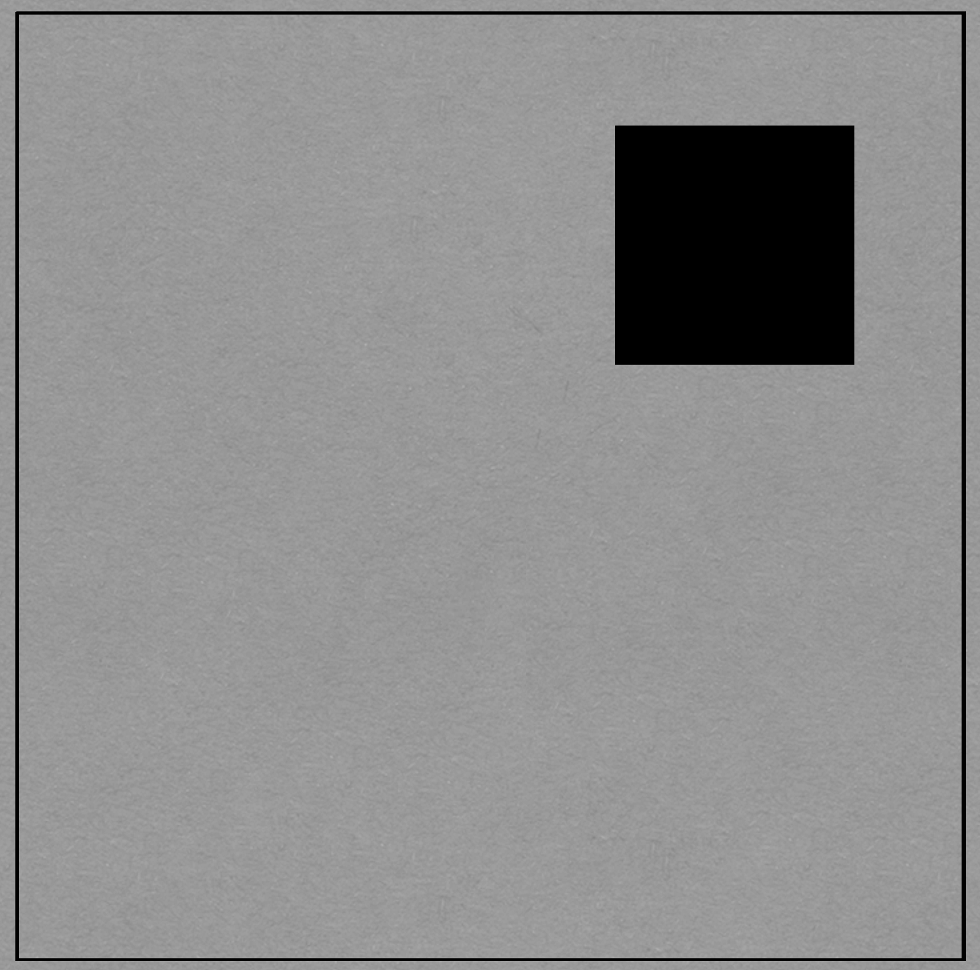
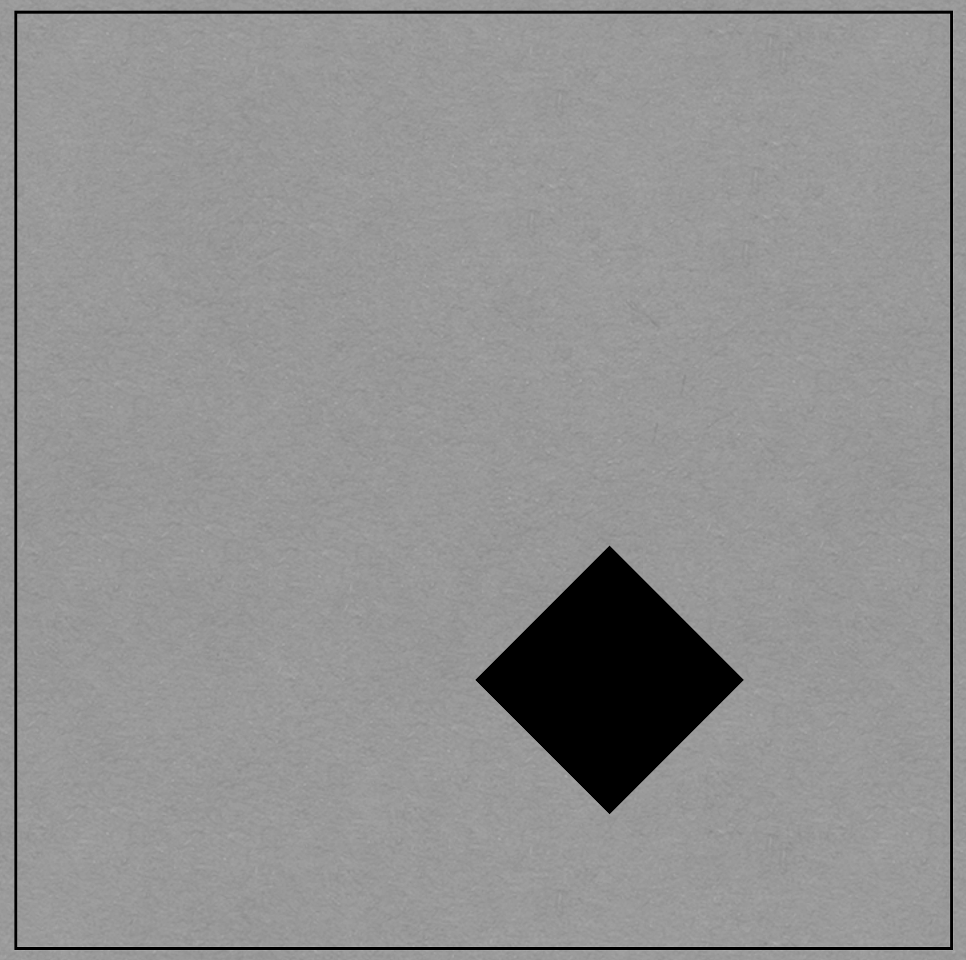


ϕ_1



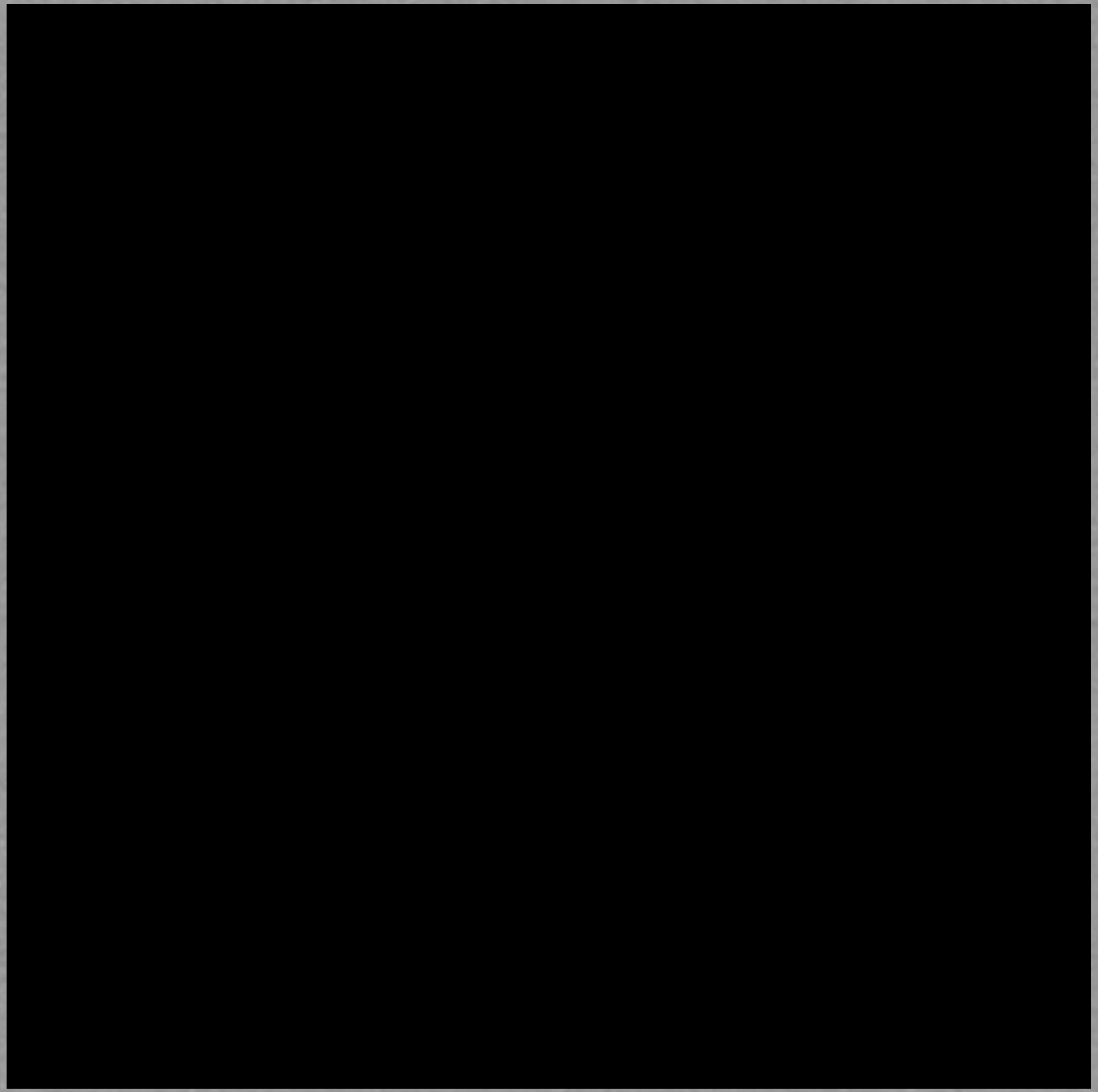
ϕ_2



ϕ_3

Iterated function systems

- One method for constructing the attractor:



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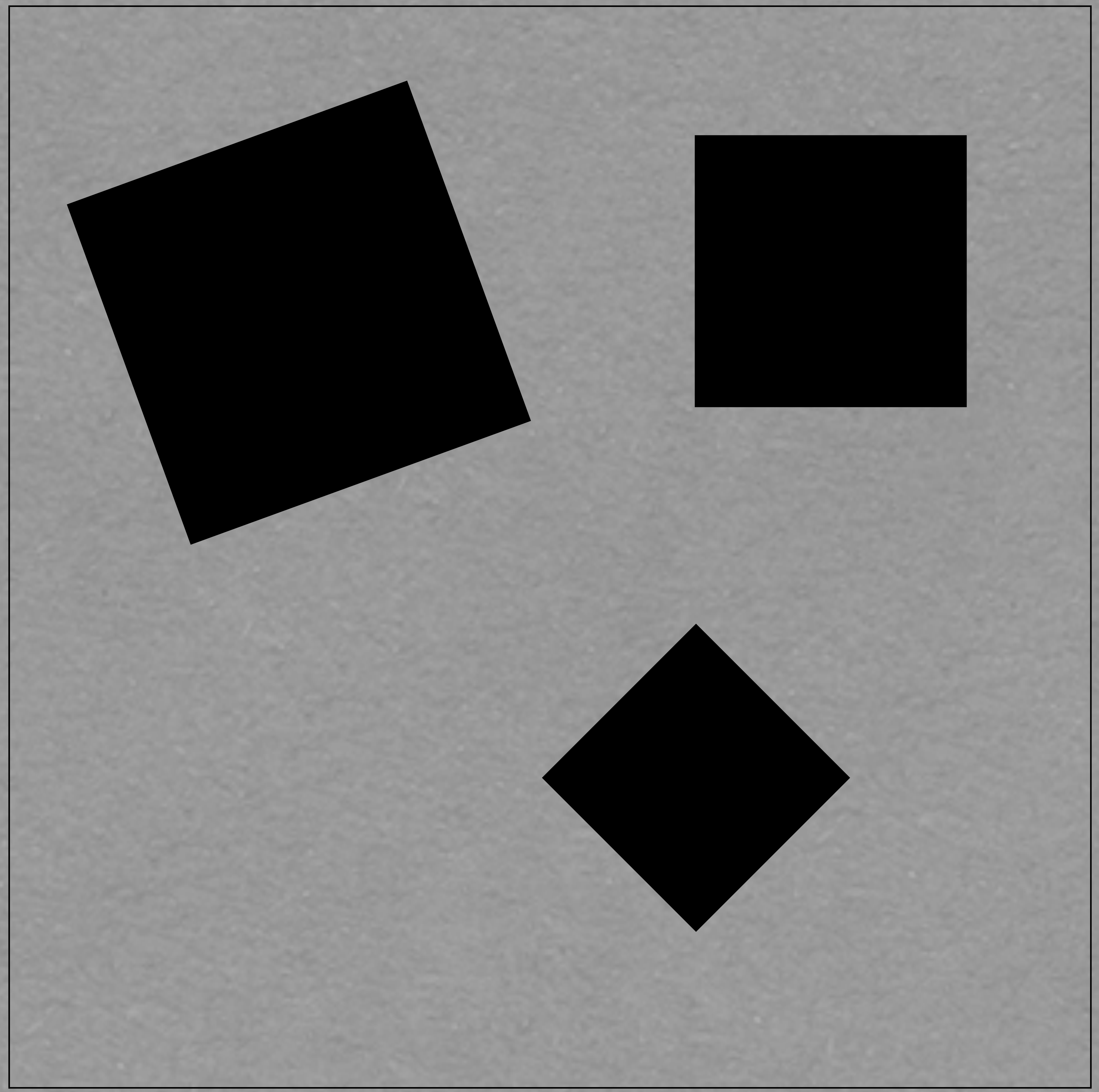

