Non-dominance of amenable group actions with zero entropy

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Generic extensions

G = discrete amenable group

(X, μ, T) = ergodic measure preserving action of G

What properties of T are preserved by a "generic" ergodic extension?

$$\begin{array}{c}(\overline{X}, \overline{\mu}, \overline{T}) \\ \downarrow \\ (X, \mu, T)\end{array}$$

Examples (Glasner-Thouvenot-Weiss 2021):

- A generic $\overline{T}$ has the same entropy as T
- If T is Bernoulli, then a generic $\overline{T}$ also is

Generic extensions

What does "generic extension" mean?

Rokhlin's skew-product theorem: every ergodic extension $(\overline{X}, \overline{\mu}, \overline{T})$ is isomorphic to a skew product system $(X \times I, \mu \times m, T_\alpha)$ defined by $T_\alpha^g(x, t) := (T^g x, \alpha(g, x)t)$, where

- $I = [0, 1]$, $m =$ Lebesgue measure
- $\alpha : G \times X \to \mathrm{Aut}(I, m)$ satisfes the **cocycle condition:** $\alpha(h, T^g x) \circ \alpha(g, x) = \alpha(hg, x)$.

The space of all cocycles carries a natural topology.

A property **P** is said to hold for a generic extension if there is a dense G_δ set of cocycles α for which T_α has property **P**.

Dominant systems

Definition

A system (X, μ, T) is **dominant** if a generic extension is *isomorphic* to (X, μ, T).

Theorem (Austin-Glasner-Thouvenot-Weiss, 2021)

1. For $G = \mathbb{Z}$: a system is dominant **if and only if** it has positive entropy.
2. For all amenable G: if (X, μ, T) has positive entropy, then it is dominant.

Theorem (L)

For all amenable G: if (X, μ, T) has zero entropy, then it is not dominant.

Slow entropy

How to distinguish between systems of zero entropy? (Recall a generic extension also has zero entropy)

Theory of **slow entropy** developed by Katok and Thouvenot

Slow entropy

Simple case: $G = \mathbb{Z}$, $X = \Lambda^{\mathbb{Z}}$, $T =$ left shift, $\mu =$ shift-invariant measure

Hamming (pseudo-)metrics on X: $d_n(x, y) = \frac{1}{2n+1} \sum_{j=-n}^{n} 1_{x_j \neq y_j}$

Covering numbers: $\mathrm{cov}(\mu, n, \epsilon) :=$ the minimum number of sets of d_n-diameter $\leq \epsilon$ required to cover a subset of X of μ-measure $\geq 1 - \epsilon$

The **slow entropy** of μ is an isomorphism invariant which measures how fast $\mathrm{cov}(\mu, n, \epsilon)$ grows as $n \to \infty$, maximized over $\epsilon > 0$.

Slow entropy

Slow entropy recovers classical entropy if we focus on the exponential scale:

$$\sup_{\epsilon>0} \limsup_{n\to\infty} \frac{1}{2n+1} \log \mathrm{cov}(\mu, n, \epsilon) \; = \; h(\mu)$$

But it can also compare different sub-exponential sequences, so it can distinguish between zero entropy systems.

Slow entropy

For a general G:

- replace $[-n, n]$ with a Følner sequence (F_n)

For a general (X, μ, T):

- Pick a finite partition
- Calculate slow entropy for the induced shift system
- Take supremum over all finite partitions

Theorem (L)

For all amenable G: if (X, μ, T) has zero entropy, then it is not dominant.

Proof outline:

- Fix a cocycle α_0
- Construct $\alpha =$ very small perturbation of α_0 such that the skew product T_α has strictly larger slow entropy than T
- Set of α such that T_α has a lot of slow entropy is open
- Conclusion: set of α such that $T_\alpha \not\precsim T$ is a dense G_δ

Construction uses hyperfiniteness of orbit equivalence relation and independent cutting and stacking