Slicing theorems for planar self-similar sets

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Classical slicing theorem

Theorem (Marstrand, 1950s)

Let $E \subseteq \mathbb{R}^2$ be Borel. Then

$$\dim(E \cap \ell) \ \leq \ \max(0, \dim(E) - 1).$$

for **almost every** line ℓ.

Here and throughout, $\dim(\cdot)$ is **Hausdorff dimension**

Classical slicing theorem

Question

What conditions on E imply that
$\dim(E \cap \ell) \leq \max(0, \dim(E) - 1)$ for **every** line ℓ?

- Conjectures by Furstenberg: when E has nice fractal structure
- Connections to intersections of Cantor sets, $(\times 2), (\times 3)$ conjecture, etc.
- What is "nice fractal structure"?

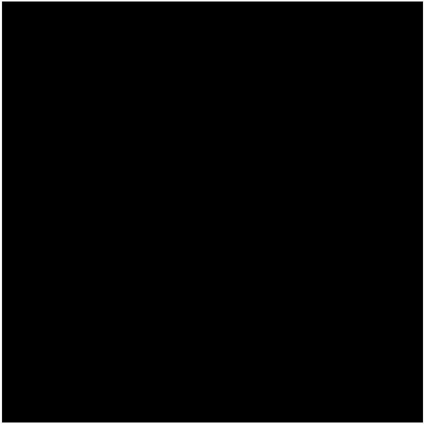
Iterated function systems

Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a finite set of contraction mappings in $\mathbb{R}^2$. Then there exists a **unique** compact set $K \subseteq \mathbb{R}^2$ such that

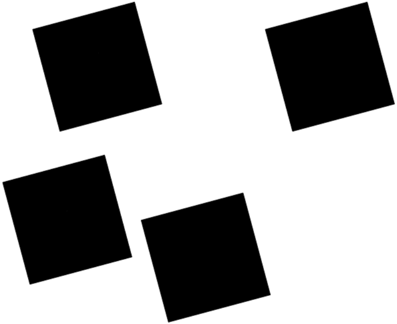
$$K = \bigcup_{1 \leq i \leq n} \varphi_i(K).$$

- Φ is an **iterated function system (IFS)**.
- K is the **attractor** of the IFS.

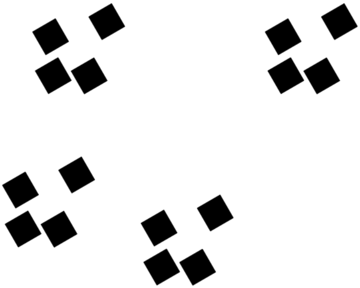
Iterated function systems



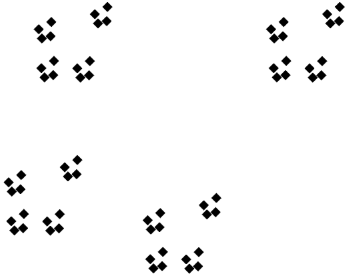
Iterated function systems



Iterated function systems



Iterated function systems



Self-similar sets

A **contracting similarity** in $\mathbb{R}^2$ is a map $\mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$\varphi(z) \;=\; \rho \cdot Az + q$$

where

- $0 < \rho < 1$
- A is a 2×2 orthogonal matrix (for simplicity, assume $A \in SO_2(\mathbb{R})$)
- $q \in \mathbb{R}^2$

If Φ consists only of contracting similarities, then the attractor K is a **self-similar set**.

Self-similar sets

Write $\varphi_i(z) = \rho_i \cdot R_{\theta_i} + q_i$ where R_θ denotes rotation by angle $2\pi\theta$, $\theta \in \mathbb{T}$.

Terminology

- **Strong separation condition (SSC):** the union in $K = \bigcup_{1 \leq i \leq n} \varphi_i(K)$ is disjoint.
- **Open set condition (OSC):** there exists an open set G such that $G \supseteq \bigcup_{1 \leq i \leq n} \varphi_i(G)$ and this union is disjoint.
- If all $\rho_i = \rho$, Φ is **homogeneous**.
- If all $\theta_i = \theta$, say Φ is **uniformly rotating**.

Recent result

Theorem (Shmerkin/Wu, 2019)

Let Φ be a self-similar IFS in $\mathbb{R}^2$ such that

1. Φ satisfies the **OSC**
2. Φ is **uniformly rotating** with angle $\theta \notin \mathbb{Q}$.
3. Φ is **homogeneous**

Then the attractor K satisfies $\dim(K \cap \ell) \leq \max(0, \dim(K) - 1)$ for every line ℓ.

