Bounds on the Sizes of Distal Cell Decompositions

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VC-Density

Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\varphi(x; y)$ an $\mathcal{L}$-formula, where x and y may be tuples of variables, with length $|x|$ and $|y|$.

Definition

- Define $\pi_\varphi(n)$ to be the maximum over all $A \subset M^{|x|}$ with $|A| = n$ of the number of subsets of A that can be written as $\varphi(M^{|x|}, b) \cap A$ for some $b \in M^{|y|}$.
- Define $\mathrm{vc}(\varphi)$ to be the infimum of all reals d such that $\pi_\varphi(n) = \mathcal{O}(n^d)$, or ∞ if none exist.

Dual VC-Density

Now let $\Phi(x; y)$ be a finite set of $\mathcal{L}$-formulas all of the form $\varphi(x; y)$, with fixed length $|x|$ and $|y|$.

Definition

- Define $S^{\Phi}(S)$ to be the set of complete Φ-types over a set $S \subseteq M^{|y|}$ of parameters, or alternately, the set of maximal consistent subsets of $\{\varphi(x; b) : \varphi \in \Phi, b \in S\} \cup \{\neg\varphi(x; b) : \varphi \in \Phi, b \in S\}$.
- Let $\pi^*_{\Phi}(n)$ be the maximum over all $S \subset M^{|y|}$ with $|S| = n$ of $\left|S^{\Phi}(S)\right|$.
- Let $\mathrm{vc}^*(\Phi)$ be the infimum of all d such that $\pi^*_{\Phi}(n) = \mathcal{O}(n^d)$, or ∞ if none exist.

Dual VC-Density Example

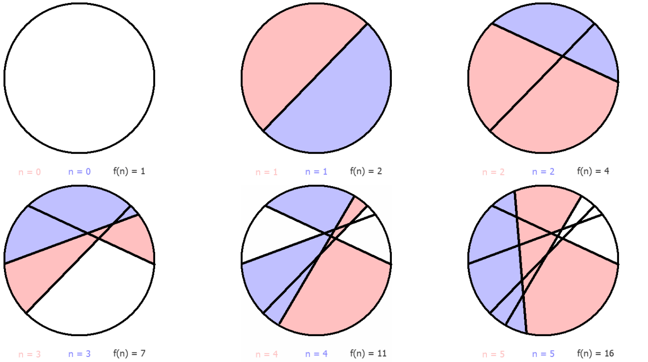

Figure: The maximum number of pieces that *n* slices cut a circle into[4]

Dual VC-Density Example

- Let $\mathcal{M} = \langle \mathbb{R}; 0, 1, +, -, \cdot, \leq \rangle$, and let $\varphi(x; y)$ be $y_1 x_1 + y_2 x_2 \leq 1$, with $\Phi = \{\varphi\}$.
- For each $b \in M^2$, $\varphi(M; b)$ is a half-plane.
- $\pi^*_\Phi(n)$ is the maximum number of pieces that n half-planes cut $\mathbb{R}^2$ into.
- By induction, $\pi^*_\Phi(n) = \binom{n}{2} + \binom{n}{1} + \binom{n}{0} = \mathcal{O}(n^2)$, and $\mathrm{vc}^*(\Phi) = 2$.

VC-Density and NIP

Fact

If $\varphi^*(y;x) \iff \varphi(x;y)$, *then* $\mathrm{vc}^*(\{\varphi\}) = \mathrm{vc}(\varphi^*)$.

Fact

$\mathrm{vc}(\varphi) = \infty \iff \mathrm{vc}^*(\varphi) = \infty$. *In this case, we call* φ *independent.*

Definition

$\mathcal{M}$ is *NIP* (has the *Non-Independence Property*) when every $\varphi(x;y)$ has $\mathrm{vc}(\varphi) < \infty$.