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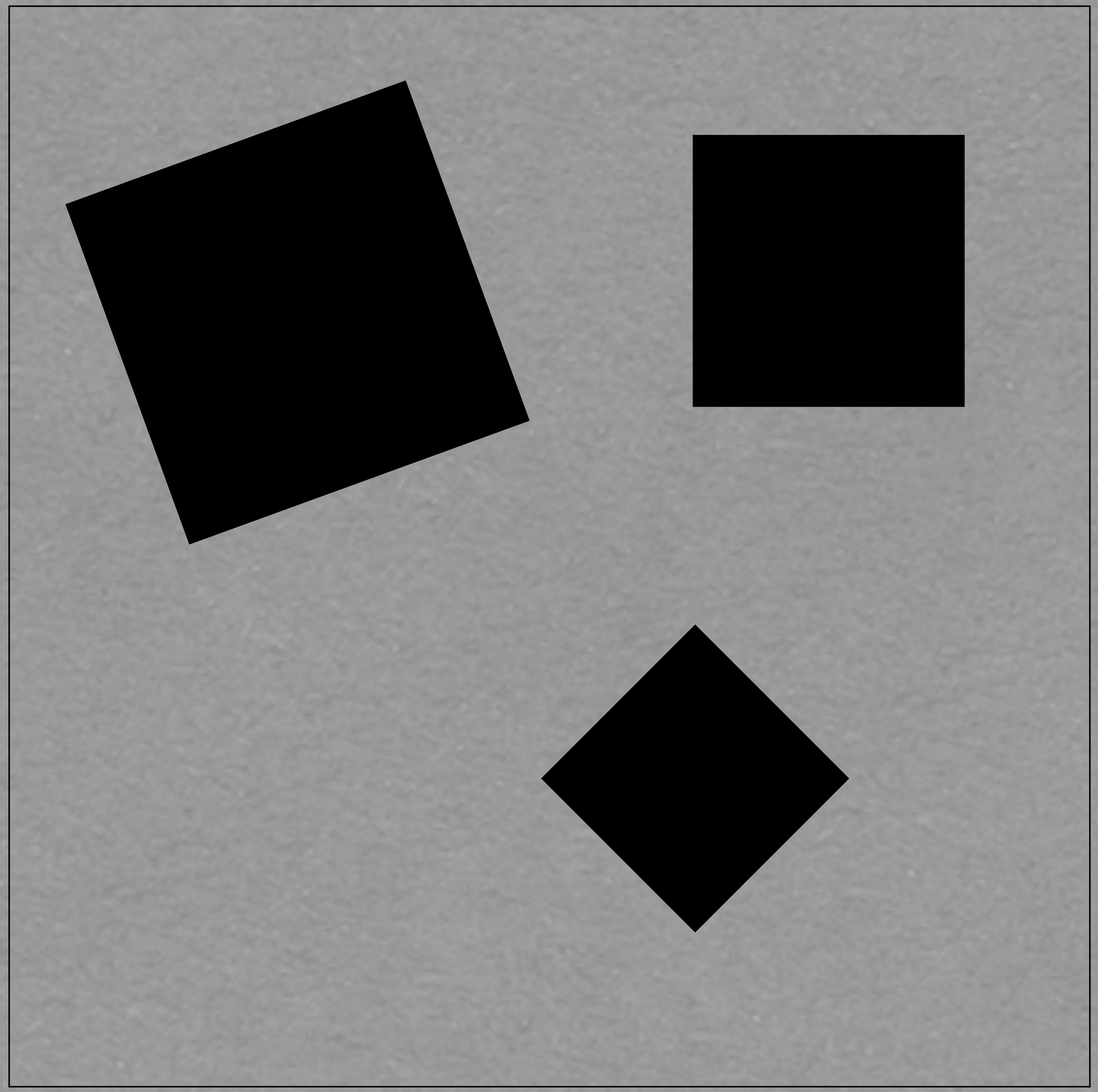

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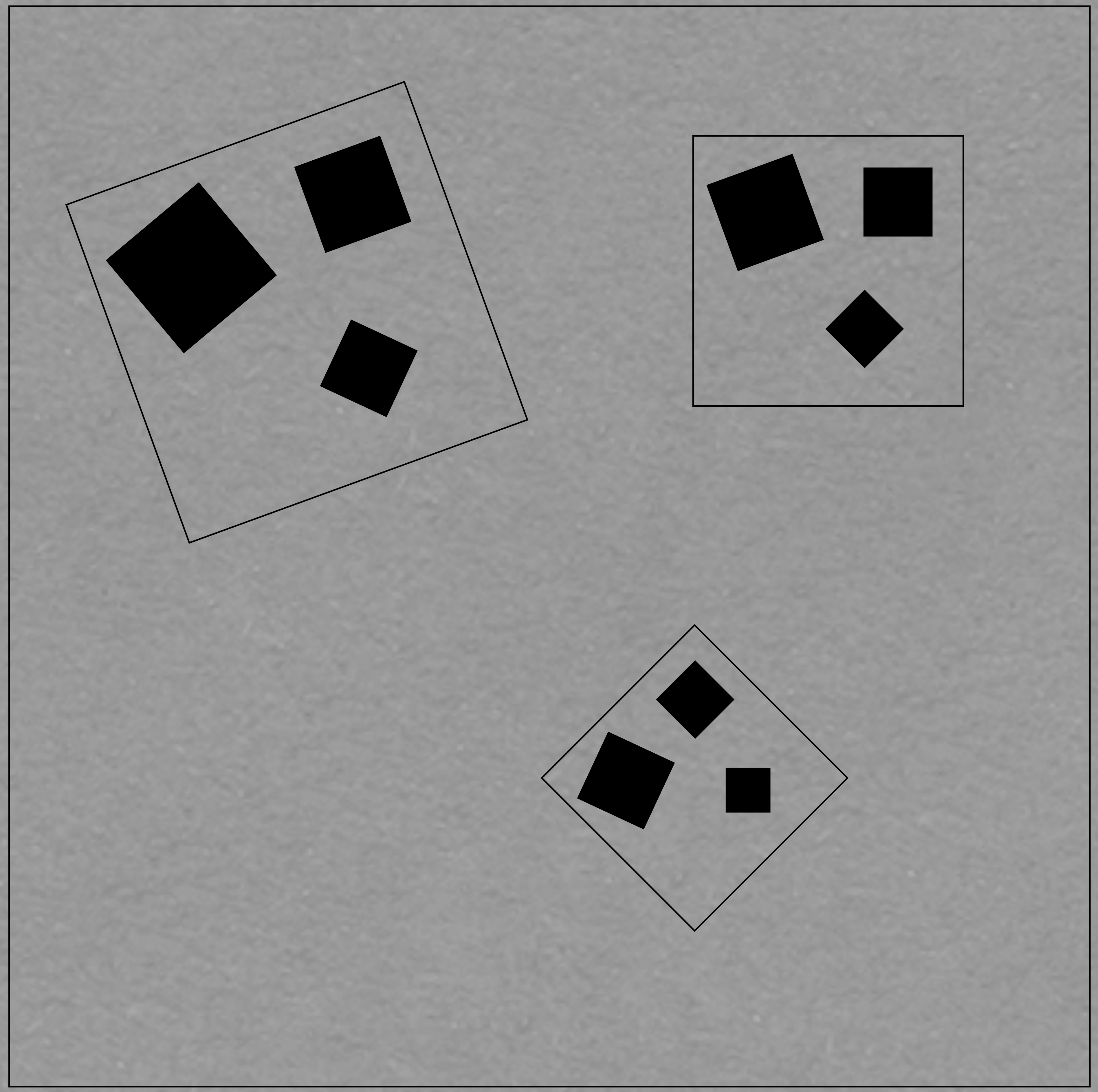

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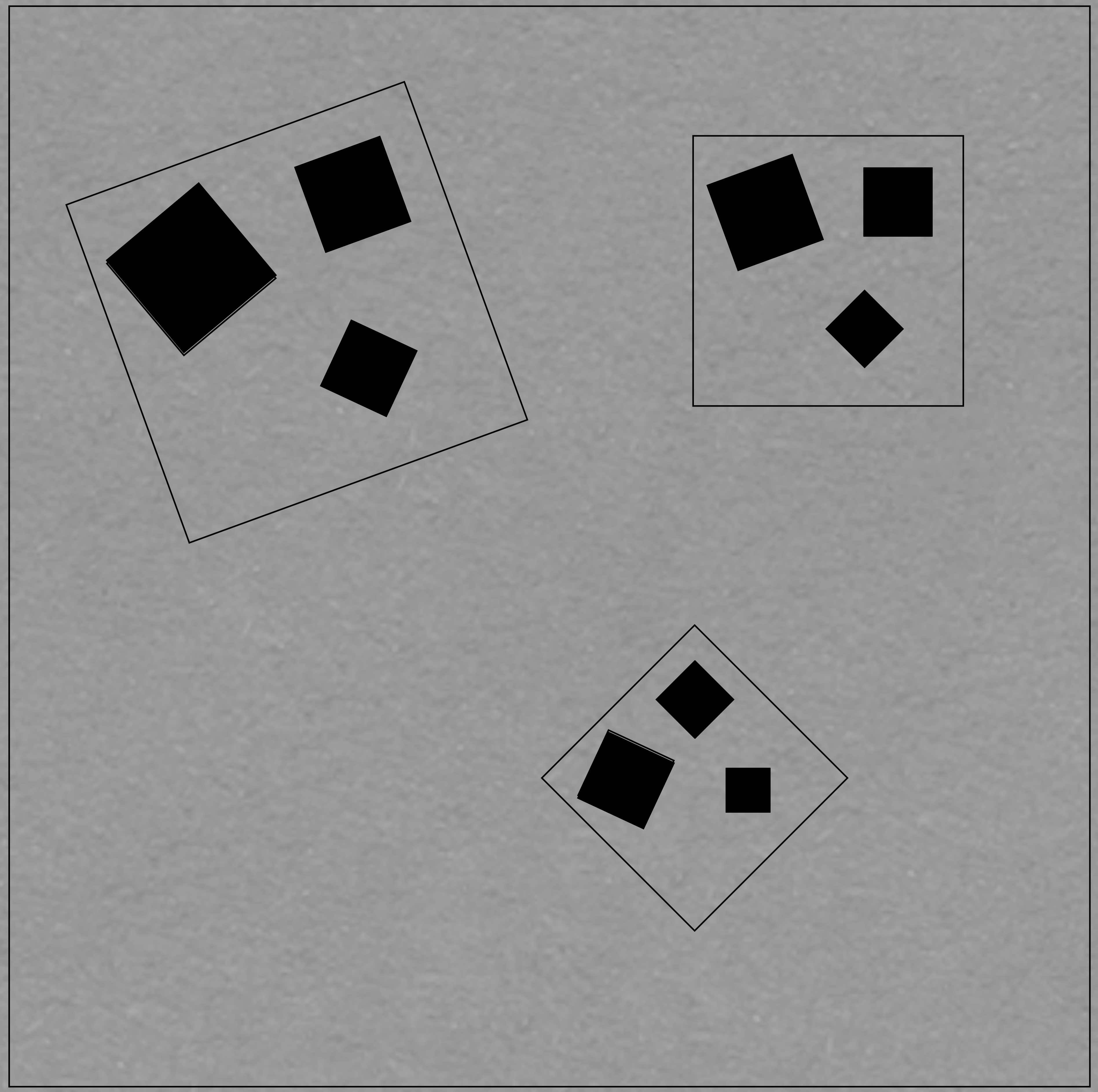

