Slicing theorems for planar self-similar sets

Adam Lott
UCLA

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Classical slicing theorem

Theorem (Marstrand, 1950s)

Let $E \subseteq \mathbb{R}^2$ be Borel. Then

$$\dim(E \cap \ell) \leq \max(0, \dim(E) - 1).$$

for almost every line $\ell$.

Here and throughout, $\dim(\cdot)$ is **Hausdorff dimension**
Question
What conditions on $E$ imply that $\dim(E \cap \ell) \leq \max(0, \dim(E) - 1)$ for every line $\ell$?

- Conjectures by Furstenberg: when $E$ has nice fractal structure
- Connections to intersections of Cantor sets, ($\times 2$), ($\times 3$) conjecture, etc.
- What is “nice fractal structure”?
Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a finite set of contraction mappings in $\mathbb{R}^2$. Then there exists a unique compact set $K \subseteq \mathbb{R}^2$ such that

$$K = \bigcup_{1 \leq i \leq n} \varphi_i(K).$$

- $\Phi$ is an iterated function system (IFS).
- $K$ is the attractor of the IFS.
Iterated function systems
Iterated function systems
Iterated function systems
Iterated function systems
Self-similar sets

A contracting similarity in $\mathbb{R}^2$ is a map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the form

$$\varphi(z) = \rho \cdot Az + q$$

where

- $0 < \rho < 1$
- $A$ is a $2 \times 2$ orthogonal matrix (for simplicity, assume $A \in SO_2(\mathbb{R})$)
- $q \in \mathbb{R}^2$

If $\Phi$ consists only of contracting similarities, then the attractor $K$ is a self-similar set.
Write $\varphi_i(z) = \rho_i \cdot R_{\theta_i} + q_i$ where $R_\theta$ denotes rotation by angle $2\pi \theta$, $\theta \in \mathbb{T}$.

**Terminology**

- **Strong separation condition (SSC):** the union in $K = \bigcup_{1 \leq i \leq n} \varphi_i(K)$ is disjoint.
- **Open set condition (OSC):** there exists an open set $G$ such that $G \supseteq \bigcup_{1 \leq i \leq n} \varphi_i(G)$ and this union is disjoint.
- If all $\rho_i = \rho$, $\Phi$ is **homogeneous**.
- If all $\theta_i = \theta$, say $\Phi$ is **uniformly rotating**.
Recent result

**Theorem (Shmerkin/Wu, 2019)**

Let $\Phi$ be a self-similar IFS in $\mathbb{R}^2$ such that

1. $\Phi$ satisfies the **OSC**
2. $\Phi$ is **uniformly rotating** with angle $\theta \not\in \mathbb{Q}$.
3. $\Phi$ is **homogeneous**

Then the attractor $K$ satisfies $\dim(K \cap \ell) \leq \max(0, \dim(K) - 1)$ for every line $\ell$.

- Independent & simultaneous proofs by Pablo Shmerkin and Meng Wu
- Tim Austin (2020) found a simpler version of Wu’s proof
- Homogeneity assumption can be removed without much extra work
Recent result

Shmerkin’s proof:
- Quantitative, uses additive combinatorics methods
- Roughly based on Hochman’s work on the exact overlaps conjecture

Wu’s proof:
- Builds on Furstenberg’s theory of magnification dynamics and CP distributions
- Clever application of Sinai’s factor theorem
- Austin’s proof also follows Furstenberg, main innovation is to avoid using Sinai’s theorem
Related work

- Shmerkin/Wu: products of $(\times 2)$—, $(\times 3)$—invariant sets
- Algom, Algom-Wu: Bedford-McMullen carpets
- Bárány-Käenmäki-Yu: more general self-affine sets
- Yu: Quantitative/uniform versions
- Yu, Shmerkin, L.: Higher dimensional versions of $(\times 2), (\times 3)$
Dynamical approach

- The attractor $K$ can be turned into a dynamical system $S : K \rightarrow K$ defined by $S|_{\varphi_i(K)} = \varphi_i^{-1}$
- Called the **attractor system**
Dynamical approach

Suppose $K \cap \ell$ is high-dimensional. Say the direction of $\ell$ is $e^{2\pi i t_0}$.

Observation

- $S(K \cap \ell)$ is a union of slices, each in the direction $e^{2\pi i (t_0 - \theta)}$, and at least one of them is also high-dimensional.
- We can iterate.
Furstenberg’s great idea: attractor system + keeping track of “slice data”

- \( X := K \times \mathbb{T} \times \text{Prob}(K) \)
- Define \( M : X \to X \) by \((z, t, \nu) \mapsto (Sz, t - \theta \mod 1, S_*(\nu_z))\), where \( \nu_z \) is defined to be \( \nu \) conditioned on the piece \( \varphi_i(K) \) that contains \( z \)
- Simulates “zooming in” to the point \( z \in K \)
Magnification dynamics

Theorem (Furstenberg, 1960s)

Suppose there is some line $\ell$ with $\dim(K \cap \ell) = \alpha > 0$. Then there is an ergodic $M$-invariant distribution $P \in \text{Prob}(X)$ (called a CP distribution) supported on

$$\{(z, t, \nu) \in X : \nu(\ell_{z,t}) = 1 \text{ and } \dim(\nu) \geq \alpha\},$$

where $\ell_{z,t}$ is the line through $z$ in direction $e^{2\pi it}$.