NIP Structures

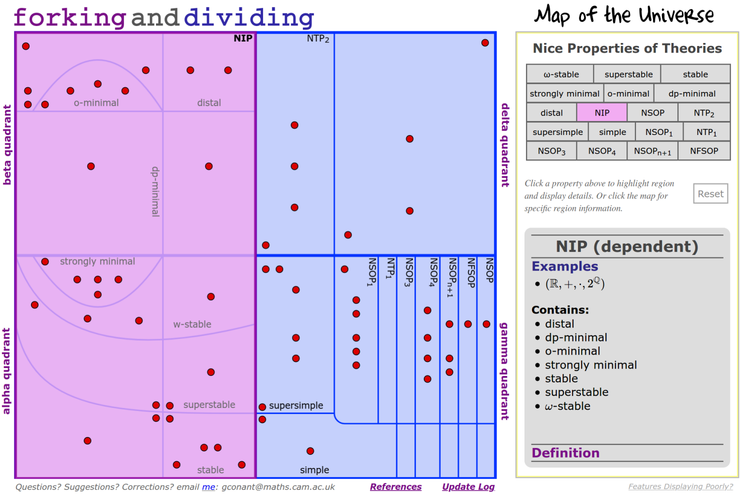

Figure: [5]

Abstract Cell Decompositions

Definition

- If A, X are sets, then X *is crossed by* A if $A \cap X \neq \emptyset$ and $X \setminus A \neq \emptyset$.
- If $X \subset M^{|x|}$, $S \subset M^{|y|}$, then X is crossed by $\Phi(x; S)$ if for some $\varphi \in \Phi$ and some $b \in S$, X is crossed by $\varphi(M; b)$.

Definition

An *abstract cell decomposition* for $\Phi(x; y)$ is a function $\mathcal{T}$ that assigns to each finite $S \subset M^{|y|}$ a set $\mathcal{T}(S)$ whose elements, called cells, are not crossed by $\Phi(x; S)$, and cover $M^{|x|}$ so that $M^{|x|} = \bigcup \mathcal{T}$.

Abstract Cell Decomposition Example

Definition (Dual VC Cell Decomposition)

- Given $\Phi(x; y)$ and a finite $S \subset M^{|y|}$, let the cells of $\mathcal{T}_{\mathrm{vc}}(S)$ correspond to the types in $S^{\Phi}(S)$:
- For each type $p(x) \in S^{\Phi}(S)$, take $\{x \in M^{|x|} : \mathcal{M} \models p(x)\}$ as a cell.

Fact

- *$\mathcal{T}_{\mathrm{vc}}$ is an abstract cell decomposition for Φ*
- *If $\mathcal{T}$ is an abstract cell decomposition for Φ, then for all S, $|\mathcal{T}(S)| \geq |\mathcal{T}_{\mathrm{vc}}(S)|$.*

Abstract Cell Decomposition Example

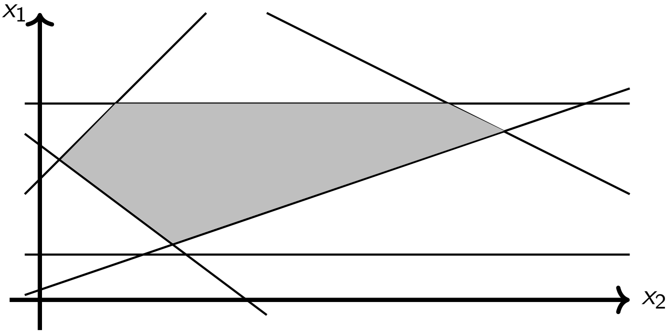

Figure: A cell of $\mathcal{T}_{\mathrm{vc}}(S)$, where $\Phi(x;y) = \{x_1y_1 + x_2y_2 \leq 1\}$, and $\mathcal{M} = \langle \mathbb{R}; 0, 1, +, -, \cdot, \leq \rangle$.[2]

Distal Cell Decompositions

Definition

A *distal* cell decomposition for $\Phi(x; y)$ is an abstract cell decomposition $\mathcal{T}$ whose cells are *uniformly definable* in the following way:

- Each cell is defined by an instance of one of a finite set $\Psi(x; y_1, \ldots, y_k)$ of formulas where $|y_1| = \cdots = |y_k| = |y|$.
- For a given S, the set of potential cells is $\Psi(S) := \{\psi(M^{|x|}; b_1, \ldots, b_k) : \psi \in \Psi, b_1, \ldots, b_k \in S\}$
- For each $\psi \in \Psi$, there is a formula $\theta_\psi(y; y_1, \ldots, y_k)$.
- For each potential cell $\Delta = \psi(M^{|x|}; b_1, \ldots, b_k)$, let $\mathcal{I}(\Delta) = \theta_\psi(M^{|y|}; b_1, \ldots, b_k)$, then exclude Δ if $\mathcal{I}(\Delta) \cap S \neq \emptyset$.
- $\mathcal{T}(S) = \{\Delta \in \Psi(S) : S \cap \mathcal{I}(\Delta) = \emptyset\}$

Distal Structures

Theorem (Chernikov, Galvin, Starchenko)

A structure $\mathcal{M}$ is distal if and only if every finite set of formulas $\Phi(x; y)$ admits a distal cell decomposition.

Distal Structures

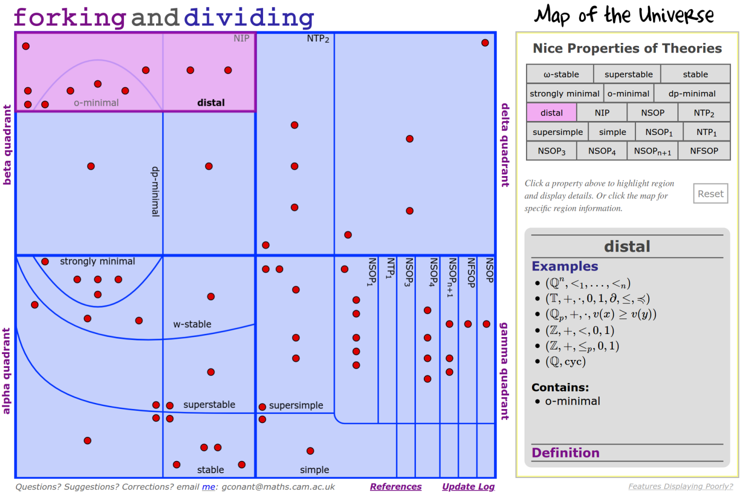

Figure: [5]

Distal Shatter Function, Exponent, Density

Definition

Given an abstract cell decomposition $\mathcal{T}$ for $\Phi(x; y)$:

- Let $\pi_{\mathcal{T}}(n) = \max_{|S|=n} |\mathcal{T}(S)|$
- Say that $\mathcal{T}$ *has exponent d* if $\pi_{\mathcal{T}}(n) = \mathcal{O}(n^d)$.
- Let the *distal density* of Φ be the infimum of all d such that there exists a distal cell decomposition for Φ of exponent d, or ∞ if no distal cell decomposition exists.

Fact

The distal density of Φ is at least the dual vc-density.

Corollary

All distal structures are NIP.

Distal Density for Certain Structures

$\mathcal{M}$	Dual VC-Density	Distal Density
o-minimal expansions of fields	$\|x\|$	$2\|x\| - 2$ (1 if $\|x\| = 1$)
weakly o-minimal structures	$\|x\|$	$2\|x\| - 1$
ordered vector spaces over ordered division rings	$\|x\|$	$\|x\|$
$\mathbb{Q}_p$ the valued field	$2\|x\| - 1$	$3\|x\| - 2$
$\mathbb{Q}_p$ in the linear reduct	$\|x\|$	$\|x\|$

(VC-Density Calculations: Aschenbrenner, Dolich, Haskell, Macpherson, Starchenko, except $\mathbb{Q}_p$ in the linear reduct by Bobkov)

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Dimension Induction

Theorem (A.)

Let $\mathcal{M}$ be a structure in which all finite sets of formulas with $|x| = 1$ admit a distal cell decomposition $\mathcal{T}_1$ where every formula ψ of $\mathcal{T}_1$ refers to at most k parameters from S, and for some $d_0 \in \mathbb{N}$, all finite sets of formulas with $|x| = d_0$ have distal density at most r. Then all finite sets Φ of formulas with $|x| = d \geq d_0$ have distal density at most $k(d - d_0) + r$.

Dimension Induction Proof

The resulting cell decomposition generalizes the "vertical decomposition" of Chazelle, Edelsbrunner, Guibas, Sharir.