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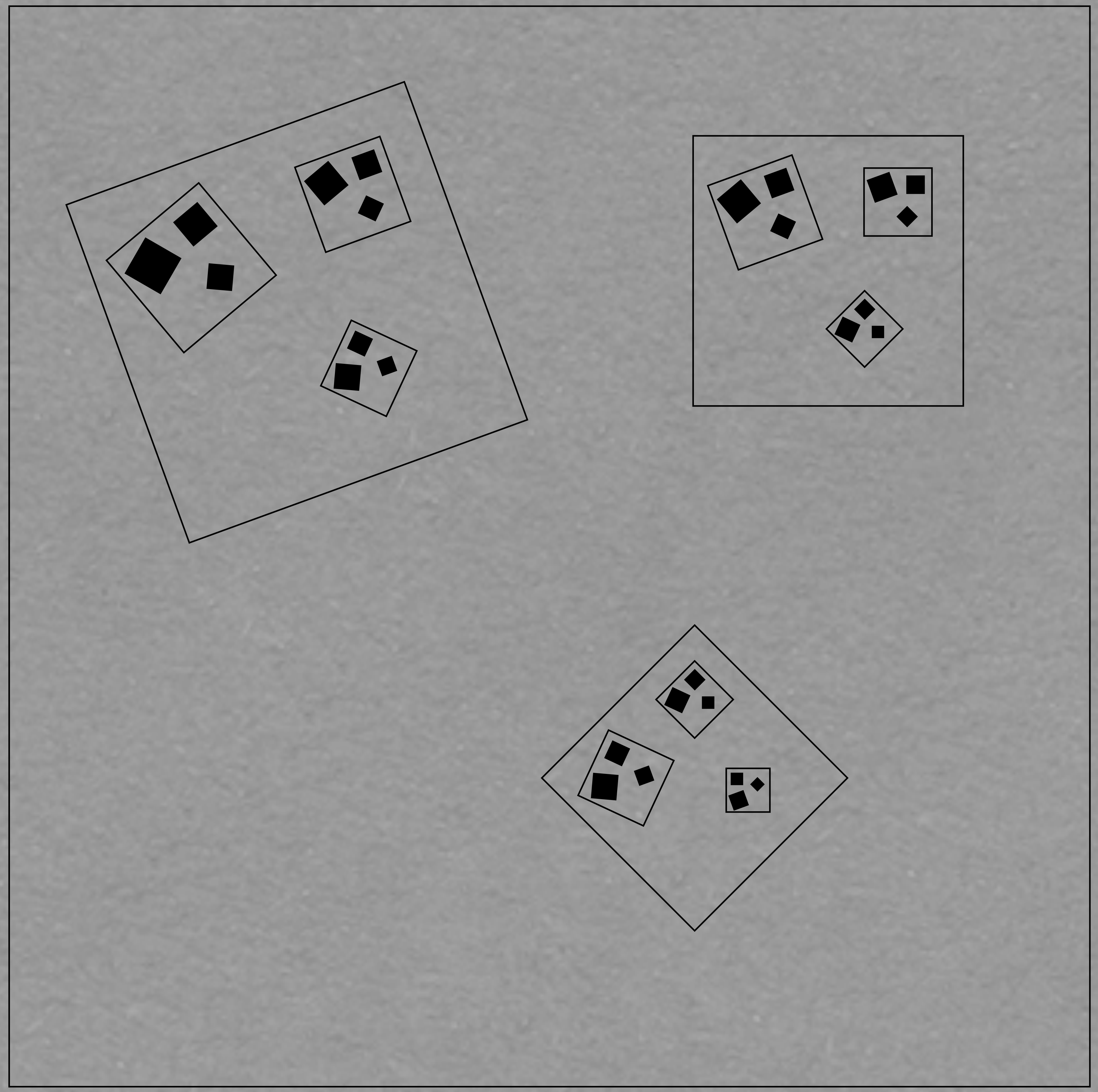

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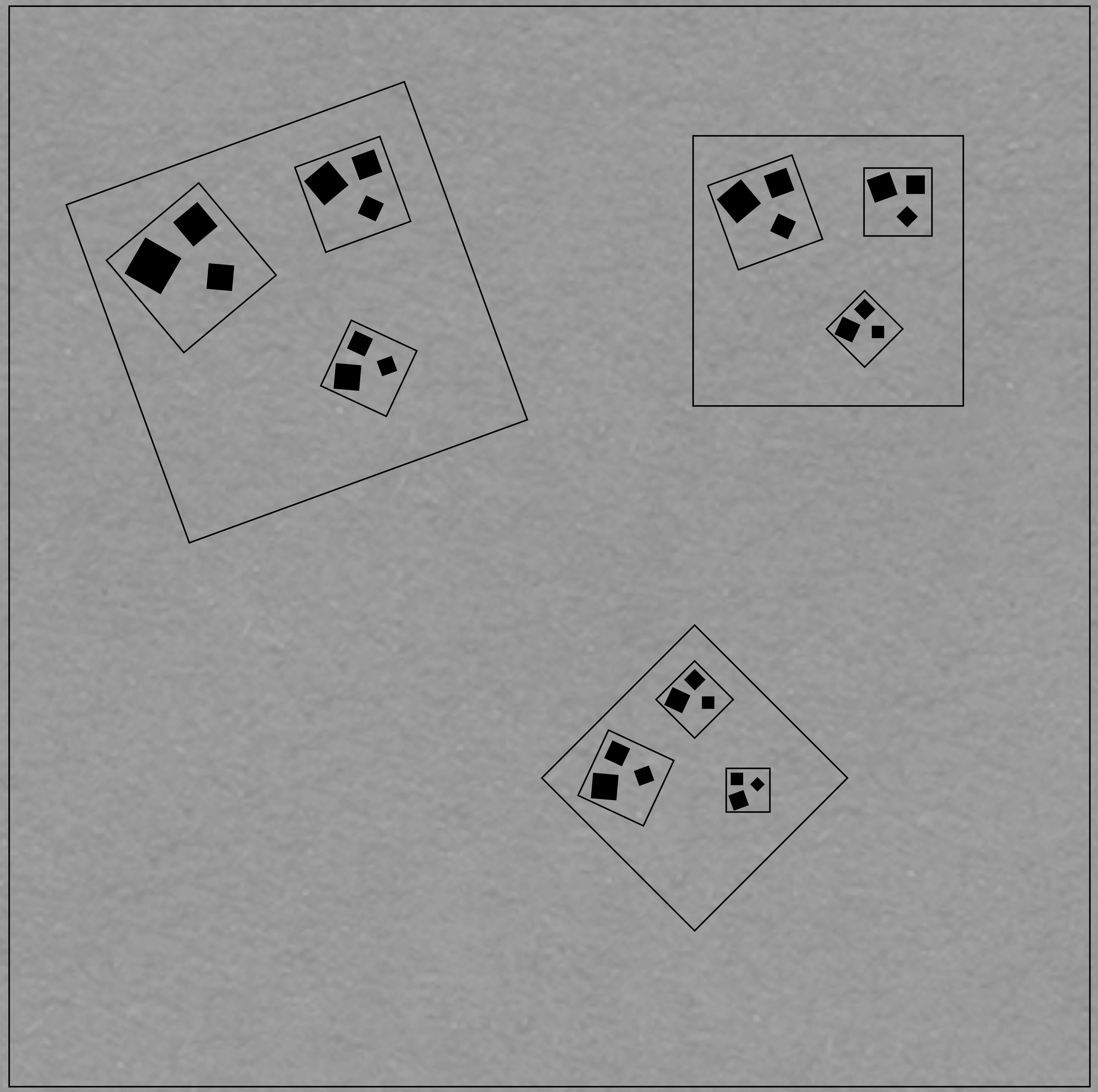