- Independent & simultaneous proofs by Pablo Shmerkin and Meng Wu
- Tim Austin (2020) found a simpler version of Wu's proof
- Homogeneity assumption can be removed without much extra work

Recent result

Shmerkin's proof:

- Quantitative, uses additive combinatorics methods
- Roughly based on Hochman's work on the exact overlaps conjecture

Wu's proof:

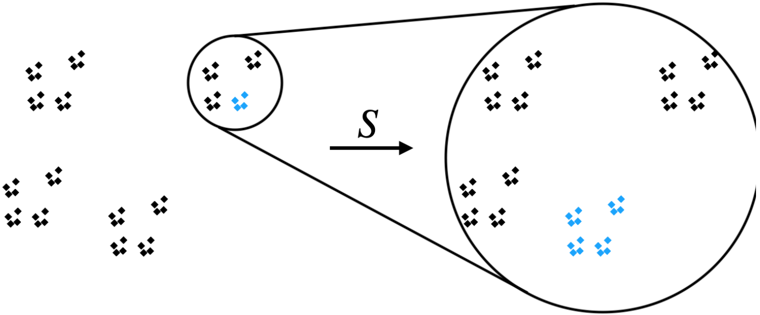
- Builds on Furstenberg's theory of magnification dynamics and CP distributions
- Clever application of Sinai's factor theorem
- Austin's proof also follows Furstenberg, main innovation is to avoid using Sinai's theorem

Related work

- Shmerkin/Wu: products of $(\times 2)-, (\times 3)-$invariant sets
- Algom, Algom-Wu: Bedford-McMullen carpets
- Bárány-Käenmäki-Yu: more general self-affine sets
- Yu: Quantitative/uniform versions
- Yu, Shmerkin, L.: Higher dimensional versions of $(\times 2), (\times 3)$

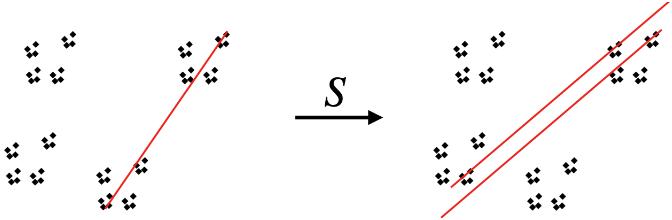
Dynamical approach

- The attractor K can be turned into a dynamical system
- $S : K \to K$ defined by $S|_{\varphi_i(K)} = \varphi_i^{-1}$
- Called the **attractor system**



Dynamical approach

Suppose $K \cap \ell$ is high-dimensional. Say the direction of ℓ is $e^{2\pi i t_0}$.



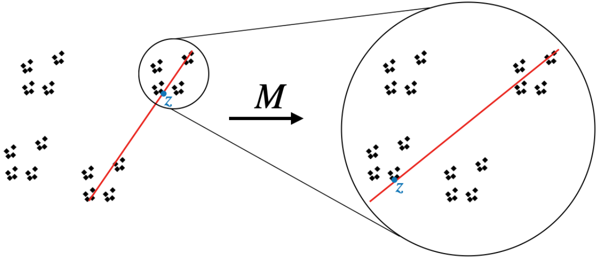
Observation

- $S(K \cap \ell)$ is a union of slices, each in the direction $e^{2\pi i(t_0-\theta)}$, and at least one of them is also high-dimensional
- We can iterate

Magnification dynamics

Furstenberg's great idea: attractor system + keeping track of "slice data"

- $X := K \times \mathbb{T} \times \text{Prob}(K)$
- Define $M : X \to X$ by $(z, t, \nu) \mapsto (Sz, t - \theta \bmod 1, S_*(\nu_z))$, where ν_z is defined to be ν conditioned on the piece $\varphi_i(K)$ that contains z
- Simulates "zooming in" to the point $z \in K$



Magnification dynamics

Theorem (Furstenberg, 1960s)

Suppose there is some line ℓ with $\dim(K \cap \ell) = \alpha > 0$. Then there is an ergodic M-invariant distribution $P \in \text{Prob}(X)$ (called a **CP distribution**) supported on

$$\{(z, t, \nu) \in X : \nu(\ell_{z,t}) = 1 \text{ and } \dim(\nu) \geq \alpha\},$$

where $\ell_{z,t}$ is the line through z in direction $e^{2\pi it}$.

Important fact: formula of the form

$$\dim(\nu) = \frac{\text{“average” entropy}}{\text{Lyapunov exponent}}$$

for P-typical measure ν

Magnification dynamics

The marginal of P on $\mathbb{T}$ is invariant for $t \mapsto t - \theta$, so it must be Lebesgue measure.

Corollary

Same assumption. Then for Lebesgue-a.e. t there is some line ℓ_t with direction $e^{2\pi i t}$ with $\dim(K \cap \ell_t) \geq \alpha$ also.

- Content of Wu/Austin proofs: how to upgrade Furstenberg's result to the full theorem
- Heuristic: K contains α-dimensional slices in a 1-dimensional set of directions "$\implies$" $\dim(K) \geq \alpha + 1$

New results

Theorem (L.)

Let Φ be a self-similar IFS in $\mathbb{R}^2$ such that

1'. Φ satisfies the **asymptotically weak separation condition (AWSC)**
2'. The rotation parts $\{\theta_1, \ldots, \theta_n\}$ are **"quasi-uniform"** and at most one θ_i is rational

Then the attractor K satisfies $\dim(K \cap \ell) \leq \max(0, \dim(K) - 1)$ for every line ℓ.