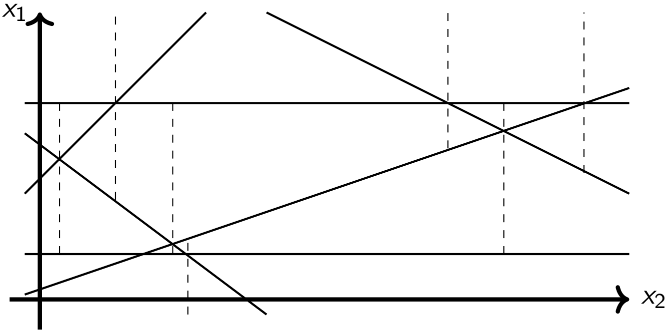

Figure: A "vertical decomposition" for lines in the plane[2]

Dimension Induction Proof

- Partition $x = x_1, \ldots, x_d$ into x_1 and $x' = (x_1, \ldots, x_d)$.
- Let $\mathcal{T}_1$ be a distal cell decomposition for $\Phi'(x_1; x', y) = \Phi(x_1, x'; y)$.
- For each formula $\psi(x_1; y_1, \ldots, y_k)$ used to define cells in $\mathcal{T}_1$, and each $b_1, \ldots, b_k \in S$, $\mathcal{T}(S)$ uses cells of the form

$$(x' \in \Delta) \wedge \psi(x_1; b'_1, \ldots, b'_k)$$

where b'_i is the tuple (x', b_i).

Dimension Induction Proof

- There are $|S|^k$ choices for $\psi(x_1; b'_1, \dots, b'_k)$.
- Each is accompanied by $\mathcal{O}(|S|^{k(d-1-d_0)+r})$ choices of Δ, taken from the distal cell decomposition for $\theta'(x'; y, b_1, \dots, b_k)$.
- $\theta'(x'; y, y_1, \dots, y_k)$ is a rearrangement of the variables of $\theta_\psi(x', y; y'_1, \dots, y'_k)$ where $y'_i = (x', y_i)$.
- We use only those cells Δ such that $\mathcal{M} \models \neg\theta'(x'; b, b_1, \dots, b_k)$ for all $b \in S$, so for all $a \in \Delta$, if $b'_i = (a, b_i)$, then $\psi(x_1; b'_1, \dots, b'_k)$ is actually used in $\mathcal{T}_1(\{a\} \times S)$.

Algorithmic Quantifier Elimination

- This construction is modeled after the "singly-exponential stratification" of semi-algebraic sets in [3].
- That cell decomposition can be used to algorithmically eliminate (existential) quantifiers, and Collins's *cylindrical algebraic decomposition* can be used to eliminate all quantifiers for semi-algebraic sets.
- Distal cell decompositions may be useful for efficient QE in several distal structures.

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(Weakly) o-minimal structures

Definition

Let $\mathcal{M}$ be a structure whose language includes $<$, which is interpreted as a linear order.

- $\mathcal{M}$ is *weakly o-minimal* if every definable subset of M^1 is a finite union of $<$-convex pieces.
- $\mathcal{M}$ is *o-minimal* if those $<$-convex pieces are points and intervals.

DCDs in weakly *o*-minimal structures

Theorem (A.)

If $\mathcal{M}$ is a weakly o-minimal structure and $\Phi(x; y)$ is a finite set of formulas, then Φ has a distal cell decomposition of exponent $2|x| - 1$.

Proof.

We construct a distal cell decomposition $\mathcal{T}_1$ in the case $|x| = 1$ with 2 parameters, with $|\mathcal{T}_1(S)| = \mathcal{O}(|S|)$, and then apply Dimension Induction. □

If $\mathcal{M}$ is an *o*-minimal expansion of a field, we can drop the exponent to $2|x| - 2$, using Chernikov, Galvin and Starchenko's bound in the plane.

o-minimal 1d case (proof sketch)

Fix $\mathcal{M}$ to be an o-minimal expansion of a group. Consider $\Phi(x; y)$ with $|x| = 1$.