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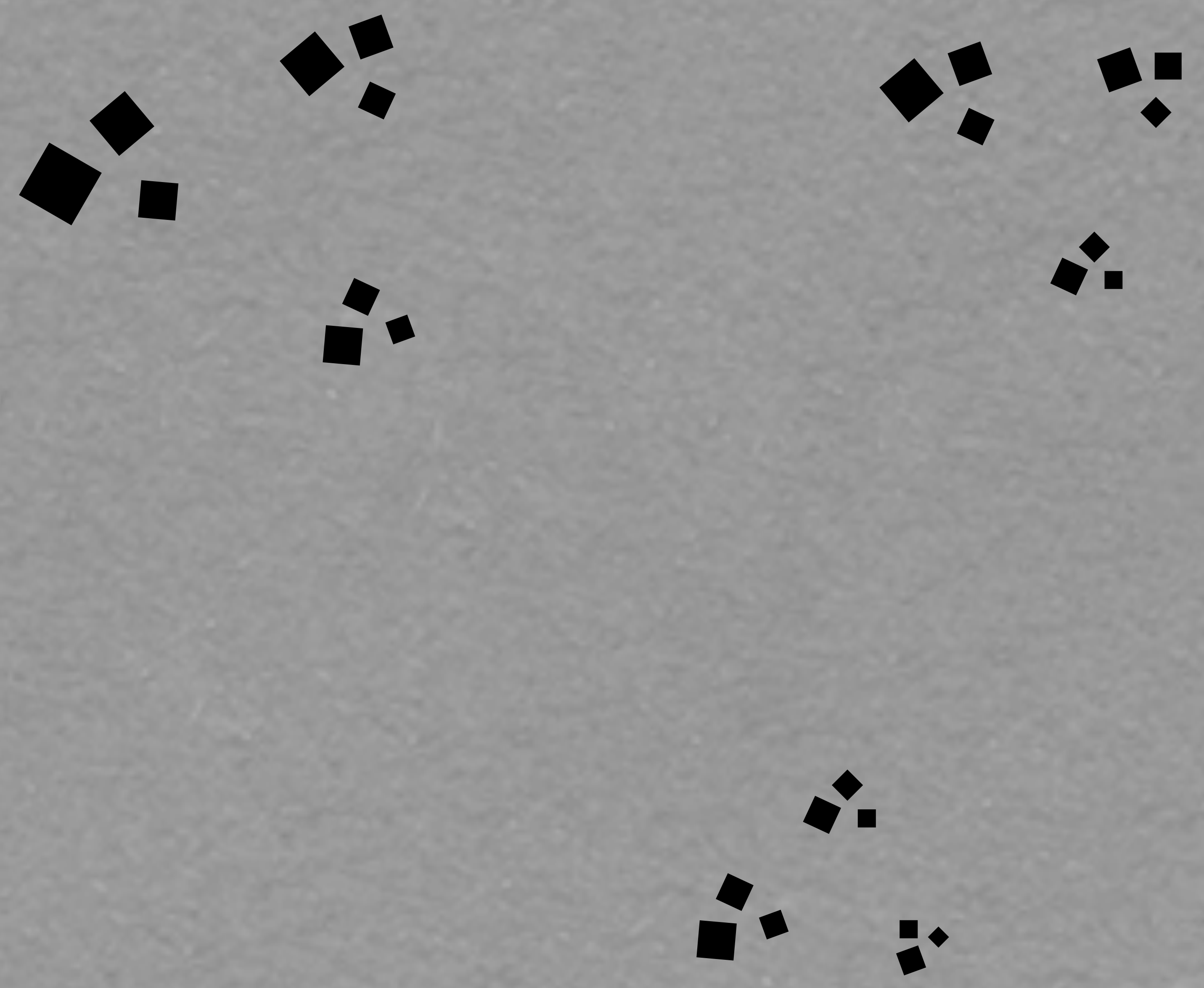
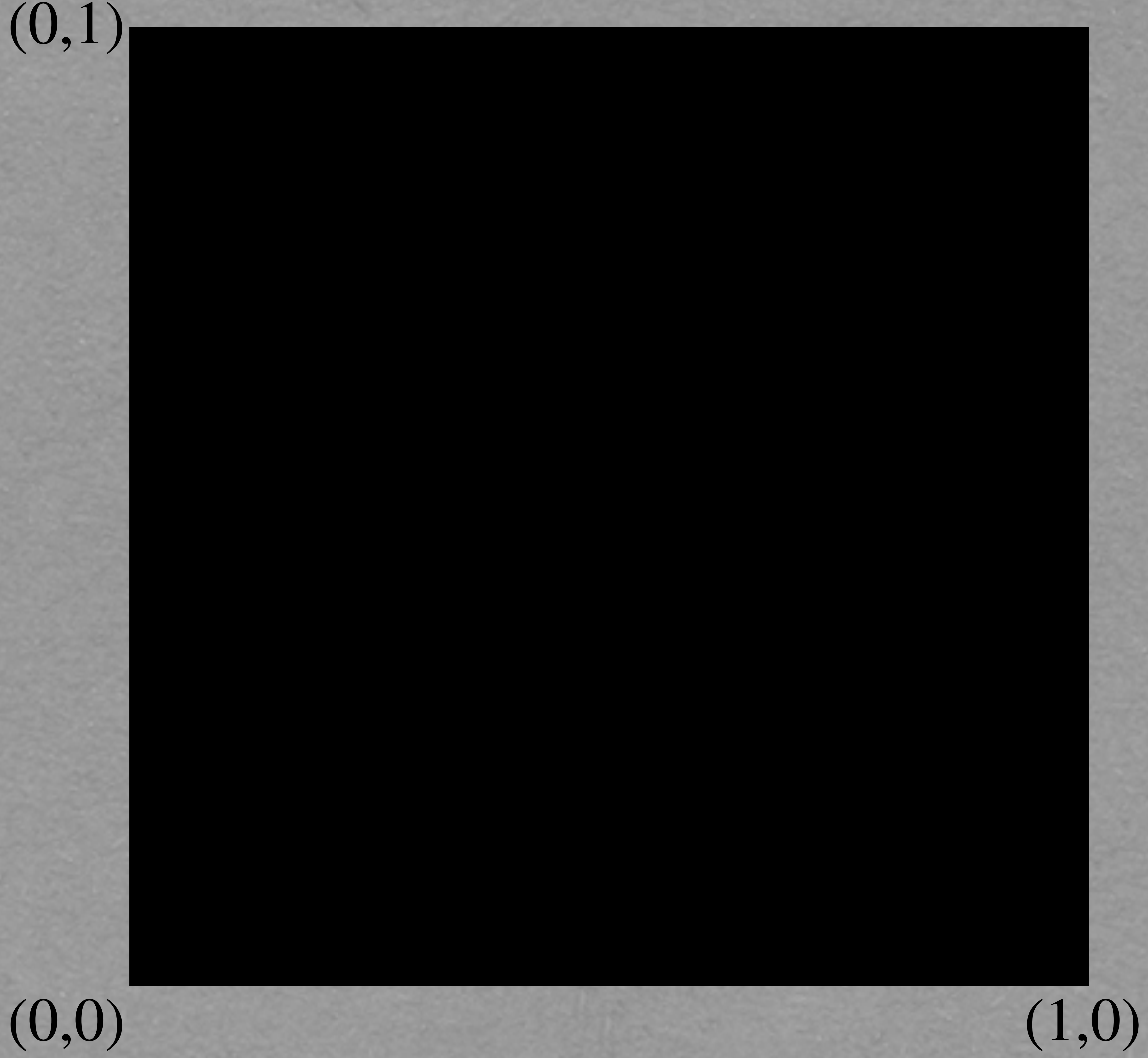


Example

- Sierpiński triangle
  - $\phi_1(x, y) = \frac{1}{2}(x, y)$
  - $\phi_2(x, y) = \frac{1}{2}(x, y) + (1/2,0)$
  - $\phi_3(x, y) = \frac{1}{2}(x, y) + (1/4,1/2)$



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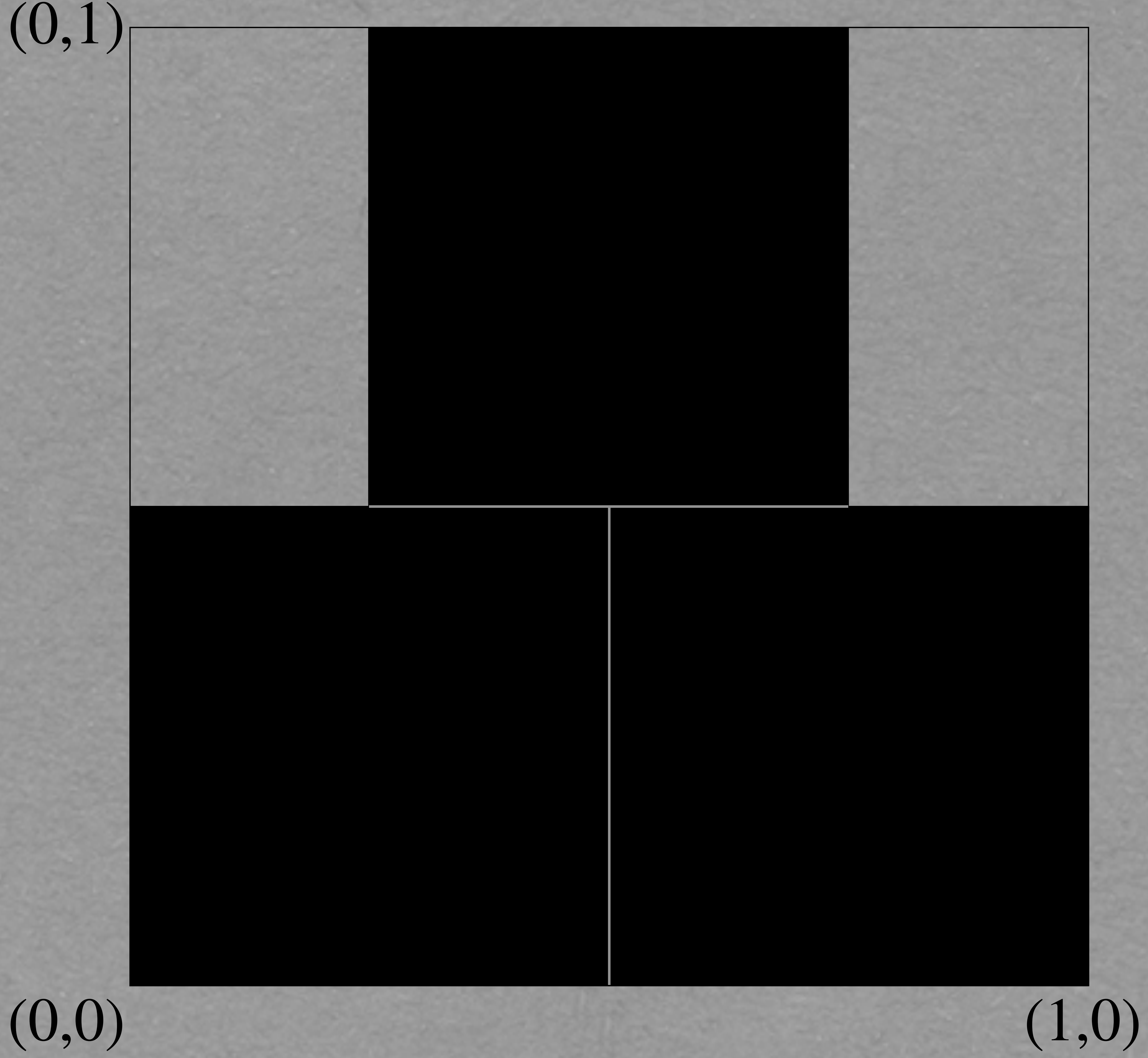

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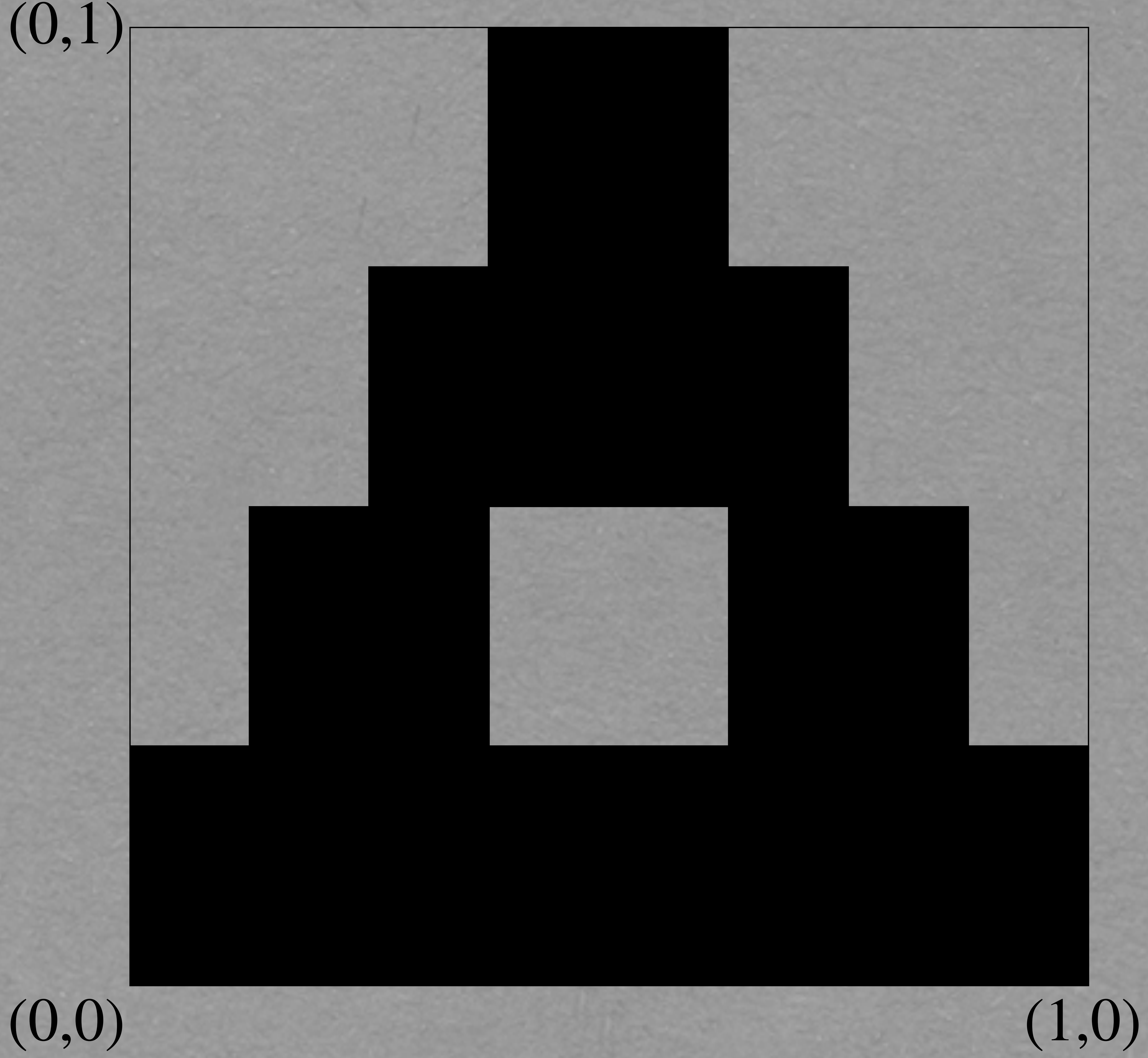