- AWSC was introduced by Feng & Hu (2009), other similar conditions studied by Lau & Ngai, etc.
  - Provably weaker than OSC
- "Quasi-uniform" roughly means that $\{1, \theta_1, \ldots, \theta_n\}$ spans a 2-dimensional vector space over $\mathbb{Q}$

New results

Theorem (L.)

Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a self-similar IFS in $\mathbb{R}^2$ satisfying the AWSC with rotation parts $\{\theta_1, \ldots, \theta_n\}$. Then for **Lebesgue-generic** $(\theta_1, \ldots, \theta_n)$, the attractor K satisfies $\dim(K \cap \ell) \leq \max(0, \dim(K) - 1/n)$ for every line ℓ.

- Can apply similar ideas for products of 1D attractors
- Likely to yield analogous result with "rotation parts" replaced by "log(contraction ratios)"
- Wu (2021): similar result, more deterministic

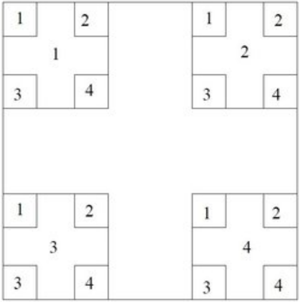
Weak separation

Symbolic representation:

- Symbolic space $\Omega = [n]^{\mathbb{N}}$
- Coding map $\pi : \Omega \to K$ defined by

$$\pi(x) = \lim_{N\to\infty} (\varphi_{x_1} \circ \cdots \circ \varphi_{x_N})(0)$$

  - Homeomorphism under SSC
- Left shift $\sigma : \Omega \to \Omega$
  - $(\Omega, \sigma) \simeq (K, S)$ under SSC



Weak separation

Symbolic version of magnification dynamics (still assuming uniform rotations for now):

$$(x, t, \nu) \ \mapsto \ (\sigma x,\ \ t - \theta \mod 1,\ \ \sigma_* \nu(\cdot \,|\, x_1))$$

Revised important fact: for symbolic ergodic CP distributions P, formula of form

$$\dim(\pi_* \nu) \ = \ \frac{\text{average } \textbf{conditional entropy over } \pi}{\text{Lyapunov exponent}}$$

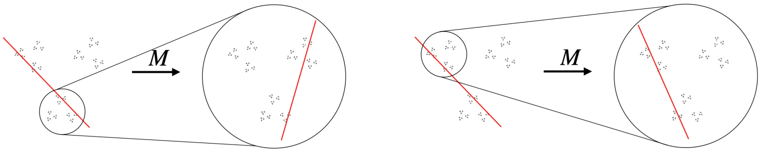
for P-typical ν

Non-uniform rotations

Let $\{\varphi_1, \ldots, \varphi_n\}$ be an IFS with rotation parts $\theta_1, \ldots, \theta_n \in \mathbb{T}$.

Adjust magnification dynamics accordingly:

$$(x, t, \nu) \mapsto (\sigma x,\ \ t - \theta_{x_1} \bmod 1,\ \ \sigma_* \nu(\cdot \,|\, x_1))$$



Non-uniform rotations

- Furstenberg's method still works $\longrightarrow$ ergodic CP distribution P supported on high-dimensional slices
- $\mathbb{T}$-marginal of P is *a priori* not an invariant measure for any system
- How to tell how smooth (high-dimensional) it is?

Non-uniform rotations

Ergodic theorem says the $\mathbb{T}$-marginal of P is obtained as the limiting distribution of "multi-rotation orbits":

Definition

Let $\theta_1, \ldots, \theta_n \in \mathbb{T}$ and fix $x \in [n]^{\mathbb{N}}$. The **multi-rotation orbit** generated by x is the sequence $\{t_n\}_{n \geq 1} \subseteq \mathbb{T}$ defined by $t_n = \theta_{x_1} + \cdots + \theta_{x_n}$.

The **limiting empirical distribution** associated to x is $\mu_x := \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{t_n}$.

- Study goes back to 1960s (Engelking)
- Goal: estimate smoothness of μ_x

Non-uniform rotations

Proposition

Suppose $\{\theta_1, \dots, \theta_n\}$ is "quasi-uniform" and at most one θ_i is rational. Let P_1 be a non-atomic ergodic shift-invariant measure on $[n]^{\mathbb{N}}$. Then $\overline{\dim}(\mu_x) = 1$ for P_1-a.e. x.

Proposition

For Lebesgue-a.e. $(\theta_1, \dots, \theta_n)$ and any $x \in [n]^{\mathbb{N}}$, $\overline{\dim}(\mu_x) \geq \frac{1}{n}$.

Similar results (for sets rather than measures) due to Feng-Xiong, Yu, Baker

Possible applications

Self-similar sets in $\mathbb{R}^d$, $d \geq 3$

- Multi-rotation orbits on S^{d-1}
- Fixed rotation orbits don't equidistribute
- Non-commutative

Limit sets of Kleinian groups

- Möbius transformations preserve circles $\longrightarrow$ can define a form of magnification dynamics for "circular slices"
- Analogue of multi-rotation orbits?