Sperner's Lemma and Brouwer's Fixed Point Theorem

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1 Graph Theory Warmup

In this worksheet, a graph will be an ordered pair (V, E), where V is a finite set we call the *vertices*, and E is a set of unordered pairs of vertices, which we call the *edges*. If $v \in V$, then we can define its *degree* in a given graph G, written $\deg_G(v)$, to be the number of edges that connect v to other vertices.

Problem 1 (The Handshake Lemma)**.** Let $G = (V, E)$ be a graph. Find a simpler formula for $\sum_{v \in V} \deg_G(v)$. Show that the number of vertices with odd degree must be even.

2 Sperner's Lemma

Let T be a graph formed by taking the triangle with vertices R, G, B, and *triangulating* it, that is, breaking it up into many small triangular pieces, like so:

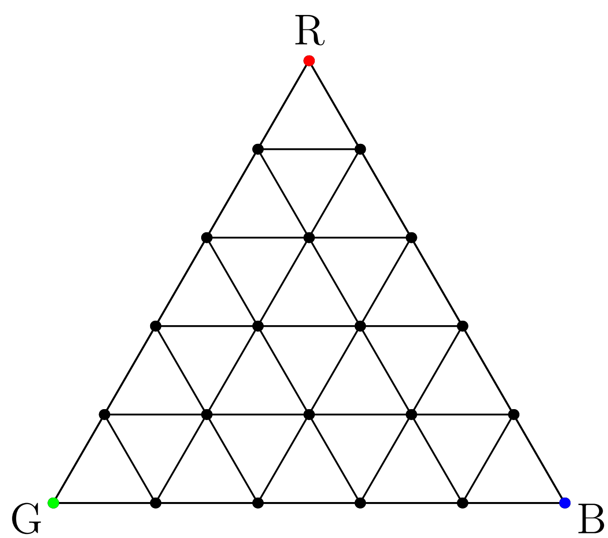

Figure 1: A sample triangulation

Now color every vertex of T red, green, or blue, subject to the following rules: Color vertex R red, G green, B blue, and every vertex on the side of the outer triangle connecting R and G either red or green, every vertex on the side connecting R and B either red or blue, and every vertex on the side connecting G and B either green or blue. The interior vertices can be colored with any of the 3 colors. This is called a *Sperner coloring*.

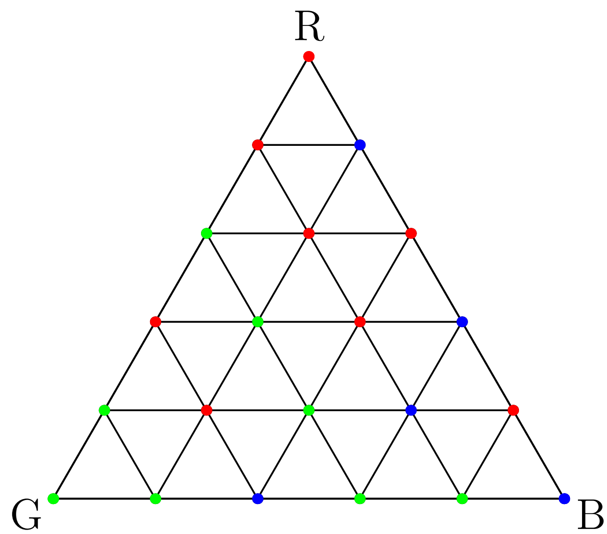

Figure 2: A Sperner coloring

The rest of this section is dedicated to proving *Sperner's Lemma*:

Lemma 2.1 (Sperner)**.** *If T is a triangulation colored with a Sperner coloring, at least one of the small trianglular faces of T will have its three vertices colored with three different colors.*

In order to prove this, we make another graph, T', whose vertices are faces of T. All of the faces inside the triangle RGB will themselves be triangles, but there will be one more face, the outside face O. Note that each pair of neighboring faces of T meet at an edge of T, whose two vertices are colored. We connect these two faces, as vertices in T', if and only if the two vertices of the edge where they meet are colored red and green. Faces which are not adjacent are not connected in T'.