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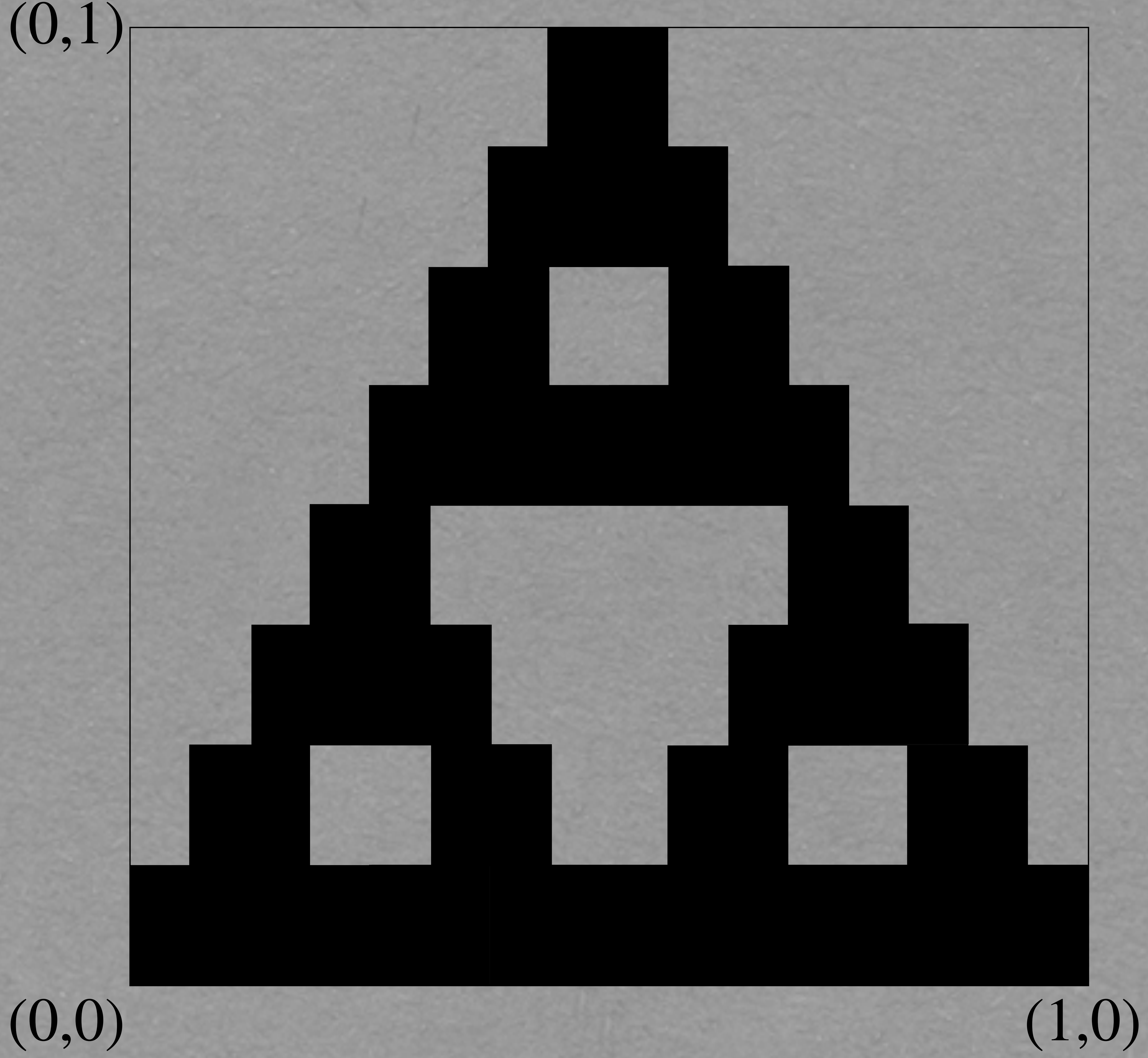

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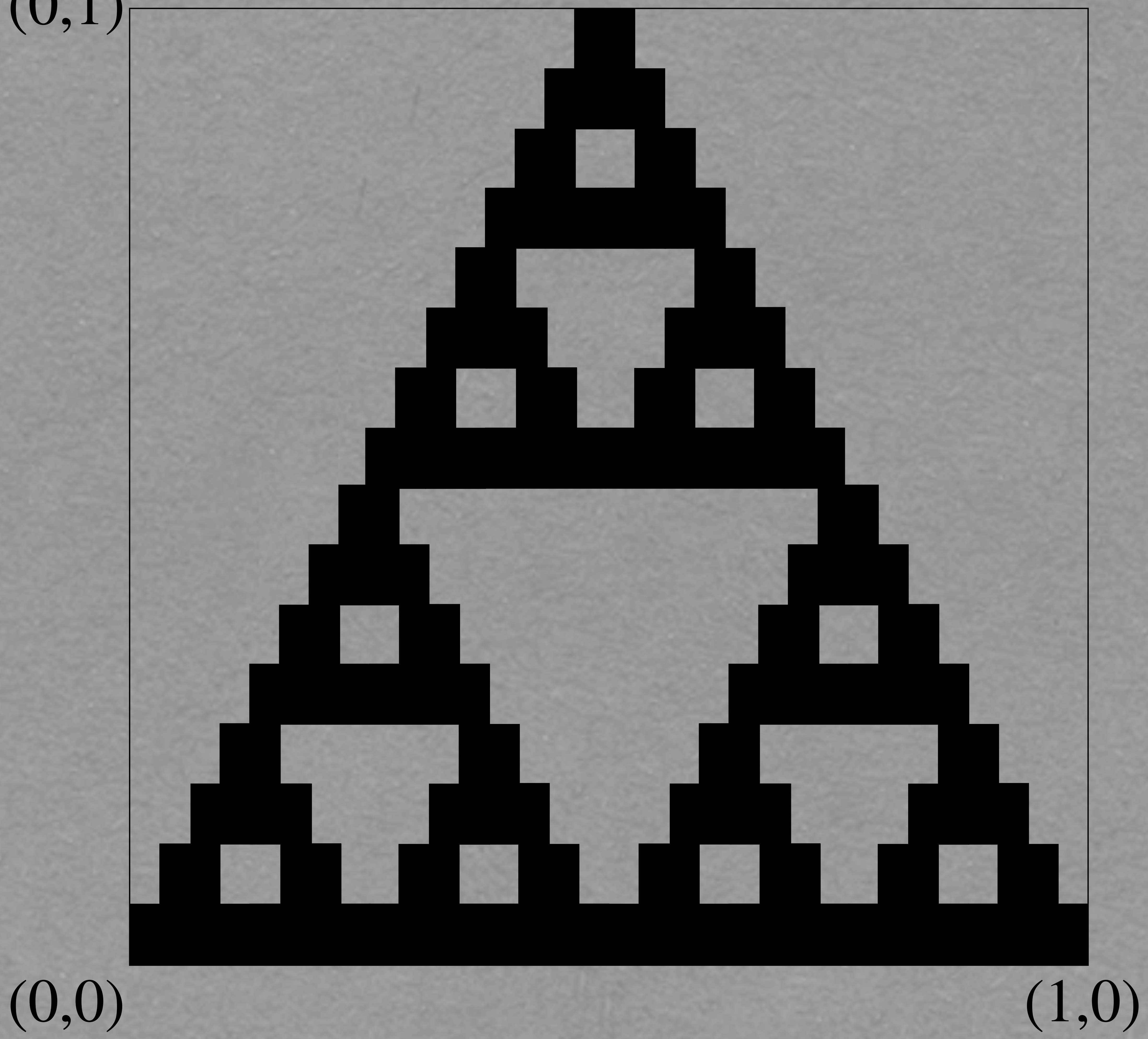


- Remarks/definitions:
  - If the union in $K = \bigcup_{1 \le i \le n} \phi_i(K)$ is disjoint, the IFS satisfies the **strong separation condition (SSC)**
  - If all of the ϕ_i are affine, the attractor K is a **self-affine set.** If all of the ϕ_i are similarity maps (scaling + isometry), K is a **self-similar set**.
- Without these conditions, very hard to analyze

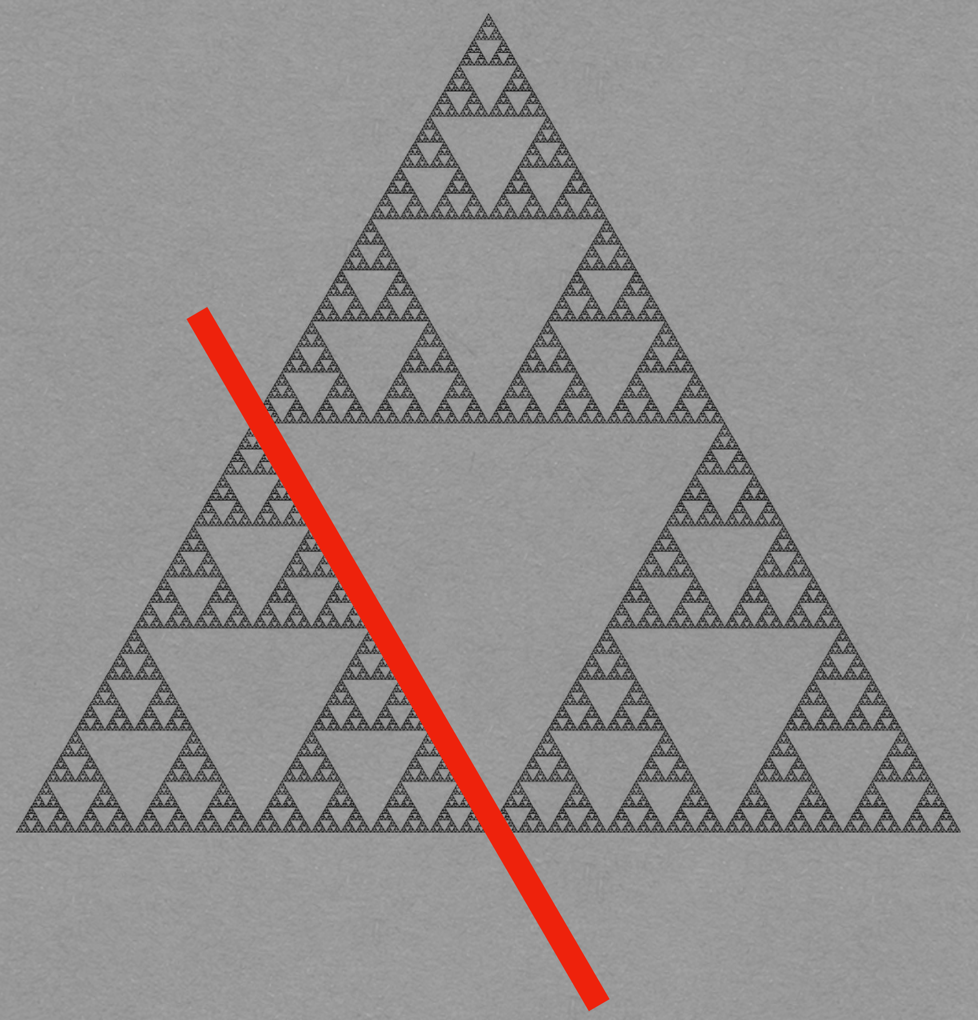
Dimension

- A natural quantity associated to a fractal set is its **Hausdorff dimension**
- Nice properties:
  - $\dim_H(\,\cdot\,) = 0,\ \dim_H(\,-\,) = 1,\ \dim_H(\blacksquare) = 2$, etc.
  - $\dim_H\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sup_{i\in\mathbb{N}} \dim_H(A_i)$
  - $\dim_H(A \times B) = \dim_H(A) + \dim_H(B)$ (under some conditions on A and B)
  - $\dim_H(f(A)) = \dim_H(A)$ if f is bi-Lipschitz

