Slicing theorems for IFS attractors

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**Notation.** For $z \in \mathbb{R}^2$, $t \in T = \mathbb{R}/\mathbb{Z}$, let $L_{z,t}$ denote the line through the point $z$ in the direction $e^{2\pi it}$.

**Theorem (Marstrand, 1950s)**

Let $A \subseteq \mathbb{R}^2$. Then for Lebesgue-a.e. $(z, t)$,

$$\dim(A \cap L_{z,t}) \leq \max(0, \dim(A) - 1).$$

- Here and throughout, $\dim(\cdot)$ is **Hausdorff dimension**
Question

What conditions on $A$ imply that $\dim(A \cap L_{z,t}) \leq \max(0, \dim(A) - 1)$ for every $z, t$?

Conjectures by Furstenberg: if $A$ has nice fractal structure, then the above should be true.
Iterated function systems

Let $\phi_1, \ldots, \phi_n$ be contraction mappings in $\mathbb{R}^2$. Then there exists a unique compact set $K \subseteq \mathbb{R}^2$ such that

$$K = \bigcup_{1 \leq i \leq n} \phi_i(K).$$

- $\{\phi_1, \ldots, \phi_n\}$ is an iterated function system (IFS).
- $K$ is the attractor of the IFS.
Iterated function systems
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**Iterated function systems**

**Definitions**

- If the union in $K = \bigcup_{1 \leq i \leq n} \phi_i(K)$ is disjoint, the IFS satisfies the **strong separation condition (SSC)**.
- If all of the $\phi_i$ are similarity maps, the attractor $K$ is a **self-similar set**.

From now on, assume the IFS has the following properties.

- **SSC**
- **Self-similarity**
- Each $\phi_i$ has the same rotation part $\theta \notin 2\pi\mathbb{Q}$
The attractor $K$ can be turned into a dynamical system. Define $S : K \rightarrow K$ by $S|_{\phi_i(K)} = \phi_i^{-1}$.

- Well defined by the SSC
Observation. Suppose there exists a line $L$ such that $\dim(K \cap L) > \dim(K) - 1$. Say the direction of $L$ is $e^{2\pi it_0}$.

- $S(K \cap L)$ is a union of slices, each in the direction $e^{2\pi i(t_0-\theta)}$, and at least one of them also has dimension $> \dim(K) - 1$.
- Iterate this procedure: for each $n$ there is a line $L_n$ in the direction $e^{2\pi i(t_0-n\theta)}$ such that $\dim(K \cap L_n) > \dim(K) - 1$. 

Let $\mu$ be a probability measure on $\mathbb{R}^d$.

- The **local dimension of $\mu$ at $x$** is
  \[
  \dim(\mu, x) = \liminf_{r \searrow 0} \frac{\log \mu(B_r(x))}{\log(1/r)}
  \]

- If the limit exists and is $\mu$-a.e. constant, the measure $\mu$ is **exact dimensional** and the common value is denoted $\dim(\mu)$.

**Correspondence between measures and sets:**

- $\dim(\mu) = \inf\{\dim(A) : \mu(A) > 0\}$
- **“Frostman’s lemma”**: If $\dim(A) \geq \alpha$, then there exists a measure $\mu$ such that $\dim(\mu) = \alpha$ and $\mu(A) = 1$. 
Magnification dynamics

Attractor system + keeping track of “slice data”

- Let $X = K \times \mathbb{T} \times \text{Prob}(K)$
- Define $M : X \rightarrow X$ by $(z, t, \nu) \mapsto (Sz, t - \theta, S_\ast(\nu_z))$, where $\nu_z$ is defined to be $\nu$ conditioned on the piece $\phi_i(K)$ that contains $z$
- Simulates “zooming in” to the point $z \in K$
Magnification dynamics

**Idea:** Use the existence of one high-dimensional slice to construct a special invariant measure.

Let $L_{z_0,t_0}$ be any slice with $\alpha := \dim(K \cap L_{z_0,t_0}) > 0$.

- Let $\nu_0 \in \text{Prob}(K)$ be supported on $L_{z_0,t_0}$ and satisfy $\dim(\nu_0) = \alpha$.

- Let $\mu_0 := \nu_0 \times \delta_{t_0} \times \delta_{\nu_0} \in \text{Prob}(X)$.

- Let $\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} M_i \mu_0$.

- $\mu := \lim_{n \to \infty} \mu_n$ is an $M$-invariant probability measure on $X$ supported on $\{(z, t, \nu) : \nu(L_{z,t}) = 1 \text{ and } \dim(\nu) \geq \alpha\}$. 
Magnification dynamics

The marginal of $\bar{\mu}$ on the $\mathbb{T}$ coordinate is invariant for the **irrational** circle rotation $t \mapsto t - \theta$, so it must be Lebesgue measure.

**Theorem (Furstenberg, 1960s)**

Suppose there is some line $L$ with $\dim(K \cap L) = \alpha > 0$. Then for Lebesgue-a.e. $t$ there is a line $L_t$ in the direction $e^{2\pi i t}$ such that $\dim(K \cap L_t) \geq \alpha$ also.

