Non-dominance of amenable group actions with zero entropy

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UMD Workshop on Dynamical Systems 1 April 2022

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G = discrete amenable group

 $(X, \mu, T) =$ ergodic measure preserving action of G

What properties of T are preserved by a "generic" ergodic extension?

$$(\overline{X}, \overline{\mu}, \overline{T})$$
 \downarrow
 (X, μ, T)

Examples (Glasner-Thouvenot-Weiss 2021):

- A generic \overline{T} has the same entropy as T
- If T is Bernoulli, then a generic \overline{T} also is

What does "generic extension" mean?

Rokhlin's skew-product theorem: every ergodic extension $(\overline{X}, \overline{\mu}, \overline{T})$ is isomorphic to a skew product system $(X \times I, \mu \times m, T_{\alpha})$ defined by $T_{\alpha}^{g}(x, t) := (T^{g}x, \alpha(g, x)t)$, where

•
$$I = [0, 1]$$
, $m =$ Lebesgue measure

• $\alpha : G \times X \to \operatorname{Aut}(I, m)$ satisfies the **cocycle condition**: $\alpha(h, T^g x) \circ \alpha(g, x) = \alpha(hg, x).$

The space of all cocycles carries a natural topology.

A property **P** is said to hold for a generic extension if there is a dense G_{δ} set of cocycles α for which T_{α} has property **P**.

Dominant systems

Definition

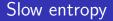
A system (X, μ, T) is **dominant** if a generic extension is *isomorphic* to (X, μ, T) .

Theorem (Austin-Glasner-Thouvenot-Weiss, 2021)

- **1** For $G = \mathbb{Z}$: a system is dominant **if and only if** it has positive entropy.
- 2 For all amenable G: if (X, μ, T) has positive entropy, then it is dominant.

Theorem (L)

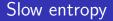
For all amenable G: if (X, μ, T) has zero entropy, then it is not dominant.



How to distinguish between systems of zero entropy? (Recall a generic extension also has zero entropy)

Theory of slow entropy developed by Katok and Thouvenot

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Simple case: $G = \mathbb{Z}$, $X = \Lambda^{\mathbb{Z}}$, T = left shift, $\mu = \text{shift-invariant}$ measure

Hamming (pseudo-)metrics on X: $d_n(x, y) = \frac{1}{2n+1} \sum_{j=-n}^n \mathbb{1}_{x_j \neq y_j}$

Covering numbers: $cov(\mu, n, \epsilon) :=$ the minimum number of sets of d_n -diameter $\leq \epsilon$ required to cover a subset of X of μ -measure $\geq 1 - \epsilon$

The **slow entropy** of μ is an isomorphism invariant which measures how fast $cov(\mu, n, \epsilon)$ grows as $n \to \infty$, maximized over $\epsilon > 0$.

Slow entropy recovers classical entropy if we focus on the exponential scale:

$$\sup_{\epsilon>0} \limsup_{n\to\infty} \frac{1}{2n+1} \log \operatorname{cov}(\mu, n, \epsilon) = h(\mu)$$

But it can also compare different sub-exponential sequences, so it can distinguish between zero entropy systems.

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Slow entropy

For a general G:

- replace [-n, n] with a Følner sequence (F_n)
- For a general (X, μ, T) :
 - Pick a finite partition
 - Calculate slow entropy for the induced shift system

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Take supremum over all finite partitions

Theorem (L)

For all amenable G: if (X, μ, T) has zero entropy, then it is not dominant.

Proof outline:

- Fix a cocycle α₀
- Construct α = very small perturbation of α₀ such that the skew product T_α has strictly larger slow entropy than T
- Set of α such that T_{α} has a lot of slow entropy is open
- Conclusion: set of α such that $T_{\alpha} \not\simeq T$ is a dense G_{δ}

Construction uses hyperfiniteness of orbit equivalence relation and independent cutting and stacking