Slices of fractals

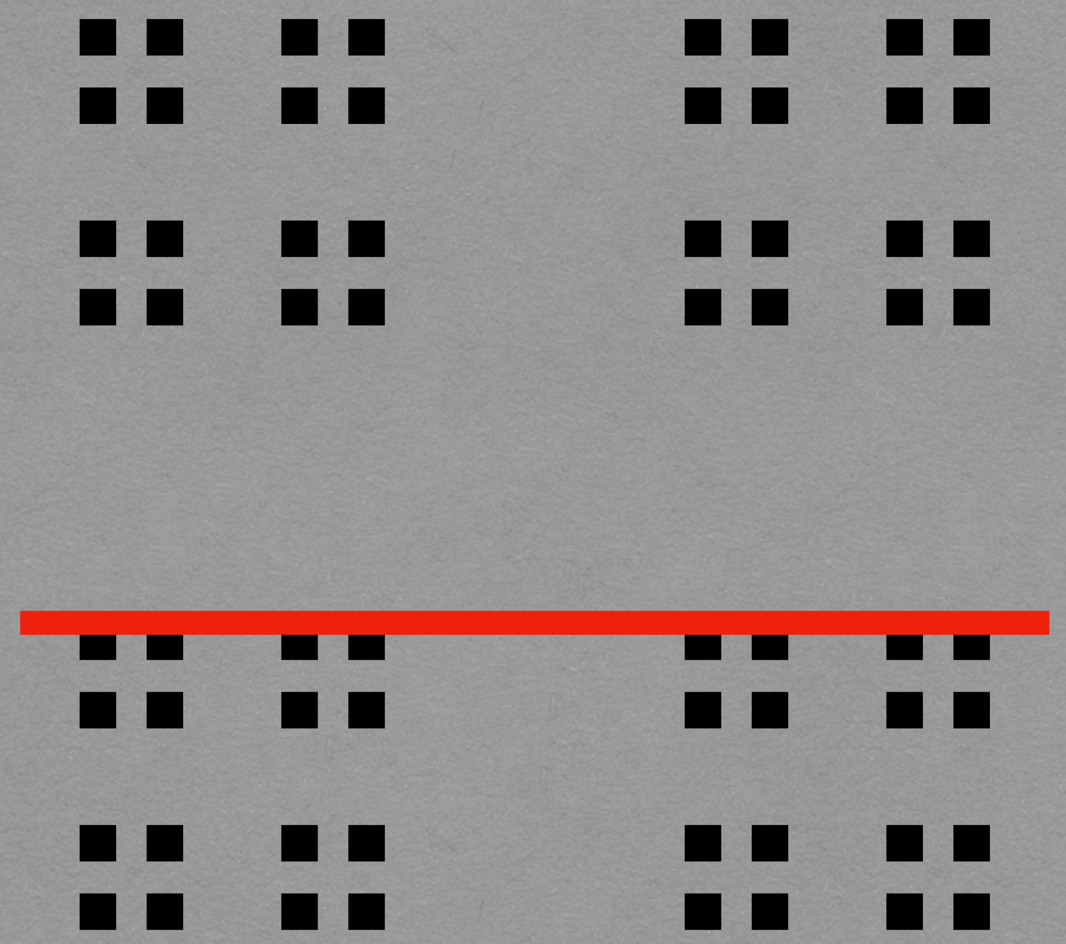
- **Theorem (Marstrand, 1950s):** Let $A \subseteq \mathbb{R}^2$. Then for a.e. line L, $\dim(A \cap L) \leq \max(0, \dim(A) - 1)$.
  - More generally, if $A \subseteq \mathbb{R}^d$, then $\dim(A \cap W) \leq \max(0, \dim(A) - \operatorname{codim}(W))$ for a.e. affine subspace W.
- Marstrand's theorem holds for any set (doesn't even have to be measurable!). If A has nice fractal structure maybe Marstrand's theorem is true for **every** line L.
  - Conjectured in various forms by Furstenberg

Obvious obstructions



$$\dim(A \cap L) = 1$$

$$\dim(A) - 1 = \frac{\log(3)}{\log(2)} - 1 \approx 0.58$$

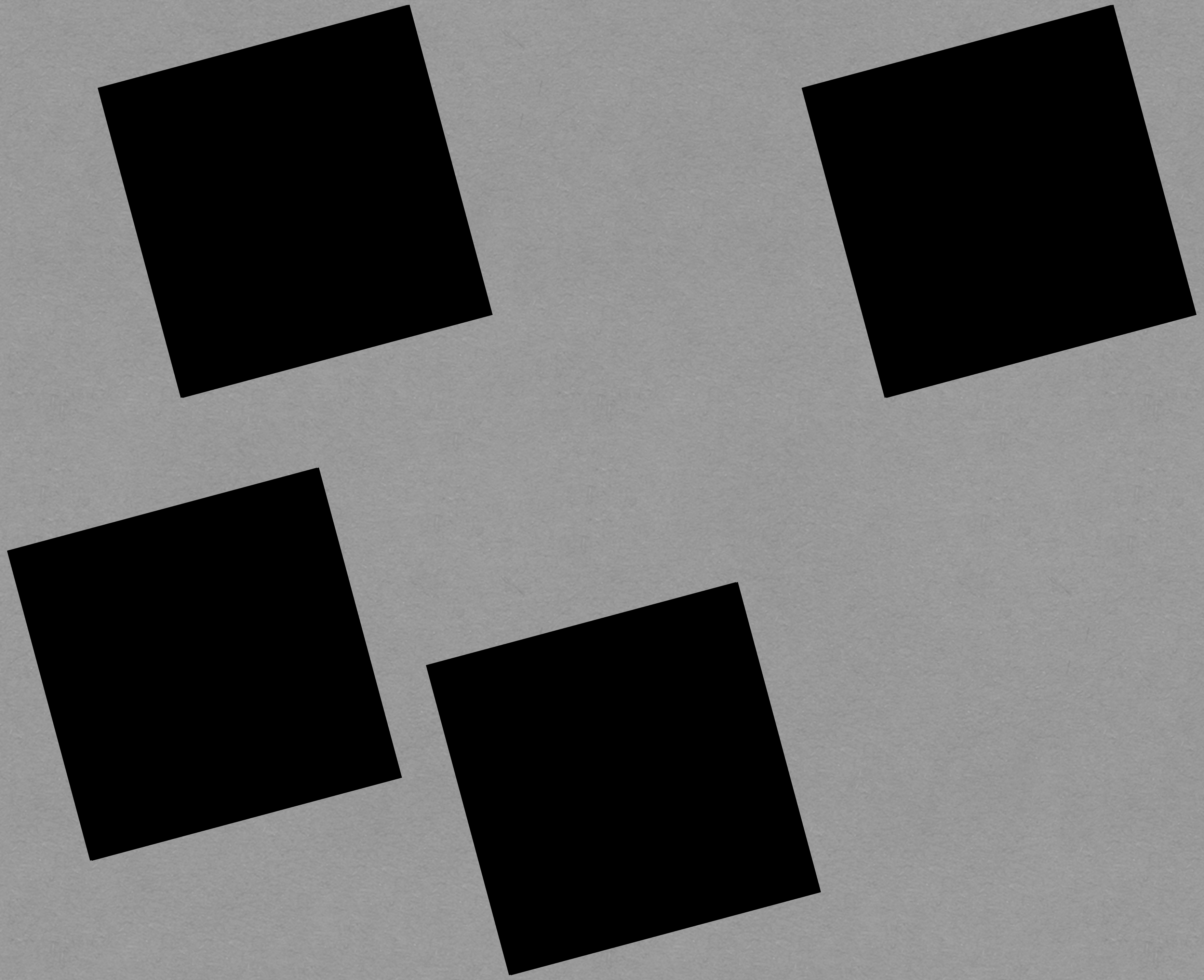


$$\dim(A \cap L) = \frac{\log(2)}{\log(3)} \approx 0.63$$

$$\dim(A) - 1 = \frac{\log(4)}{\log(3)} - 1 \approx 0.26$$

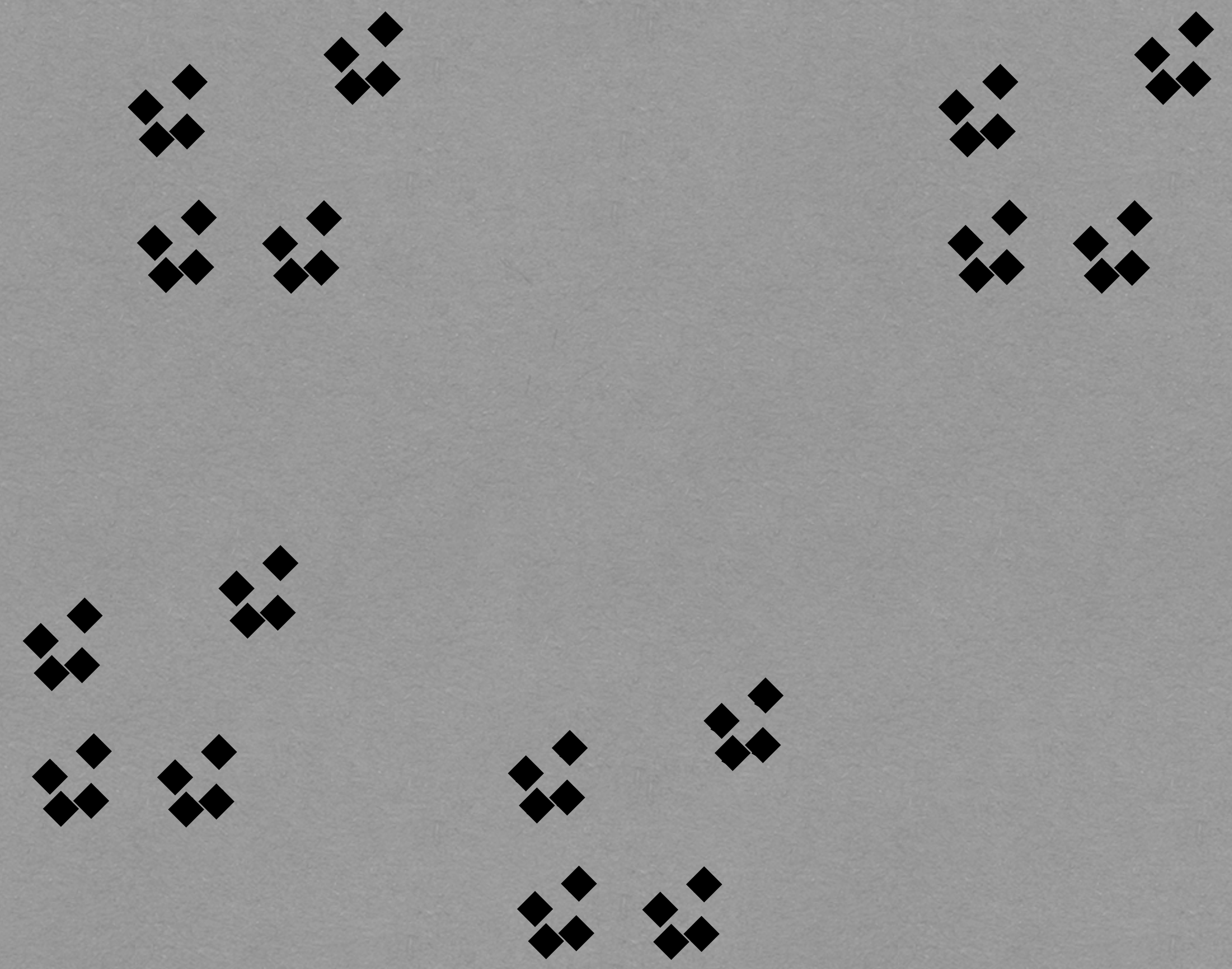
- From now, make the following assumptions:
  - Each ϕ_i is a **similarity**
  - The attractor K satisfies the **SSC**
  - Each ϕ_i has the **same rotation part** $\theta \notin 2\pi\mathbb{Q}$
- For example:

[redacted]



