The Fraïssé Limit of Matrix Algebras with the Rank Metric

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November 4, 2017

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Motivation

- In the 1930s, von Neumann developed a version of projective geometry where (normalized) dimensions of subspaces can take any real value in $[0, 1]$
- Subspaces in projective geometry correspond to ideals of matrix rings, rank gives dimension
- For continuous geometry, we need matrices with rank $\in [0, 1]$

Basic Construction

- Fix finite field $\mathbb{F}_q$.
- Given $m, n \in \mathbb{N}$, $m|n$, define an embedding $\phi_{nm} : M_m(\mathbb{F}_q) \to M_n(\mathbb{F}_q)$, where $\phi_{nm}(X) = X \otimes I_{n/m}$
- Put a metric on each $M_n(\mathbb{F}_q)$, by letting $d(A, B) = \frac{\mathrm{rank}(A-B)}{n}$.
- ϕ_{nm} respects the metric, so it is an embedding of *metric rings*.

Basic Construction

Definition

Let $n_0, n_1, \dots \in \mathbb{N}$ be a *factor sequence* when $n_i | n_{i+1}$ and $\lim_{i\to\infty} n_i = \infty$.

- For any factor sequence $n_0, n_1, \dots,$ we get an inductive sequence

$$M_{n_0}(\mathbb{F}_q) \overset{\phi_{n_1 n_0}}{\hookrightarrow} M_{n_1}(\mathbb{F}_q) \overset{\phi_{n_2 n_1}}{\hookrightarrow} M_{n_2}(\mathbb{F}_q) \overset{\phi_{n_3 n_2}}{\hookrightarrow} \dots$$

- Let the direct limit of this sequence (as rings) be $M_0(\mathbb{F}_q)$. This inherits a metric.

Basic Construction

Theorem (von Neumann, Halperin)

If $M(\mathbb{F}_q)$ is the metric completion of $M_0(\mathbb{F}_q)$, then the definition of $M(\mathbb{F}_q)$ does not depend on the factor sequence used.

Proof.

Heavily uses the theory of von Neumann regular rings. Somewhat arcane, not too informative.[2] □

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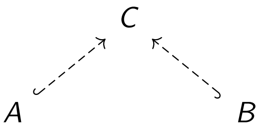
Fraïssé Classes

Definition

A *Fraïssé Class* is a countable class of finitely-generated structures in the same language satisfying the Joint Embedding Property and the Amalgamation Property.



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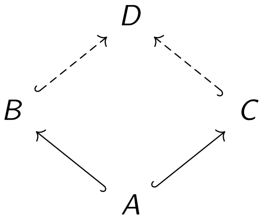

AP

Basic Examples of Fraïssé Classes

- Finite linear orderings
- Finite graphs

Fraïssé Limits

Theorem

Each Fraïssé class $\mathcal{K}$ has a unique Fraïssé limit, a countably-generated structure into which every element of $\mathcal{K}$ embeds, which is also $\mathcal{K}$-homogeneous.

Definition

A structure F is $\mathcal{K}$-homogeneous if for every $A \in \mathcal{K}$, every embedding $\phi : A \hookrightarrow F$ extends to an automorphism $\psi : F \xrightarrow{\sim} F$.

Basic Examples of Fraïssé Limits

Class	Limit
Finite linear orderings	$\mathbb{Q}$
Finite graphs	The random graph

Metric Fraïssé Theory

- In the case of metric structures, we require that each structure be a complete metric space
- AP can be replaced with Near AP, where diagram commutes up to ε
- Homogeneity also up to ε

Reproof of von Neumann/Halperin

Theorem

Given any factor sequence, the resulting $M(\mathbb{F}_q)$ is a metric Fraïssé limit of $\mathcal{K} = \{M_n(\mathbb{F}_q) : n \in \mathbb{N}\}$. Thus $M(\mathbb{F}_q)$ is unique.

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Extreme Amenability

Definition

A group G is *extremely amenable* if any action $\phi : G \curvearrowright X$ on a compact Hausdorff space X has a fixed point

Ramsey Property

Definition

Let $\binom{B}{A}$ be the set of embedded copies of A in B. Then a class $\mathcal{K}$ has the *Ramsey Property* when for any $A \leq B \in \mathcal{K}$, and any $k \in \mathbb{N}$, there is some $C \in \mathcal{K}$ such that any k-coloring of $\binom{C}{A}$ has a monochromatic $\binom{B'}{A}$, with $B' \cong B$.

- For linear orders, this is equivalent to Ramsey's Theorem.
- This is replaced with an approximate version in the metric case

Kechris Pestov Todorcevic Correspondence

Theorem

A Fraïssé class $\mathcal{K}$ has the Ramsey Property if and only if the automorphism group of its Fraïssé limit is extremely amenable.[3]

Carderi and Thom

Theorem

The unit group of $M(\mathbb{F}_q)$ is extremely amenable. A corresponding lemma, reminiscent of the Ramsey Property, holds for the set $\{SL_n(\mathbb{F}_q) : n\}$[1]

$M(\mathbb{F}_q)$

Theorem

$\mathrm{Aut}(M(\mathbb{F}_q))$ *is extremely amenable, and the Ramsey Property holds for* $\mathcal{K}$.

If $A = M_a(\mathbb{F}_q), B = M_b(\mathbb{F}_q), C = M_c(\mathbb{F}_q)$, then we only need

$$c \geq 64\varepsilon^{-2}(\log(2) + \max(b^2 \log(q), \log(6\lceil \varepsilon^{-1} \rceil))$$

where ε^{-1} is analogous to number of colors.

References

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I. Halperin. "Von Neumann's Manuscript on Inductive Limits of Regular Rings". In: *Canadian Journal of Mathematics* 20 (1968). URL: http://dx.doi.org/10.4153/CJM-1968-045-0.

A. S. Kechris, V. G. Pestov, and S. Todorcevic. "Fraïssé Limits, Ramsey Theory, and topological dynamics of automorphism groups". In: *Geometric & Functional Analysis GAFA* 15.1 (Feb. 2005), pp. 106–189. ISSN: 1420-8970. DOI: 10.1007/s00039-005-0503-1.