- $\mathcal{M}$ has definable choice
- Definable choice gives us definable functions $h_1, \ldots, h_k : M^{|y|} \to M$, such that for each $b \in M^{|y|}$,

$$h_1(b) \leq h_2(b) \leq \cdots \leq h_k(b)$$

  and for any $a_1, a_2 \in (h_i(b), h_{i+1}(b))$, and $\varphi \in \Phi$,

$$\mathcal{M} \models \varphi(a_1, b) \leftrightarrow \varphi(a_2, b)$$

- The cells of $\mathcal{T}_1(S)$ are defined by $x = h_i(b)$ or $h_i(b_1) < x < h_j(b_2)$, and we need $\mathcal{O}(|S|)$ of them.

Linear o-minimal case

Theorem

Let $\mathcal{M}$ be an ordered vector space over an ordered division ring A, in the language $\mathcal{L} = \{0, +, -, <, c\cdot_{c \in A}\}$. let $\Phi(x; y)$ be a finite set of formulas in the language of $\mathcal{M}$. Then Φ has a distal cell decomposition of exponent $|x|$.

Linear o-minimal case: Proof sketch

- We may assume Φ consists of formulas $\ell(x) \leq c(y)$, where ℓ, c are linear polynomials.
- The subsets of $M^{|x|}$ realizing types $p(x) \in S^{\Phi}(S)$ take the form
$$\bigcap_{\ell,c} \{\ell(x) \in I_{\ell,c}\}$$
where $I_{\ell,c}$ is an interval with endpoints $c(b_1)$ and $c(b_2)$ for some $b_1, b_2 \in S$.
- Thus $\mathcal{T}_{\mathrm{vc}}$ is a distal cell decomposition, which has exponent $|x|$.

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The Valued Field Structure

Let K be a P-minimal field (such as $\mathbb{Q}_p$) in the language $\mathcal{L}_{\mathrm{Mac}}$:

Definition

$\mathcal{L}_{\mathrm{Mac}} := \mathcal{L}_{\mathrm{ring}} \cup \{|, P_n : n \in \mathbb{N}\}$.

Describe the valuation by interpreting $x|y \iff v(x) \leq v(y)$.
Interpret P_n so that $P_n(x) \iff \exists y, y^n = x$.

Theorem (Macintyre)

K has quantifier-elimination.

Distal Cell Decomposition for K

Theorem

Let $\Phi(x; y)$ be a finite set of $\mathcal{L}_{\mathrm{Mac}}$-formulas. Then Φ admits a distal cell decomposition with exponent $3|x| - 2$.

Lemma

If $|x| = 1$, then Φ admits a distal cell decomposition with 3 parameters and exponent 1.

Basics for P-minimal K

Lemma

Suppose $n > 1$, and let $x, y, a \in K$ with $v(y - x) > 2v(n) + v(y - a)$. Then $(x - a)(y - a)^{-1} \in P_n^\times$.

Theorem

Let $\Phi(x; y)$ be a finite set of formulas, with $|x| = 1$, closed under negation. There exists $n > 0$ and finite sets F, C of definable functions $K^{|y|} \to K$ and a set Λ of n coset representatives of $P_n^\times < K^\times$ such that for each $\varphi \in \Phi$ and $b \in K^{|y|}$, $\varphi(K; b)$ is a finite union of cells defined by

$$v(f(b))\square_{i1}v(x - c(b))\square_{i2}v(g(b)) \wedge P_n(\lambda(x - c(b)))$$

where each $\square$ is $<, \leq$, or no condition, $f, g \in F, c \in C$, and $\lambda \in \Lambda$.

Now with multiple parameters

The cells in the previous slide are intersections of differences of two balls with a translated coset of $P_n^\times$.
Fix $S \subset K^{|y|}$.

- Let $B_r(t) = \{x \in K : v(x-t) > r\}$.
- Define $\mathcal{B}_F := \{B_{f(b)}(c(b) : b \in S, f \in F, c \in C\}$
- $\mathcal{B}_C := \{B_{v(c_1(b_1)-c_2(b_2))}(c_1(b_1) : b_1, b_2 \in S, c_1, c_2 \in C\}$
- Define *subintervals* to be the atoms in the boolean algebra generated by the balls $\mathcal{B} = \mathcal{B}_C \cup \mathcal{B}_F$.