**Idea:** if $K$ has high-dimensional slices in many different directions, it forces $\dim(K)$ to be big (c.f. the Kakeya problem)
Theorem (Shmerkin/Wu, 2019)
Let \( \{\phi_1, \ldots, \phi_n\} \) be a self-similar IFS satisfying the SSC. Further assume that each \( \phi_i \) has the same rotation part \( \theta \notin 2\pi\mathbb{Q} \). Then \( \dim(K \cap L) \leq \max(0, \dim(K) - 1) \) for every line \( L \).

- Independent & simultaneous proofs by Pablo Shmerkin and Meng Wu (appeared in back-to-back Annals issues)
- Shmerkin’s proof is not based on magnification dynamics
- Wu’s proof uses a complicated argument based on Sinai’s factor theorem to upgrade the previous observation to get the correct upper bound
- Austin (2020) found a simpler version of Wu’s proof
Non-uniform rotations

**Goal:** drop the assumption that each $\phi_i$ has the same irrational rotation part

Two main places that assumption was used:
- Definition of magnification dynamics
- Identifying the marginal of $\overline{\mu}$ on the $T$ coordinate
Non-uniform rotations

Let \( \{\phi_1, \ldots, \phi_n\} \) be an IFS with rotation parts \( \theta_1, \ldots, \theta_n \in T \).

Let \( \theta(z) = \theta_i \) for \( z \in \phi_i(K) \)

New magnification dynamics \( \tilde{M} : X \rightarrow X \) are defined by

\[
(z, t, \nu) \mapsto (Sz, t - \theta(z), S_\ast(\nu_z))
\]
Assuming the existence of one high-dimensional slice, the same process works – construct an \( \tilde{M} \)-invariant measure \( \tilde{\mu} \) supported on measures supported on high-dimensional slices.

- Because \( \tilde{M} \) is a skew product in the \( T \) coordinate, the \( T \)-marginal of \( \tilde{\mu} \) is (a priori) not an invariant measure for any system.
- Need to analyze its regularity “by hand.”
By the ergodic theorem, the $\mathbb{T}$-marginal of $\tilde{\mu}$ is obtained as the limiting distribution of “multi-rotation orbits”:

**Definition**

Let $\theta_1, \ldots, \theta_n \in \mathbb{T}$ and fix $\omega \in \{\theta_1, \ldots, \theta_n\}^\mathbb{N}$. The **multi-rotation orbit** generated by $\omega$ is the sequence $\{x_n\}_{n \geq 1} \subseteq \mathbb{T}$ defined by $x_n = \omega_1 + \cdots + \omega_n$.

The **limiting empirical distribution** associated to $\omega$ is $\nu_\omega := \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{x_n}$.

**Goal:** show that for typical $\omega$, $\nu_\omega$ is not too singular
Non-uniform rotations

Proposition

Suppose \( \{\theta_1, \ldots, \theta_n\} \) satisfy

- At most one \( \theta_j \) is rational
- \( \{\theta_1, \ldots, \theta_n\} \subseteq \text{span}_{\mathbb{Q} \geq 0} \{1, \gamma\} \) for some \( \gamma \notin \mathbb{Q} \).

Let \( \mu \neq \delta_\omega \) be an ergodic shift-invariant measure on \( \{\theta_1, \ldots, \theta_n\}^\mathbb{N} \). Then \( \mu\)-a.s., \( \nu_\omega \) is not singular to Lebesgue measure.

Proof idea: Compare multi-rotation orbit to a single orbit of rotation by \( \gamma \) and use ergodic theorem to control frequency of overlaps.

Corollary

Under these assumptions on the IFS rotations, the attractor \( K \) satisfies \( \dim(K \cap L) \leq \max(0, \dim(K) - 1) \) for every line \( L \).
Non-uniform rotations

**Proposition**

For Lebesgue-a.e. \((\theta_1, \ldots, \theta_n)\) (including all algebraic \(\theta_j\)) and any \(\omega \in \{\theta_1, \ldots, \theta_n\}^\mathbb{N}\), \(\nu_\omega\) has Hausdorff dimension \(\geq \frac{1}{n}\).

**Proof idea:** Estimate the typical growth rate of

\[
\Phi_{\theta_1, \ldots, \theta_n}(r) := \min \{ k_1 + \cdots + k_n : k_j \geq 0, \| k_1 \theta_1 + \cdots + k_n \theta_n \| < r \}
\]

as \(r \to 0\) to control the \(\nu_\omega\)-mass of balls of radius \(r\).

**Corollary**

If an IFS has rotation parts \((\theta_1, \ldots, \theta_n)\) from the “good” set above, then the attractor \(K\) satisfies

\[
\dim(K \cap L) \leq \max(0, \dim(K) - 1/n)
\]

for any line \(L\).
More questions

- The general case where \( \{1, \theta_1, \ldots, \theta_n\} \) are linearly independent over \( \mathbb{Q} \)
  - Difficulty: there exist examples of \( \theta_1, \theta_2 \in \mathbb{T} \) and \( \mu \in \text{Prob}_\sigma^e(\{\theta_1, \theta_2\}^\mathbb{N}) \) such that \( \mu \)-typical multi-rotation orbits are supported on a closed set of dimension 0.

- Higher dimensions
  - Study multi-rotation orbits on the unit sphere instead of unit circle