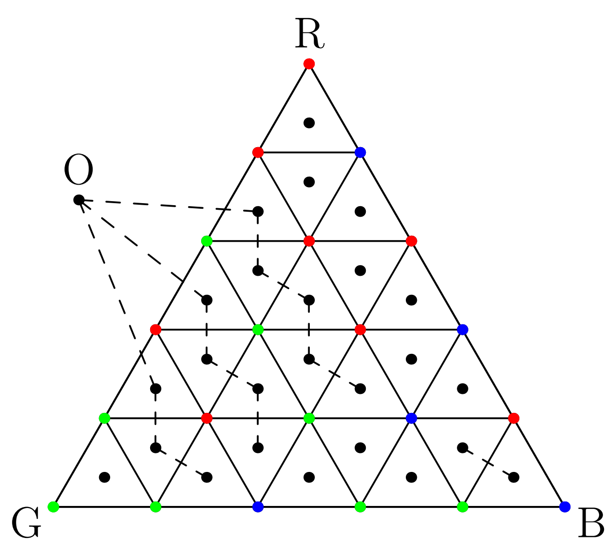

Figure 3: T', with vertices in black and edges dashed

Problem 2. In T', show that the degree of O is odd.

Problem 3. Show that the number of small triangles whose vertices are colored red, green, blue is odd, and thus greater than 0. This proves Sperner's Lemma. (Hint: Problem 1)

Problem 4. What should the 1-dimensional version of Sperner's Lemma look like, or the 3-dimensional version? Prove your 1D version.

Problem 5. Prove your 3D version of Sperner's Lemma, and see if you can extend it to arbitrary dimensions.

3 Bonus Material: Fan's Lemma and Tucker's Lemma

In this section, instead of triangulating a triangle, and coloring the vertices of the triangulation with colors, we will "triangulate" a disk, and label our vertices with integers from $\{-2, -1, 1, 2\}$.

To triangulate a disk, distribute vertices along the outer circle, and connect them to vertices inside the disk with lines, and connect those vertices to each other with lines, so that each component of the circle is either a triangle, or a "triangle" consisting of two lines and a circular arc. We will call a triangulation "boundary-symmetric" if the vertices on the outer circle are distributed in a way that has 180° rotational symmetry. A labelling of this triangulation will assign each vertex a nonzero integer, and we'll call a labelling of a boundary-symmetric triangulation *odd* if every pair of vertices exactly across the circle from each other (each vertex on the circle will belong to such a pair) is labelled with integers adding to 0.

The subject of this section is *Tucker's Lemma*, which says that every odd labelling of a boundary-symmetric triangulation has a *complementary edge*: an edge whose vertices add to 0. Here's a picture from Wikipedia of the whole setup, with the complementary edge highlighted.

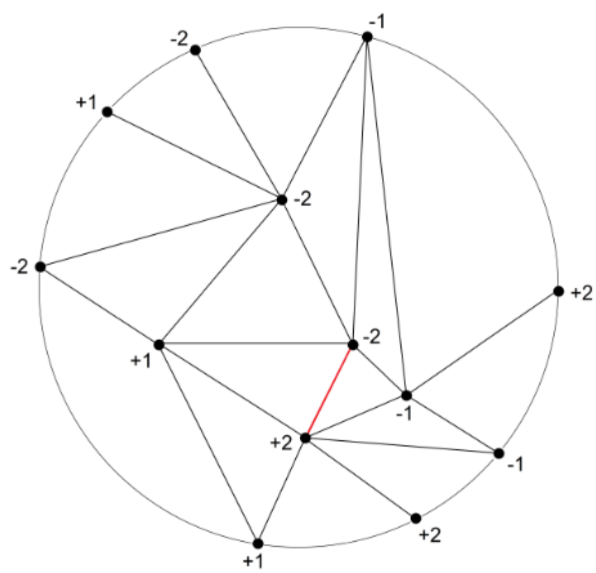

Figure 4: An odd labelling of a boundary-symmetric triangulation with a complementary edge highlighted.

Assume that we have such a triangulation. These problems will show us how to find a complementary edge.

Problem 6. Assume that there are no complementary edges on the boundary.