Counting Subintervals

- $\mathcal{B}$ forms a tree under inclusion with $\mathcal{O}(|S|)$ leaves corresponding to $\{c(b) : c \in C, b \in S\}$.
- There are $\mathcal{O}(|S|)$ balls in $\mathcal{B}$, and thus $\mathcal{O}(|S|)$ subintervals, which correspond to edges in the tree.

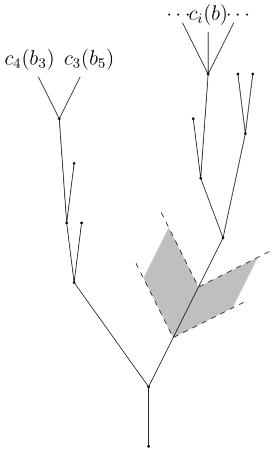

Figure: A subinterval[1]

Subinterval Type

We now want to break up the subintervals into pieces where for each $c(b)$ with $c \in c, b \in S$, the $P_n^\times$-coset of $(x - c(b))$ is constant.

Given a subinterval $B_{\alpha_L}(t) \setminus B_{\alpha_U}(t)$, two points a_1, a_2 in that subinterval are defined to have the same *subinterval type* if one of the following conditions is satisfied:

- $\alpha_L + 2v(n) \leq v(a_i - t) \leq \alpha_U - 2v(n)$ for $i = 1, 2$ and $(a_1 - t)(a_2 - t)^{-1} \in P_n^\times$
- $\neg(\alpha_L + 2v(n) \leq (a_i - t) \leq \alpha_U - 2v(n))$ for $i = 1, 2$ and $v(a_1 - t) = v(a_2 - t) \leq v(a_1 - a_2) - 2v(n)$

The Distal Cell Decomposition

- Each cell of the distal cell decomposition consists of all points in a given subinterval with a given subinterval type.
- Each cell is definable over α_L, α_U, t, where the subinterval is $B_{\alpha_L}(t) \setminus B_{\alpha_U}(t)$. As $t = c(b_1)$, and α_L and α_U are either of the form $v(f(b_1))$ or $v(c_1(b_1) - c_2(b_2))$, so each cell is definable from 3 parameters of S.
- Each subinterval divides into a bounded number of subinterval types, so there are $\mathcal{O}(|S|)$ cells.

The Linear Reduct of $\mathbb{Q}_p$

Consider $\mathbb{Q}_p$ in the language $\mathcal{L}_{\mathrm{aff}}$:

Definition

$\mathcal{L}_{\mathrm{aff}} := \{0, 1, +, -, \{c\cdot\}_{c \in \mathbb{Q}_p}, |, \{Q_{m,n}\}_{m,n \in \mathbb{N}}\}$.

We interpret these symbols as the normal vector space structure, with $|$ describing the valuation as above, and $\mathbb{Q}_p \models Q_{m,n}(a) \iff a \in \bigcup_{k \in \mathbb{Z}} p^{km}(1 + p^n \mathbb{Z}_p)$.

Theorem (Leenknegt)

This structure on $\mathbb{Q}_p$ has quantifier elimination. It is also a reduct of the standard $\mathcal{L}_{\mathrm{Mac}}$−structure on $\mathbb{Q}_p$.

Distal Cell Decompositions in the Linear Reduct

Theorem (Bobkov)

If $\Phi(x; y)$ is a finite set of $\mathcal{L}_{\mathrm{aff}}$-formulas, then $\mathrm{vc}^*(\Phi) \leq |x|$.

Theorem (A.)

Φ admits a distal cell decomposition of exponent $|x|$.

Linear $\mathbb{Q}_p$ Cell Decomposition: One Parameter

Lemma

Any $\mathcal{L}_{\text{aff}}$-formula $\varphi(x; y)$ is equivalent to a boolean combination of formulas of the form

$$v(p_1(x) - c_1(y)) < v(p_2(x) - c_2(y))$$

or

$$p_1(x) - c_1(y) \in \lambda Q_{m,n}$$

where p_1, p_2 are homogeneous linear polynomials, c_1, c_2 are linear polynomials, $m, n \in \mathbb{N}$, and Λ is a finite set of coset representatives of $Q_{m,n}$.