Attractor system

- The attractor K can be turned into a dynamical system
- Define $S : K \to K$ to be the local inverse to the ϕ_i , i.e. $S(z) = \phi_i^{-1}(z)$ for $z \in \phi_i(K)$ (well-defined by SSC)



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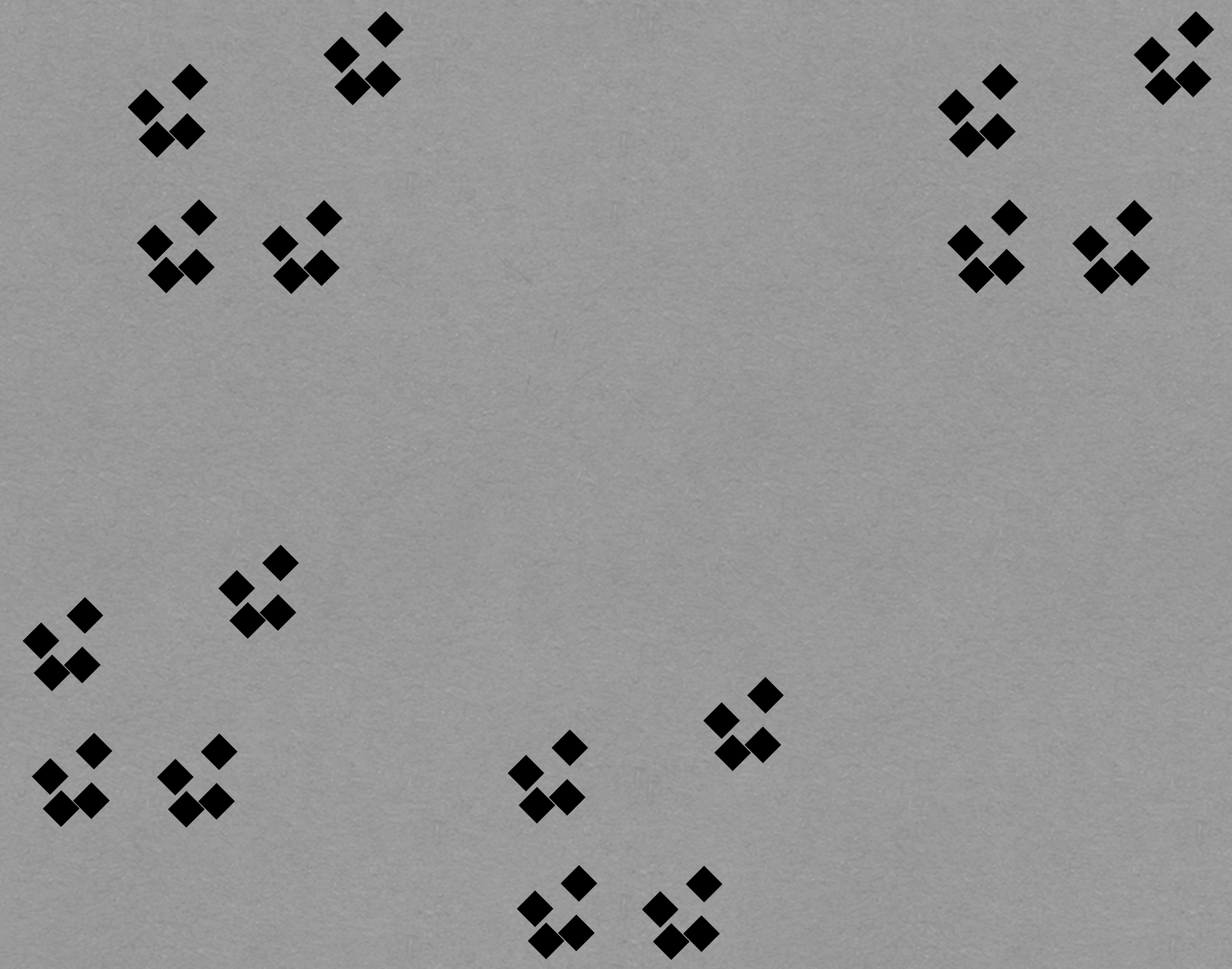
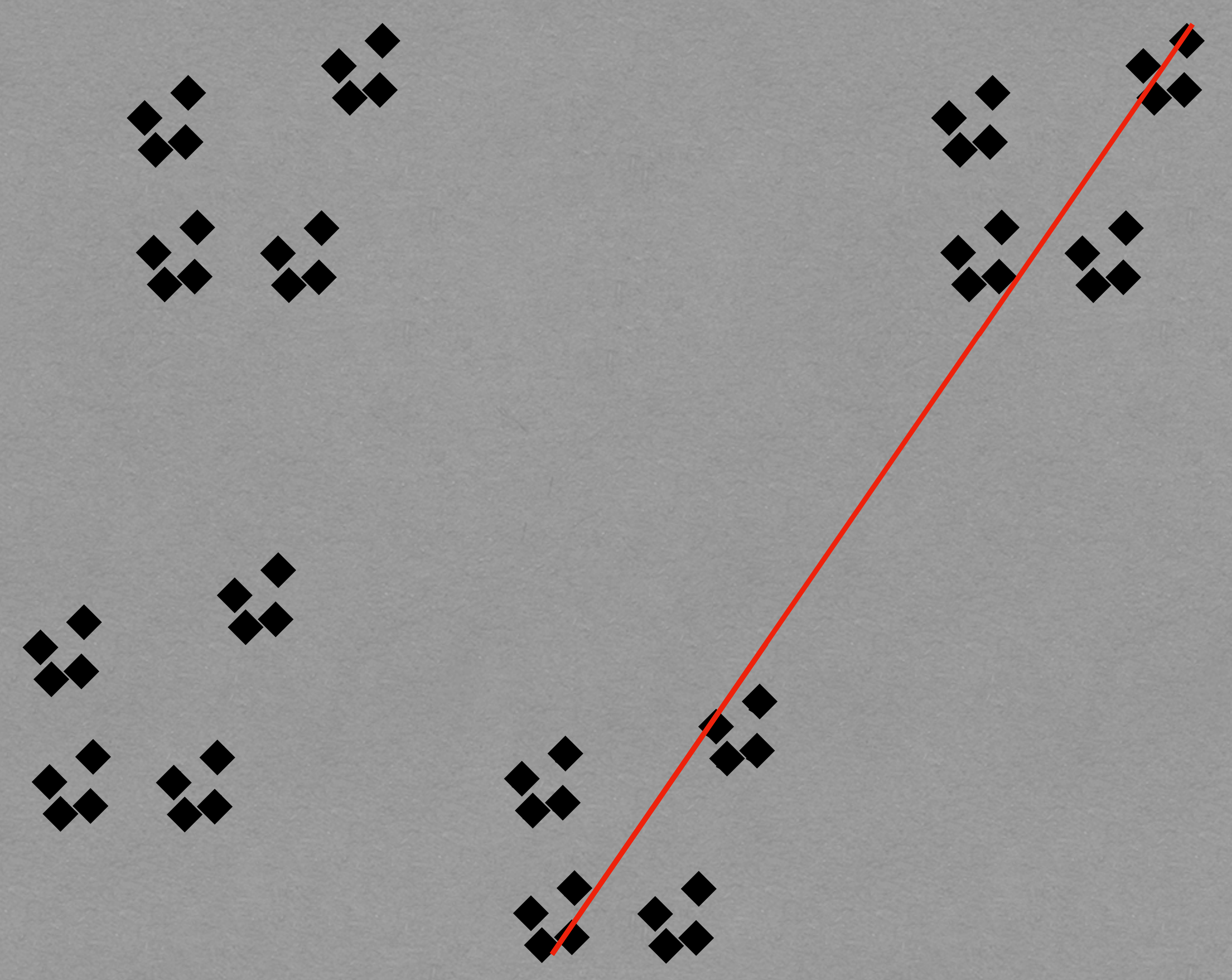


An observation

- Suppose there exists L with $\dim(K \cap L) > \dim(K) - 1$.
  - L points in the direction of $e^{2\pi i t_0}$



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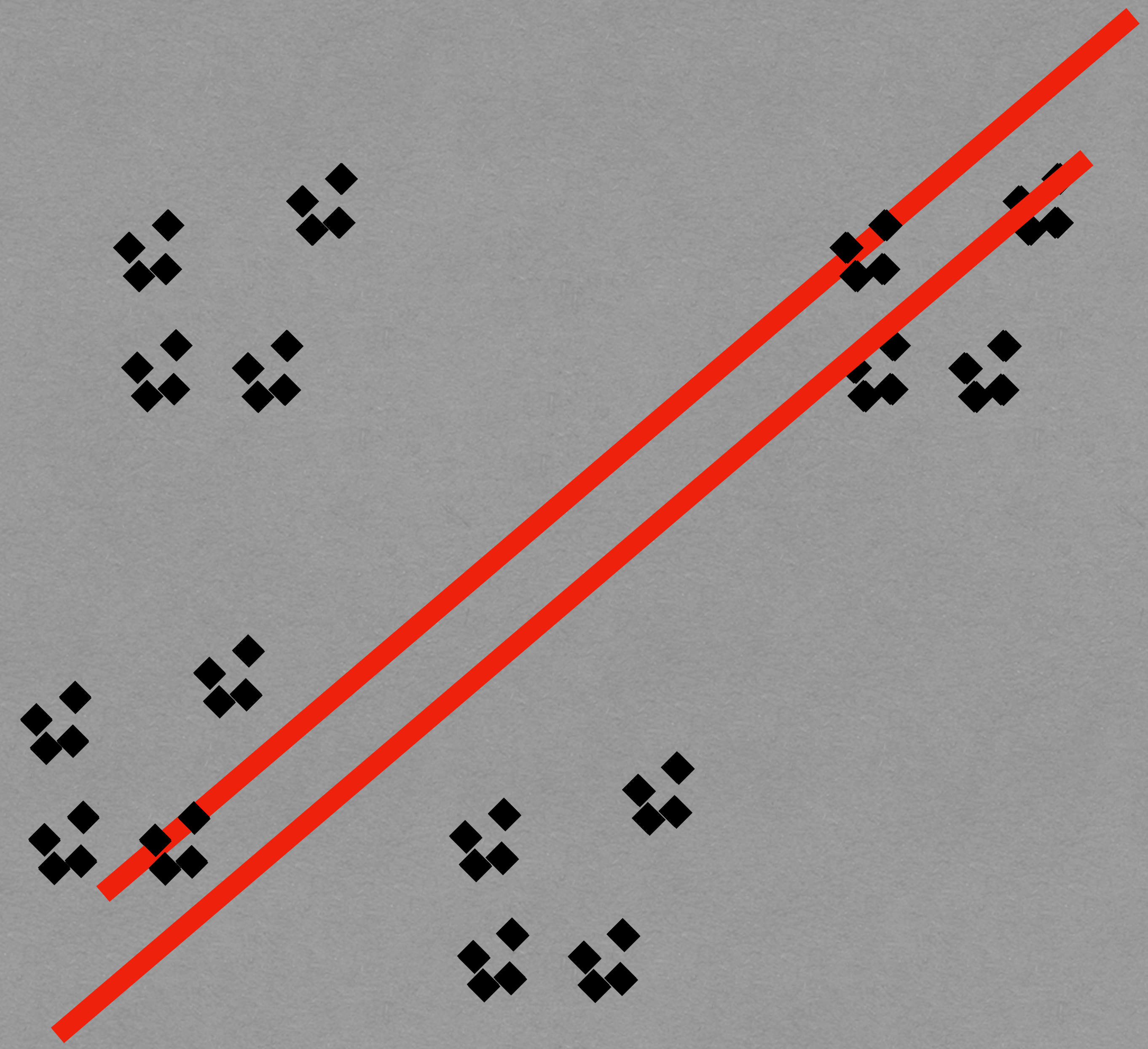

