Slicing theorems for IFS attractors

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Classical slicing theorems

Notation. For $z \in \mathbb{R}^2$, $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, let $L_{z,t}$ denote the line through the point z in the direction $e^{2\pi i t}$.

Theorem (Marstrand, 1950s)

Let $A \subseteq \mathbb{R}^2$. Then for Lebesgue-a.e. (z, t),

$$\dim(A \cap L_{z,t}) \;\leq\; \max(0, \dim(A) - 1).$$

- Here and throughout, $\dim(\cdot)$ is **Hausdorff dimension**

Classical slicing theorems

Question

What conditions on A imply that
$\dim(A \cap L_{z,t}) \leq \max(0, \dim(A) - 1)$ for **every** z, t?

Conjectures by Furstenberg: if A has nice fractal structure, then the above should be true.

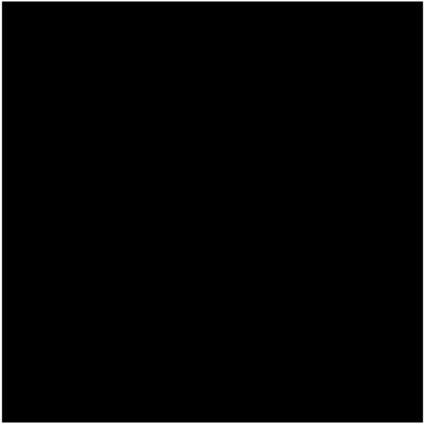
Iterated function systems

Let $\phi_1, \ldots, \phi_n$ be contraction mappings in $\mathbb{R}^2$. Then there exists a **unique** compact set $K \subseteq \mathbb{R}^2$ such that

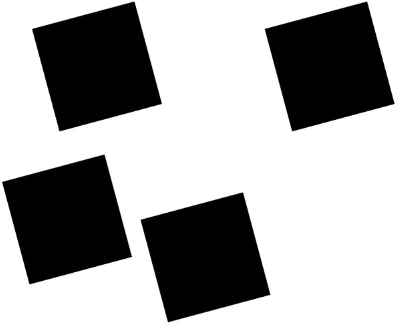
$$K = \bigcup_{1 \leq i \leq n} \phi_i(K).$$

- $\{\phi_1, \ldots, \phi_n\}$ is an **iterated function system (IFS)**.
- K is the **attractor** of the IFS.

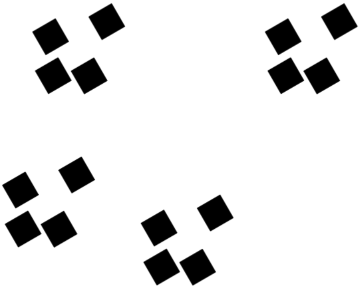
Iterated function systems



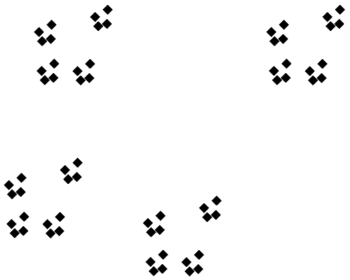
Iterated function systems



Iterated function systems



Iterated function systems



Iterated function systems

Definitions

- If the union in $K = \bigcup_{1 \le i \le n} \phi_i(K)$ is disjoint, the IFS satisfies the **strong separation condition (SSC)**.
- If all of the ϕ_i are similarity maps, the attractor K is a **self-similar set**.

Iterated function systems

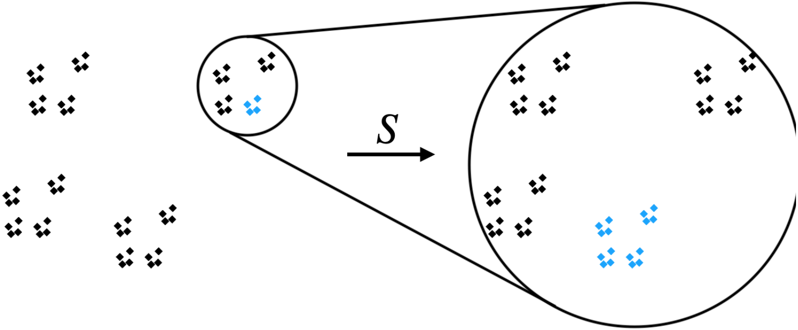
From now on, assume the IFS has the following properties.

- SSC
- Self-similarity
- Each ϕ_i has the same rotation part $\theta \notin 2\pi\mathbb{Q}$

Attractor system

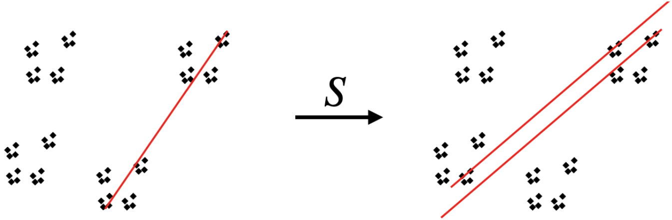
The attractor K can be turned into a dynamical system. Define $S : K \to K$ by $S|_{\phi_i(K)} = \phi_i^{-1}$.