Important fact: formula of the form

$$\dim(\nu) = \frac{\text{"average" entropy}}{\text{Lyapunov exponent}}$$

for $P$-typical measure $\nu$. 
Magnification dynamics

The marginal of $P$ on $\mathbb{T}$ is invariant for $t \mapsto t - \theta$, so it must be Lebesgue measure.

**Corollary**

Same assumption. Then for Lebesgue-a.e. $t$ there is some line $\ell_t$ with direction $e^{2\pi it}$ with $\dim(K \cap \ell_t) \geq \alpha$ also.

- Content of Wu/Austin proofs: how to upgrade Furstenberg’s result to the full theorem
- Heuristic: $K$ contains $\alpha$-dimensional slices in a 1-dimensional set of directions “$\iff$” $\dim(K) \geq \alpha + 1$
New results

Theorem (L.)

Let \( \Phi \) be a self-similar IFS in \( \mathbb{R}^2 \) such that

1' \( \Phi \) satisfies the **asymptotically weak separation condition (AWSC)**

2' The rotation parts \( \{\theta_1, \ldots, \theta_n\} \) are “quasi-uniform” and at most one \( \theta_i \) is rational

Then the attractor \( K \) satisfies \( \dim(K \cap \ell) \leq \max(0, \dim(K) - 1) \) for every line \( \ell \).

- AWSC was introduced by Feng & Hu (2009), other similar conditions studied by Lau & Ngai, etc.
  - Provably weaker than OSC
- “Quasi-uniform” roughly means that \( \{1, \theta_1, \ldots, \theta_n\} \) spans a 2-dimensional vector space over \( \mathbb{Q} \)
**Theorem (L.)**

Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a self-similar IFS in $\mathbb{R}^2$ satisfying the AWSC with rotation parts $\{\theta_1, \ldots, \theta_n\}$. Then for Lebesgue-generic $(\theta_1, \ldots, \theta_n)$, the attractor $K$ satisfies

$$\dim(K \cap \ell) \leq \max(0, \dim(K) - 1/n)$$

for every line $\ell$.

- Can apply similar ideas for products of 1D attractors
- Likely to yield analogous result with “rotation parts” replaced by “log(contraction ratios)”
- Wu (2021): similar result, more deterministic
Weak separation

Symbolic representation:

- Symbolic space $\Omega = [n]^\mathbb{N}$
- Coding map $\pi : \Omega \to K$
  
  defined by

$$\pi(x) = \lim_{N \to \infty} (\varphi_{x_1} \circ \cdots \circ \varphi_{x_N})(0)$$

- Homeomorphism under SSC
- Left shift $\sigma : \Omega \to \Omega$
  
  $$(\Omega, \sigma) \simeq (K, S)$$ under SSC
Weak separation

Symbolic version of magnification dynamics (still assuming uniform rotations for now):

$$(x, t, \nu) \mapsto (\sigma x, t - \theta \mod 1, \sigma_* \nu(\cdot | x_1))$$

Revised important fact: for symbolic ergodic CP distributions $P$, formula of form

$$\dim(\pi_* \nu) = \frac{\text{average conditional entropy over } \pi}{\text{Lyapunov exponent}}$$

for $P$-typical $\nu$
Non-uniform rotations

Let \( \{ \varphi_1, \ldots, \varphi_n \} \) be an IFS with rotation parts \( \theta_1, \ldots, \theta_n \in \mathbb{T} \).

Adjust magnification dynamics accordingly:

\[
(x, t, \nu) \mapsto (\sigma x, \ t - \theta x_1 \mod 1, \ \sigma \nu(\cdot | x_1))
\]
Non-uniform rotations

- Furstenberg’s method still works $\rightarrow$ ergodic CP distribution $P$ supported on high-dimensional slices
- $\mathbb{T}$-marginal of $P$ is a priori not an invariant measure for any system
- How to tell how smooth (high-dimensional) it is?
Ergodic theorem says the $\mathbb{T}$-marginal of $P$ is obtained as the limiting distribution of “multi-rotation orbits”:

**Definition**

Let $\theta_1, \ldots, \theta_n \in \mathbb{T}$ and fix $x \in [n]^\mathbb{N}$. The multi-rotation orbit generated by $x$ is the sequence $\{t_n\}_{n \geq 1} \subseteq \mathbb{T}$ defined by $t_n = \theta_{x_1} + \cdots + \theta_{x_n}$.

The **limiting empirical distribution** associated to $x$ is $\mu_x := \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{t_n}$.

- Study goes back to 1960s (Engelking)
- Goal: estimate smoothness of $\mu_x$
Non-uniform rotations

Proposition

Suppose \( \{\theta_1, \ldots, \theta_n\} \) is “quasi-uniform” and at most one \( \theta_i \) is rational. Let \( P_1 \) be a non-atomic ergodic shift-invariant measure on \( [n]^\mathbb{N} \). Then \( \dim(\mu_x) = 1 \) for \( P_1 \)-a.e. \( x \).

Proposition

For Lebesgue-a.e. \( (\theta_1, \ldots, \theta_n) \) and any \( x \in [n]^\mathbb{N} \), \( \dim(\mu_x) \geq \frac{1}{n} \).

Similar results (for sets rather than measures) due to Feng-Xiong, Yu, Baker
Possible applications

Self-similar sets in $\mathbb{R}^d$, $d \geq 3$

- Multi-rotation orbits on $S^{d-1}$
- Fixed rotation orbits don’t equidistribute
- Non-commutative

Limit sets of Kleinian groups

- Möbius transformations preserve circles $\rightarrow$ can define a form of magnification dynamics for “circular slices”
- Analogue of multi-rotation orbits?