Show that there are an odd number of edges on the boundary whose labels are 1 and -2 (we call these *decreasing edges*), and an odd number of edges on the boundary whose labels are -1 and $+2$ (we call these *increasing edges*).

Problem 7. Show that any (generalized) triangle either has a complementary edge or has an even number of increasing and an even number of decreasing edges.

Problem 8. Finish the proof of Tucker's Lemma.

3.1 Fan's Lemma and Higher Dimensions

Tucker's lemma extends to higher dimensions. To state the higher-dimensional version, we define a *d-simplex* to be a d-dimensional shape with $d+1$ vertices, each of which are connected with edges, and each 3 of which are connected with a triangular face, and each 4 of which are connected with a tetrahedral hyperface, and so on. For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

Theorem 3.1 (Tucker's Lemma (all dimensions))**.** *Fix $d \in \mathbb{N}$. "Triangulate" the unit ball in $\mathbb{R}^d$ with d-simplices, in such a way that the vertices on the boundary (a $d-1$-dimensional sphere) are symmetrical about the origin. Then if the vertices are labelled with $\{-n, -n+1, \dots, -1, 1, 2, \dots, n\}$, such that any two opposite points on the boundary have labels that sum to 0, there must exist a complementary edge somewhere in the triangulation.*

In order to prove this version, we will actually want to prove Fan's Lemma, a generalization. In this generalization, we refer to *alternating* simplices. In a generalized triangulation labelled with nonnegative integers, a *positive alternating d-simplex* is a d-simplex whose vertices are labelled with $k_1, -k_2, k_3, \dots$ for integers $k_1 < \dots < k_d$, and a *negative alternating d-simplex* is a d-simplex whose vertices are labelled with $-k_1, k_2, -k_3, \dots$ where $k_1 < \dots < k_d$. These together form the alternating d-simplices.

Theorem 3.2 (Fan's Lemma)**.** *Let the unit ball in $\mathbb{R}^d$ be triangulated in a boundary-symmetric way, and label the vertices of the triangulation with nonnegative integers such that there are no complementary edges, and the labels on any two opposite points on the boundary add to 0. Then there is an odd number of alternating d-simplices.*

Problem 9. Prove Tucker's Lemma from Fan's Lemma.

Now we'll try to prove Fan's Lemma by induction on the dimension.

Problem 10. Let $d = 1$. Now a "triangulation" of the unit ball is just a partition of a line segment into intervals. Explain what Fan's Lemma means in this context, and prove it.

Problem 11. Let $d = 2$. Adapt your proof of 2D Tucker's Lemma to prove Fan's Lemma.

Problem 12. Assume Fan's Lemma holds in d dimensions, show it holds in $d+1$ dimensions (or at least try to prove it for $d = 3$).

Hint: Split the boundary into two hemispheres. The triangulation of each of these can be thought of as a triangulation of the d-dimensional ball, to which we can apply Fan's Lemma. One will have an odd number of positive alternating d-simplices and an even number of negative alternating d-simplices, the other will have the opposite.

4 Brouwer's Fixed Point Theorem

First, let's acquaint ourselves with a particular space, which we will think of as a subset of $\mathbb{R}^3$:

Problem 13. Let Δ be the set of points (x, y, z) in three-dimensional space where $x + y + z = 1$, and $x \geq 0, y \geq 0, z \geq 0$. Sketch a picture of Δ, and describe its shape.

The Euclidean distance makes Δ a metric space, meaning that for any two points $x, y \in \Delta$, there is a natural definition of $d(x, y)$, the distance between x and y. We can use this distance to come up with a formal definition of what it means for a sequence of points in Δ to have a limit:

Definition 1. Let $x_0, x_1, x_2, \dots \in \Delta$. Then x is the *limit* of $x_0, x_1, \dots$ when for any positive real $\epsilon > 0$, there is some N such that if $n > N$, then $d(x_n, x) < \epsilon$.

We will need the following fact about Δ, but I won't prove it here. Consider it a bonus exercise for later.