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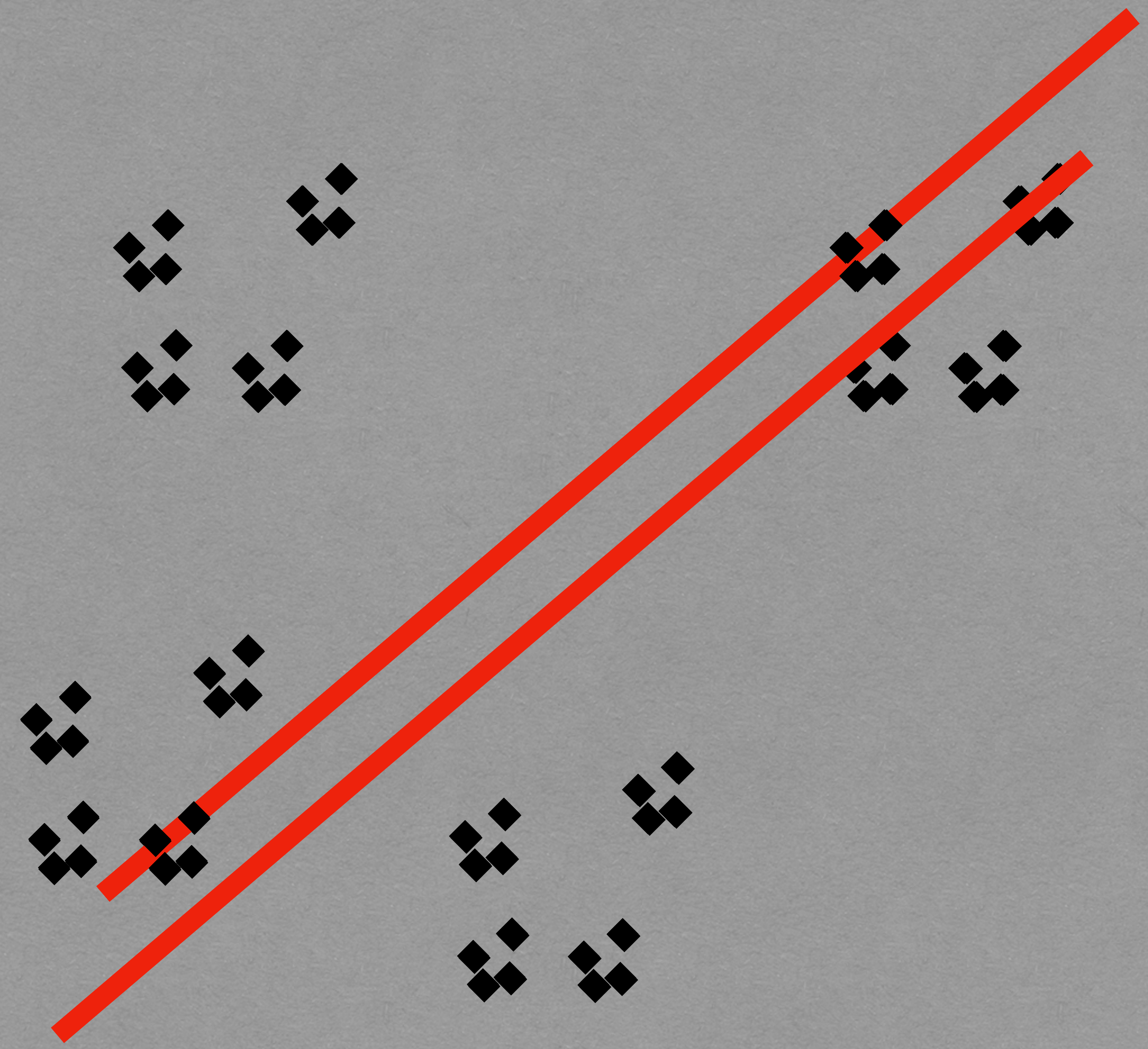


  - $S(K \cap L)$ is a union of several slices, each in the direction $e^{2\pi i(t_0 - \theta)}$, and at least one of them also has dimension $> \dim(K) - 1$

An observation

- Iterate this procedure: for every $n \geq 0$ there is a line L_n in the direction $e^{2\pi i(t_0 - n\theta)}$ such that $\dim(K \cap L_n) > \dim(K) - 1$
  - One bad slice $\longrightarrow$ a lot of bad slices whose slopes are dense in S^1
- Furstenberg (1960s) was able to push this observation much further with two key insights:
  - Introduce **randomness** into the picture
  - Work with dimensions of **measures** rather than dimensions of **sets**

Dimension of measures

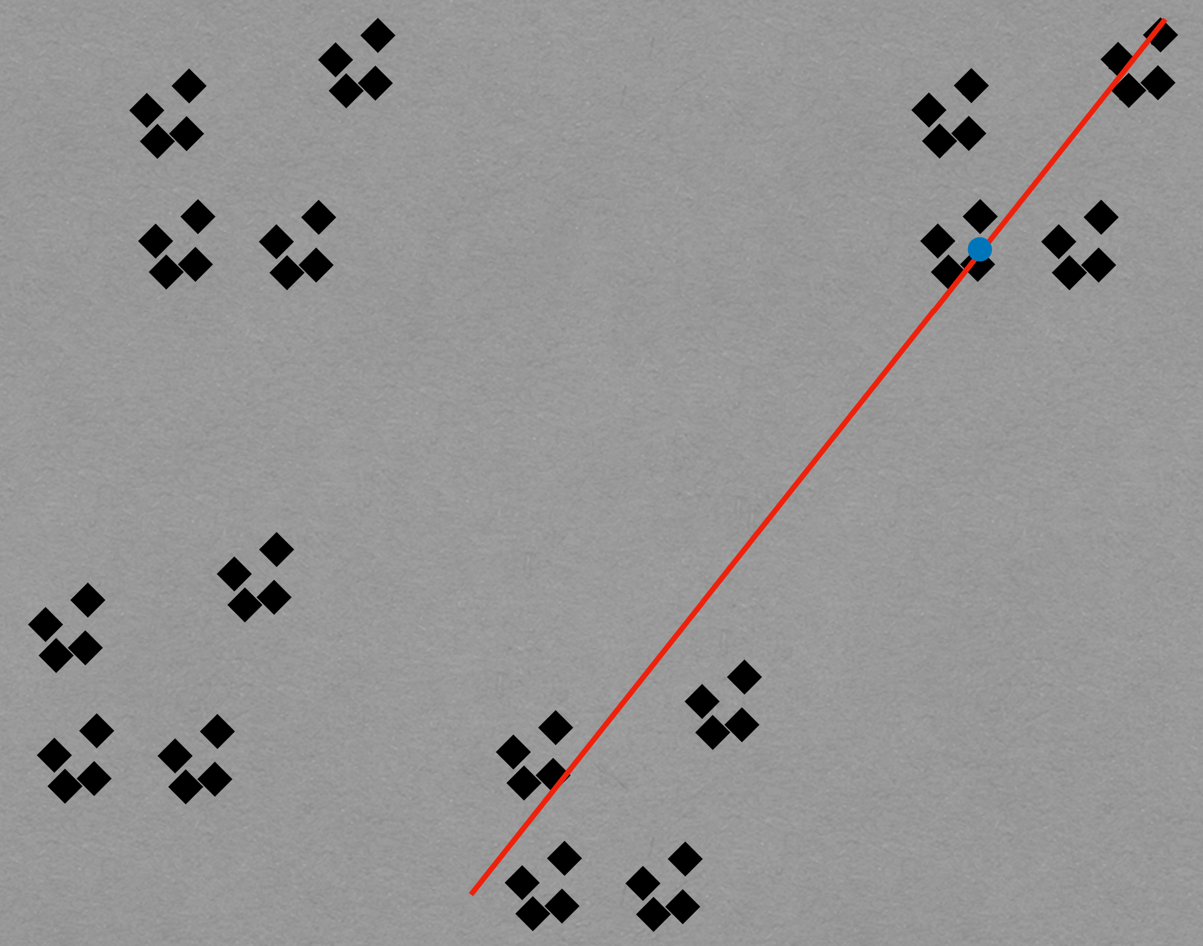
- Say a probability measure μ on $\mathbb{R}^d$ is α**-regular** if $\mu(B_r(x)) \lesssim r^\alpha$ as $r \searrow 0$ for all x.
  - Example: d-dimensional Lebesgue measure is d-regular
  - Example: a point mass is 0-regular (and not α-regular for any larger α)

Dimension of measures

- Strong relationship between regularity of measures and dimensions of sets they can "see"
  - If μ is α-regular and $\dim(A) < \alpha$, then $\mu(A) = 0$.
  - **"Frostman's lemma":** If $\dim(A) \geq \alpha$, then there exists an α-regular μ supported on A.
- Morally, there is a rough correspondence
  "$\dim(\mu)$" $:= \sup\{\alpha : \mu$ is α-regular$\}$ "$=$" $\inf\{\dim(A) : \mu(A) > 0\}$

Magnification dynamics

- Idea: run the attractor system, but keep track of more data
- "Zooming in" dynamics on $K \times \mathbb{T} \times \mathrm{Prob}(K)$:
  - $M : (z, t, \nu) \mapsto (Sz, t - \theta, S_*(\nu_z))$, where $\nu_z := \nu$ conditioned on the piece $\phi_i(K)$ that contains the point z



Furstenberg's construction

- Suppose there is a slice L in the direction $e^{2\pi i t_0}$ with $\dim(K \cap L) =: \alpha > \dim(K) - 1$
  - Frostman's lemma $\longrightarrow$ let ν_0 be an α-regular measure supported on $K \cap L$
- Let $\bar{\mu}_0 = \nu_0 \times \delta_{t_0} \times \delta_{\nu_0} \in \mathrm{Prob}(K \times \mathbb{T} \times \mathrm{Prob}(K))$ ("introduce randomness")
- Let $\bar{\mu}$ be a weak-* limit point of the sequence $\bar{\mu}_n := \frac{1}{n} \sum_{j=0}^{n-1} M_*^j \bar{\mu}_0$ (Krylov-Bogolyubov machine)
  - $\bar{\mu}$ is an M-invariant measure on $K \times \mathbb{T} \times \mathrm{Prob}(K)$ supported on $\{(z, t, \nu) : \nu$ is α-dimensional and $\nu(L_{z,t}) = 1\}$

The line through the point z in the direction $e^{2\pi i t}$

Furstenberg's construction

- The projection of $\bar{\mu}$ onto the $\mathbb{T}$ coordinate is invariant for the **irrational** circle rotation $t \mapsto t - \theta$
  - Must be Lebesgue measure
- Conclusion:
  - **Theorem (Furstenberg):** Suppose there exists a line L with $\dim(K \cap L) > \alpha$. Then for Lebesgue-a.e. $t \in \mathbb{T}$ there exists some line L_t in the direction $e^{2\pi i t}$ such that $\dim(K \cap L_t) > \alpha$ also.

- The conclusion of Furstenberg's theorem should translate to an upper bound on α in terms of $\dim(K)$ (think Kakeya problem)
  - Turned out to be hard to get the "correct" value
- **Theorem (2019):** Let $\{\phi_1, \ldots, \phi_n\}$ be a **similarity** IFS in $\mathbb{R}^2$ satisfying the **SSC** and such that each ϕ_i has **common rotation part** $\theta \notin 2\pi\mathbb{Q}$. Let K be the attractor. Then $\dim(K \cap L) \leq \max(0, \dim(K) - 1)$ for **every** line L.
  - Independent & simultaneous proofs by Pablo Shmerkin and Meng Wu (appeared in back-to-back Annals issues)

Generalizations/extensions

- For what other sets K does Marstrand's theorem hold for **every** line/affine subspace?
- $K = A \times B$, where $A, B \subseteq \mathbb{T}$ are $(\times 2), (\times 3)$-invariant respectively (not including lines parallel to axes)
  - Also proven by Shmerkin/Wu
- $K =$ attractor of IFS with different irrational rotation parts
  - ????
- $K =$ attractor of IFS in higher dimensions
  - ????

Generalizations/extensions

- Projections:
  - **Theorem (Marstrand):** Let $A \subseteq \mathbb{R}^2$. Then for a.e. line L, $\dim(\Pi_L A) = \max(1, \dim(A))$, where Π_L is orthogonal projection onto the line L.
- Can also apply magnification dynamics to this problem
- Hochman & Shmerkin (2012) used similar ideas to prove the above holds for **every** line L, for many nice fractals A.