Linear $\mathbb{Q}_p$ Cell Decomposition

By that lemma, we may assume that Φ consists of the formulas

$$v(p_i(x) - c_i(y)) < v(p_j(x) - c_j(y))$$

or

$$p_i(x) - c_i(y) \in \lambda Q_{m,n}$$

for $i, j \in I = \{1, \ldots, |I|\}, \lambda \in \Lambda$, all of these sets finite. Now we attempt to apply our existing cell decomposition ideas to each component of the vector $(p_1(x), \ldots, p_{|I|}(x))$.

Linear $\mathbb{Q}_p$ Cell Decomposition

- Define subintervals and subinterval type as before, with $F = \{c_1 - c_2 : c_1, c_2 \in C\}$.
- If x is in the subinterval $B_{\alpha_L} \setminus B_{\alpha_U}(t)$, define $T - \mathrm{val}(x)$ to be $v(x - t)$.
- Define each cell all the points in a given subinterval with a given subinterval type and with a fixed ordering of

$$\{T - \mathrm{val}(p_1(x)), \ldots, T - \mathrm{val}(p_{|I|}(x))\}$$

- As before, each subinterval is broken into a bounded number of pieces. It suffices to count the number of subintervals used - a priori $|S|^{|I|}$.

Counting the Subintervals

- Let Sg be the image of the function $f : \mathbb{Q}_p^{|x|} \to \mathbb{Q}_p^{|I|}$ given by $f(x) = (p_1(x), \ldots, p_{|I|}(x))$.
- If x is in the subinterval $B_{\alpha_L} \setminus B_{\alpha_U}(t)$, define $T - \mathrm{fl}(x)$ to be α_L.
- First divide Sg into $\mathcal{O}(1)$ pieces based on the order of the set

$$\{T - \mathrm{val}(p_1(x)), \ldots, T - \mathrm{val}(p_{|I|}(x))\}$$
$$\cup\{T - \mathrm{fl}(p_1(x)), \ldots, T - \mathrm{fl}(p_{|I|}(x))\}$$

- We will show that each piece Sg' splits into $\mathcal{O}(|S|^{|x|})$ subintervals.

Counting the Subintervals

- For a given piece Sg', rename $p_1, \ldots, p_{|I|}$ so that for all $(a_1, \ldots, a_n) \in \mathrm{Sg}'$,

$$T - \mathrm{fl}(a_1) \geq \cdots \geq T - \mathrm{fl}(a_{|I|})$$

- Now let $J \subset I$ be lexicographically minimal such that $\{p_j : j \in J\}$ is a basis for $\mathrm{Span}\{p_i : i \in I\}$.
- Then we can reproduce the subintervals of $(a_1, \ldots, a_n)$ from the subintervals of $(a_j : j \in J)$ and only a fixed number of coefficients of each $a_i : i \notin I$.

Final Lemma

Lemma (Bobkov)

Suppose we have a finite collection of vectors $\{\vec{p}_i\}_{i \in I}$ with each $\vec{p}_i \in \mathbb{Q}_p^{|x|}$. Suppose $J \subseteq I$ and $i \in I$ satisfy $\vec{p}_i \in \mathrm{span}\{\vec{p}_j\}_{j \in J}$, and we have $\vec{c} \in \mathbb{Q}_p, \alpha \in v(\mathbb{Q}_p)$ with $v(\vec{p}_j \cdot \vec{c}) > \alpha$ for all $j \in J$. Then $v(\vec{p}_i \cdot \vec{c}) > \alpha - \gamma$ for some $\gamma \in v(\mathbb{Q}_p), \gamma \geq 0$. Moreover γ can be chosen independently from $J, j, \vec{c}, \alpha$ depending only on $\{\vec{p}_i\}_{i \in I}$.

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Distal Cutting Lemma

Definition

Given a real r and a family $A_1, \ldots, A_n$ of subsets of a set X, an *r-cutting* is a family $X_1, \ldots, X_t$ of sets such that

- $\bigcup_{i=1}^{t} X_i = X$.
- Each X_i is crossed by at most $\frac{n}{r}$ of the sets $A_1, \ldots, A_n$.