Theorem 4.1. *The space* Δ *is* complete*:*

Let $x_0, x_1, x_2, \dots \in \Delta$ *be a* Cauchy sequence*, that is, a sequence such that for every* $\epsilon > 0$*, there is some* N *such that if* $m, n > N$*, then* $d(x_m, x_n) < \epsilon$*. Then* $x_0, x_1, x_2, \dots$ *has some limit* x*.*

Problem 14. Check that the converse holds: if a sequence has a limit, it must be a Cauchy sequence.

We can also use the definition of limits to formally define what it means for a function to be *continuous*. If you look in a topology book, this is actually the definition of a *sequentially continuous* function, but that turns out to be the same as continuous in this case.

Definition 2. Let $f : \Delta \to \Delta$ be a function. We say that f is *continuous* when for every sequence $x_0, x_1, x_2, \dots$ that has a limit x, $f(x)$ is also a limit for $f(x_0), f(x_1), \dots$.

(If you already know a formal definition of continuity, and it isn't this, then here's an exercise for later: prove they're equivalent in any metric space.)

In this section, you will show that Sperner's Lemma is equivalent to Brouwer's Fixed Point Theorem about continuous functions from $\Delta \to \Delta$. (Tucker's Lemma is similarly equivalent to another famous topological fact, featured in today's Bonus Material.)

Theorem 4.2. *Every continuous function* $f : \Delta \to \Delta$ *has a fixed point.*

This is a topological fact, so in fact it is true of any shape topologically similar (*homeomorphic*) to Δ, including any flat disk or rectangle, or other roughly 2D shapes like Antarctica, as viewed from a satellite hovering over the South pole. In that last case, imagine that I take a map of Antarctica, crumple it up, and drop it somewhere in the frozen wastes on a sled journey. Then this map represents a function, taking points on campus and drawing them on the map, which itself has a geographical location. Brouwer's fixed point theorem says there must be some point on the map which represents its exact location in the real world.

Problem 15. Prove Sperner's Lemma from Brouwer's Fixed Point Theorem.

Hint: let Δ be the triangle you've triangulated. Pick a point to send every red vertex to, pick a point to send every green vertex to, pick a point to send every blue vertex to, and define the continuous function very simply on the edges and triangles, according to those constraints. Don't worry about a formal proof that your function is continuous. Design the function such that a fixed point can only come from the *interior* of a three-color triangle.

In the rest of this section, we will prove Brouwer's Fixed Point Theorem using Sperner's Lemma.

In order to talk in more detail about the function f, we generally refer to its coordinate functions, functions $f_x, f_y, f_z : \Delta \to [0, 1]$ such that for any point $P \in \Delta$, $f_x(P)$ is the x-coordinate of $f(P)$, $f_y(P)$ is the y-coordinate, and $f_z(P)$ is the z-coordinate. It's a fact that f_x, f_y, and f_z are continuous if and only if f is.

Problem 16. Let $f : \Delta \to \Delta$ be a continuous function, and f_x, f_y, f_z its coordinate functions. For $P = (x, y, z) \in \Delta$, show that either $f_x(P) \leq x$, $f_y(P) \leq y$, or $f_z(P) \leq z$. Then show that if all three inequalities are true, that P is a fixed point.

The remainder of the problems are designed to show that there is a point that satisfies all 3 inequalities.

Problem 17. Let T be a triangulation of Δ. Label $(1, 0, 0)$ as R, $(0, 1, 0)$ as G, and $(0, 0, 1)$ as B. Show that you can color T with a Sperner coloring, where if a vertex $P = (x, y, z)$ is red, then $f_x(P) \leq x$, if it is green, then $f_y(P) \leq y$, and if it is blue, then $f_z(P) \leq z$. Call a Sperner coloring with this property a *Brouwer coloring*. (I don't think this is standard notation, but I want to call it something.)

Problem 18. Show that for every n, you can find a triangulation T_n of Δ where all edges have side length at most $\frac{1}{2^n}$ times the side length of Δ. Then use this to show that there are points $R_n, G_n, B_n \in \Delta$ such that the distance between any two of them is at most $\frac{1}{2^n}$, and $f_x(R_n)$ is less than or equal to the x-coordinate of R_n, $f_y(G_n)$ is less than or equal to the y-coordinate of G_n, and $f_z(B_n)$ is less than or equal to the z-coordinate of B_n.