- Well defined by the SSC



Attractor system

Observation. Suppose there exists a line L such that $\dim(K \cap L) > \dim(K) - 1$. Say the direction of L is $e^{2\pi i t_0}$.



- $S(K \cap L)$ is a union of slices, each in the direction $e^{2\pi i(t_0 - \theta)}$, and at least one of them also has dimension $> \dim(K) - 1$.
- Iterate this procedure: for each n there is a line L_n in the direction $e^{2\pi i(t_0 - n\theta)}$ such that $\dim(K \cap L_n) > \dim(K) - 1$.

Dimension of measures

Let μ be a probability measure on $\mathbb{R}^d$.

- The **local dimension of** μ **at** x is

$$\dim(\mu, x) \;=\; \liminf_{r \searrow 0} \frac{\log \mu(B_r(x))}{\log(1/r)}$$

- If the limit exists and is μ-a.e. constant, the measure μ is **exact dimensional** and the common value is denoted $\dim(\mu)$.

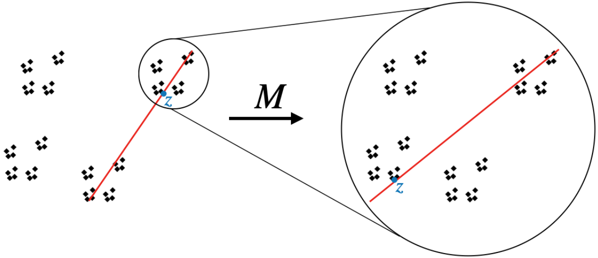
Correspondence between measures and sets:

- $\dim(\mu) = \inf\{\dim(A) : \mu(A) > 0\}$
- **"Frostman's lemma":** If $\dim(A) \geq \alpha$, then there exists a measure μ such that $\dim(\mu) = \alpha$ and $\mu(A) = 1$.

Magnification dynamics

Attractor system + keeping track of "slice data"

- Let $X = K \times \mathbb{T} \times \text{Prob}(K)$
- Define $M : X \to X$ by $(z, t, \nu) \mapsto (Sz, t - \theta, S_*(\nu_z))$, where ν_z is defined to be ν conditioned on the piece $\phi_i(K)$ that contains z
- Simulates "zooming in" to the point $z \in K$



Magnification dynamics

Idea: Use the existence of one high-dimensional slice to construct a special invariant measure.

Let L_{z_0,t_0} be any slice with $\alpha := \dim(K \cap L_{z_0,t_0}) > 0$.

- Let $\nu_0 \in \mathrm{Prob}(K)$ be supported on L_{z_0,t_0} and satisfy $\dim(\nu_0) = \alpha$
- Let $\overline{\mu}_0 := \nu_0 \times \delta_{t_0} \times \delta_{\nu_0} \in \mathrm{Prob}(X)$
- Let $\overline{\mu}_n := \frac{1}{n} \sum_{i=0}^{n-1} M_*^i \overline{\mu}_0$
- $\overline{\mu} := \lim_{n\to\infty} \overline{\mu}_n$ is an M-invariant probability measure on X supported on $\{(z,t,\nu) : \nu(L_{z,t}) = 1 \text{ and } \dim(\nu) \geq \alpha\}$

Magnification dynamics

The marginal of $\overline{\mu}$ on the $\mathbb{T}$ coordinate is invariant for the **irrational** circle rotation $t \mapsto t - \theta$, so it must be Lebesgue measure.

Theorem (Furstenberg, 1960s)

Suppose there is some line L with $\dim(K \cap L) = \alpha > 0$. Then for Lebesgue-a.e. t there is a line L_t in the direction $e^{2\pi it}$ such that $\dim(K \cap L) \geq \alpha$ also.

Idea: if K has high-dimensional slices in many different directions, it forces $\dim(K)$ to be big (c.f. the Kakeya problem)

Magnification dynamics

Theorem (Shmerkin/Wu, 2019)

Let $\{\phi_1, \ldots, \phi_n\}$ be a **self-similar** IFS satisfying the **SSC**. Further assume that each ϕ_i has the same rotation part $\theta \notin 2\pi\mathbb{Q}$. Then $\dim(K \cap L) \leq \max(0, \dim(K) - 1)$ for every line L.