Theorem (Chernikov, Galvin, Starchenko)

Let $\varphi(x; y) \in \mathcal{L}$ be a formula admitting a distal cell decomposition $\mathcal{T}$ of exponent d.
Then for any finite $H \subseteq M^{|y|}$ of size n and any real r satisfying $1 < r < n$, the family $\{\varphi(M; a) : a \in H\}$ admits an r-cutting $X_1, \ldots, X_t$ with $t = \mathcal{O}(r^d)$.
Moreover, the X_is are intersections of at most two cells from $\mathcal{T}(H)$.

Zarankiewicz's Problem: Special Case

Theorem

Let $\mathcal{M}$ be a structure and $t \geq 2$. Assume that $E(x, y) \subseteq M^{|x|} \times M^{|y|}$ is a relation defined by a formula $\theta(x; y) \in \mathcal{L}$ that admits a distal cell decomposition of exponent t, and the graph $E(x, y)$ does not contain any $K_{s,u}$. Then for any finite $P \subseteq M^{|x|}, Q \subseteq M^{|y|}$, $|P| = m, |Q| = n$, we have:

$$|E(P, Q)| = \mathcal{O}\left(m^{\frac{(t-1)s}{ts-1}} n^{\frac{t(s-1)}{ts-1}} + m + n\right).$$

Szemerédi-Trotter in $\mathbb{Q}_p$

Now fix $\mathcal{M} = \mathbb{Q}_p$, in $\mathcal{L}_{\mathrm{Mac}}$.

Theorem

Assume that $E(x,y) \subseteq \mathbb{Q}_p^{|x|} \times \mathbb{Q}_p^{|y|}$ is a relation defined by a formula $\theta(x;y)$ with $|x| = 2$, and the graph $E(x,y)$ does not contain any $K_{s,u}$.

Then for any finite $P \subseteq \mathbb{Q}_p^{|x|}, Q \subseteq \mathbb{Q}_p^{|y|}$, $|P| = m, |Q| = n$, we have:

$$|E(P,Q)| = \mathcal{O}\left(m^{\frac{3s}{4s-1}} n^{\frac{4(s-1)}{4s-1}} + m + n\right).$$

If $E(x,y)$ is the incidence relation of x on the line parametrized by y, then it omits $K_{2,2}$, so we can take $s = 2$.

Erdős Distinct Distances Problem

Let P be a set of n points in $\mathbb{R}^2$, and let $D(P) = \{|a - b| : a, b \in P\}$.

Theorem (Guth and Katz)

$$D(P) = \Omega\left(\frac{|P|}{\log |P|}\right)$$

Distinct Distances in $\mathbb{Q}_p^2$

- Define the *distance* between two points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{Q}_p^2$ by

$$d(a, b) = (a_1 - b_1)^2 + (a_2 - b_2)^2.$$

- (This is not a metric, as it is $\mathbb{Q}_p$-valued)

Theorem (A.)

Given finite $P \subseteq \mathbb{Q}_p^2$, let $D(P) = \{d(a, b) : a, b \in P\} \subseteq \mathbb{Q}_p$. Then $D(P) = \Omega(|P|^{5/8})$.

Distinct Distances in $\mathbb{Q}_p^2$: Proof

Let $\varphi(x; y, z) := (d(x, y) = z)$, so that $\varphi(\mathbb{Q}_p^2; b, c)$ defines a "circle" centered at b. Like circles in $\mathbb{R}^2$, any two share at most two points:

Lemma

For $(b, c) \neq (b', c')$, $\varphi(\mathbb{Q}_p^2; b, c) \cap \varphi(\mathbb{Q}_p^2; b', c')$ consists of at most two points, implying that the bigraph defined by φ contains no $K_{3,2}$.

- Let $P \subset \mathbb{Q}_p^2$ be finite, $Q = \{(b, d(a, a')) : a, a', b \in P\}$.
- $|Q| = D(P)|P|$.
- $E(P, Q)$, the bigraph defined by φ on P, Q, has

$$|P|^2 = |E(P, Q)| \leq \alpha \left(|P|^{9/11} |Q|^{8/11} + |P| + |Q| \right).$$

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