Problem 19. Deduce that there is a sequence $R_1, R_2, R_3, \ldots$ of points in Δ with $f_x(R_n)$, where each $R_n = (x, y, z)$ in the sequence satisfies $f_x(R_n) \leq x$, a sequence $G_1, G_2, G_3, \ldots$ where each $G_n = (x, y, z)$ satisfies $f_y(G_n) \leq y$, and a sequence $B_1, B_2, B_3, \ldots$ where each $B_n = (x, y, z)$ satisfies $f_z(B_n) \leq z$, and for each n, the distances between R_n, G_n, B_n are at most $\frac{1}{2^n}$.

Assume that $R_1, R_2, R_3, \ldots$ converges to some point P. Show P is a fixed point of f.

Problem 20. Technically, our sequence $R_1, R_2, R_3, \ldots$ may not converge, but we can find an infinite *subsequence* of it that does, by skipping some points. If the subsequence $R_{n_1}, R_{n_2}, R_{n_3}, \ldots$ converges to P, where $n_1 < n_2 < \ldots$, then show that the sequences $R_{n_1}, R_{n_2}, R_{n_3}, \ldots$, $G_{n_1}, G_{n_2}, G_{n_3}, \ldots$, $B_{n_1}, B_{n_2}, B_{n_3}, \ldots$ still satisfy the requirements of the previous problem, so P is a fixed point.

Problem 21. Show that Δ is *sequentially compact*:

Let $P_1, P_2, P_3, \ldots$ be any sequence of points in Δ. Find an increasing sequence $n_1 < n_2 < n_3 < \ldots$ that makes $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$ a Cauchy sequence, so it converges to some point P. (Hint: Use triangulations, and ask for more hints if you need them.)

Problem 22. Finish the proof of Brouwer's Fixed Point Theorem.

Brouwer's FPT actually holds in higher dimensions, if we replace Δ with Δ^n, the subset of $\mathbb{R}^{n+1}$ consisting of the points with nonnegative coordinates that sum to 1, or B^n, the unit ball in $\mathbb{R}^n$.

Problem 23. Prove that any function $f : \Delta^n \to \Delta^n$ has a fixed point.

5 Bonus Material: The Borsuk-Ulam Theorem

Just as Sperner's Lemma is equivalent to Brouwer's Fixed Point Theorem, Tucker's Lemma is a combinatorial version of the Borsuk-Ulam Theorem:

Theorem 5.1. *Let B^n be the unit "ball" in $\mathbb{R}^n$. (For instance, B^2 is the 2-dimensional unit disk in $\mathbb{R}^2$.) Let $f : B^n \to \mathbb{R}^n$ be a continuous function, and assume that for every x on the boundary "sphere" of B^n, we have $f(-x) = -f(x)$.*

There must exist some $x \in B^n$ such that $f(x) = 0$.

Problem 24. Use Borsuk-Ulam to prove Tucker's Lemma.

Hint: As with the proof of Brouwer's FPT from Sperner's Lemma, you want to define the value of your function $f : B^n \to \mathbb{R}^n$ on the vertices of your triangulation first. Then describe how you might extend it to values on the edges and inside the simplices, so that any complementary edge must have a 0 of f on it?

Problem 25. Use Tucker's Lemma to prove Borsuk-Ulam.

Hint: Like in the proof of Brouwer's FPT from Sperner's Lemma, draw finer and finer boundary-symmetric triangulations, and use your continuous function to label them. Then from your sequence of ever-shorter complementary edges, find some $i \in \{1, \ldots, n\}$ and two sequences, one of points labelled $+i$ and one of labelled $-i$, that converge to the same point, which will be your 0.

Problem 26. Use Borsuk-Ulam to prove Brouwer's FPT. In this case, prove that any function $f : B^n \to B^n$ has a fixed point.

Hint: Assume you have a function $f : B^n \to B^n$ with no fixed points. For each point $x \in B^n$, draw a ray from $f(x)$ through x, and extend it towards the boundary of B^n. Let that point be $g(x)$.

(It is also possible to use Fan's Lemma to prove Sperner's Lemma, which is the combinatorial analog of this result.)