- Independent & simultaneous proofs by Pablo Shmerkin and Meng Wu (appeared in back-to-back Annals issues)
- Shmerkin's proof is not based on magnification dynamics
- Wu's proof uses a complicated argument based on Sinai's factor theorem to upgrade the previous observation to get the correct upper bound
- Austin (2020) found a simpler version of Wu's proof

Non-uniform rotations

Goal: drop the assumption that each ϕ_i has the same irrational rotation part

Two main places that assumption was used:

- Definition of magnification dynamics
- Identifying the marginal of $\overline{\mu}$ on the $\mathbb{T}$ coordinate

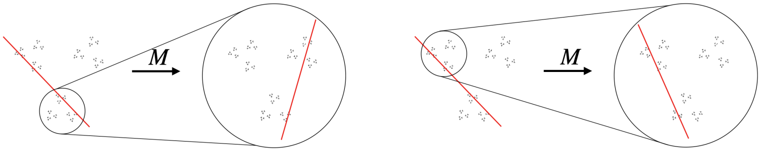
Non-uniform rotations

Let $\{\phi_1, \ldots, \phi_n\}$ be an IFS with rotation parts $\theta_1, \ldots, \theta_n \in \mathbb{T}$.

Let $\theta(z) = \theta_i$ for $z \in \phi_i(K)$

New magnification dynamics $\widetilde{M} : X \to X$ are defined by

$$(z, t, \nu) \mapsto (Sz, t - \theta(z), S_*(\nu_z))$$



Non-uniform rotations

Assuming the existence of one high-dimensional slice, the same process works – construct an $\widetilde{M}$-invariant measure $\widetilde{\mu}$ supported on measures supported on high-dimensional slices

- Because $\widetilde{M}$ is a skew product in the $\mathbb{T}$ coordinate, the $\mathbb{T}$-marginal of $\widetilde{\mu}$ is (a priori) not an invariant measure for any system
- Need to analyze its regularity “by hand”

Non-uniform rotations

By the ergodic theorem, the $\mathbb{T}$-marginal of $\widetilde{\mu}$ is obtained as the limiting distribution of "multi-rotation orbits":

Definition

Let $\theta_1, \ldots, \theta_n \in \mathbb{T}$ and fix $\omega \in \{\theta_1, \ldots, \theta_n\}^{\mathbb{N}}$. The **multi-rotation orbit** generated by ω is the sequence $\{x_n\}_{n \geq 1} \subseteq \mathbb{T}$ defined by $x_n = \omega_1 + \cdots + \omega_n$.

The **limiting empirical distribution** associated to ω is $\nu_\omega := \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{x_n}$.

Goal: show that for typical ω, ν_ω is not too singular

Non-uniform rotations

Proposition

Suppose $\{\theta_1, \ldots, \theta_n\}$ satisfy

- At most one θ_j is rational
- $\{\theta_1, \ldots, \theta_n\} \subseteq \mathrm{span}_{\mathbb{Q}\geq 0}\{1, \gamma\}$ for some $\gamma \notin \mathbb{Q}$.

Let $\mu \neq \delta_\omega$ be an ergodic shift-invariant measure on $\{\theta_1, \ldots, \theta_n\}^{\mathbb{N}}$. Then μ-a.s., ν_ω is not singular to Lebesgue measure.

Proof idea: Compare multi-rotation orbit to a single orbit of rotation by γ and use ergodic theorem to control frequency of overlaps.

Corollary

Under these assumptions on the IFS rotations, the attractor K satisfies $\dim(K \cap L) \leq \max(0, \dim(K) - 1)$ for every line L.

Non-uniform rotations

Proposition

For Lebesgue-a.e. $(\theta_1, \ldots, \theta_n)$ (including all algebraic θ_j) and any $\omega \in \{\theta_1, \ldots, \theta_n\}^{\mathbb{N}}$, ν_ω has Hausdorff dimension $\geq \frac{1}{n}$.

Proof idea: Estimate the typical growth rate of
$\Phi_{\theta_1,\ldots,\theta_n}(r) := \min\{k_1 + \cdots + k_n : k_j \geq 0, \|k_1\theta_1 + \cdots + k_n\theta_n\| < r\}$
as $r \to 0$ to control the ν_ω-mass of balls of radius r.

Corollary

If an IFS has rotation parts $(\theta_1, \ldots, \theta_n)$ from the "good" set above, then the attractor K satisfies
$\dim(K \cap L) \leq \max(0, \dim(K) - 1/n)$ for any line L.

More questions

- The general case where $\{1, \theta_1, \ldots, \theta_n\}$ are linearly independent over $\mathbb{Q}$
  - Difficulty: there exist examples of $\theta_1, \theta_2 \in \mathbb{T}$ and $\mu \in \mathrm{Prob}_\sigma^e(\{\theta_1, \theta_2\}^{\mathbb{N}})$ such that μ-typical multi-rotation orbits are supported on a closed set of dimension 0.
- Higher dimensions
  - Study multi-rotation orbits on the unit **sphere** instead of unit circle