Model Theory

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1 Monday

1.1 Intro

In the first few weeks of Mathcamp, particularly if you took a bunch of classes with names like "Introduction to Widget Theory," you've probably been bombarded with definitions of various kinds of mathematical structures. Algebra has groups, rings, fields, vector spaces and modules, combinatorics has like 8 definitions of a graph, and there are all these orderings: posets, tosets, and lattices. If that wasn't enough, you can combine these, getting ordered groups, ordered rings, and ordered fields. In this class, we're going to see that these are all manifestations of the same idea, and we're going to study all of them at once.

I'm going to be throwing you a lot of definitions today, so here's the overview version. Throughout, I'll be going back to the example of the natural numbers $\mathbb{N}$.

Our first component is a *language*, basically a set of symbols like $\{0, 1, +, *, \leq\}$. These are the building blocks from which we'll build axioms and other statements. We will combine these with logical symbols $(=, \neg, \wedge, \vee, \exists, \forall)$ and variables to make *formulas*, like $\phi(x, y) := x^2 + y^2 \leq 1$.

To give these formulas any actual meaning, we look at *structures*. A structure will be a set, together with some extra information that tells you how to interpret the symbols in the language. (For instance, a structure in the language $\{0, 1, +, *, \leq\}$ will consist of a set with two special elements 0 and 1, an "addition" and a "multiplication" of its elements, and some kind of "ordering".) In the context of a particular structure, all the symbols in a formula have meaning, so we can evaluate a formula (given values for the variables) as either true or false. A *sentence* will then be a formula without variables that need to be given values, and a theory is just a set of sentences. We can then ask if each sentence in a theory is true of a particular structure, if so, we call that structure a *model* of that theory.

This strategy lets us generalize all of the above definitions, because groups are just the models of the theory of groups, and graphs are just the models of the theory of graphs, and so on.

1.2 Languages

In order to study axiom systems formally, we need to first define formal languages, so that we can write the axioms down carefully.

Definition 1. A *first-order language* $\mathcal{L}$ consists of these three collections of symbols:

- A set of constant symbols $\mathcal{C}$
- A set of function symbols $\mathcal{F}$, and a positive integer n_f for each symbol $f \in \mathcal{F}$
- A set of relation symbols $\mathcal{R}$, and a positive integer n_R for each symbol $R \in \mathcal{R}$

The number n_f assigned to a function symbol f indicates how many variables that function takes as inputs, and the number n_R for a relation symbol R works the same way. These are the basic symbols that we'll use to build axioms, but we're also going to be allowed to use variable symbols, the "=" relation, and some logical symbols that I'll get to in a moment.

Here's an example that uses each kind of symbol:

Example 1. If we want to talk about ordered rings, we might use the language $\mathcal{L} := \{0, 1, +, *, \leq\}$. Here 0 and 1 are constant symbols, $+$ and $*$ are *binary* function symbols (meaning $n_+ = n_* = 2$) and $\leq$ is a binary relation symbol.

1.3 Structures

Now that we have all of these symbols, it's time to give them meaning. If I have a symbol, such as $+$, I could interpret it a bunch of different ways, as addition of natural numbers, integers rationals, reals, complexes, integers mod n, what have you. A first-order structure will give us the context behind these symbols.

Definition 2. An $\mathcal{L}$-structure $\mathcal{M}$ consists of the following things:

- A set M, called the *universe* of $\mathcal{M}$
- For each constant symbol $c \in \mathcal{C}$, an element $c^{\mathcal{M}} \in M$
- For each function symbol $f \in \mathcal{F}$, a function $f^{\mathcal{M}} : M^{n_f} \to M$
- For each relation symbol $R \in \mathcal{R}$, a relation $R^{\mathcal{M}} \subseteq M^{n_R}$

These specific constants, functions, and relations $c^{\mathcal{M}}$, $f^{\mathcal{M}}$, and $R^{\mathcal{M}}$ are called the *interpretations* of c, f, and R.

Example 2. Here's an example using the language $\mathcal{L} = \{0, 1, +, *, \leq\}$ from earlier: $\mathbb{N}$.

What do we need to define to turn this into a structure?

We can build an $\mathcal{L}$-structure where $M = \mathbb{N}, 0^{\mathbb{N}} = 0, 1^{\mathbb{N}} = 1$, (the constants are elements of the universe), $+^{\mathbb{N}}$ and $*^{\mathbb{N}}$ are the usual addition and multiplication operations (binary functions), and $\leq$ is the usual order (a function that takes in two inputs from $\mathbb{N}$ and outputs true or false).

1.4 Terms and Formulas

Now that we have these basic symbols, let's start putting them together. The constant symbols and variable symbols are supposed to represent elements of our structure, but we can build more complicated expressions to represent elements of our structure, called *terms*.

Definition 3. If $\mathcal{L}$ is a first-order language, the set of all $\mathcal{L}$-terms (or just "terms" for short) is the smallest set of strings of symbols such that:

- Constant symbols and variable symbols are terms.
- If $f \in \mathcal{F}$ is a function symbol, and $t_1, \ldots, t_{n_f}$ are terms, then $f(t_1, \ldots, t_{n_f})$ is also a term.

Going back to our example language, we can now construct some terms like $0, x, x + 1$, and $x*x+y*y$, but what do these terms actually *mean?* If $\mathcal{M}$ is an $\mathcal{L}$-structure, then we can give these terms interpretations in $\mathcal{M}$. We already know how to interpret individual constant and function symbols, and terms are built out of constant and function symbols, so there's only one good way to do this: if a term consists of a single constant or function symbol, we already know how to interpret it, great. If our term is $f(t_1, \ldots, t_{n_f})$, then we interpret it as $f^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_{n_f}^{\mathcal{M}})$. This'll end up being a function $M^k \to M$ for some k.

Terms are great, but they output more elements of our structure, and in order to talk about axioms, we need something that outputs "true" or "false." The easiest way to do this is by combining terms with relations.

Definition 4. An *atomic formula* is a string formed in one of the following two ways:

- If t_1, t_2 are terms, then $t_1 = t_2$ is an atomic formula.
- If $R \in \mathcal{R}$ is a relation symbol and $t_1, \ldots, t_{n_R}$ are terms, then $R(t_1, \ldots, t_{n_R})$ is an atomic formula.

So now with atomic formulas, we can compare our terms, with examples like $x * x + y * y = 1$ or $0 \leq x$. The interpretations of these end up as functions $M^k \to \{\text{true}, \text{false}\}$, which we often prefer to think of as subsets of M^k. (How?)

Often we want to get just a single "true" or "false" out of a formula, and to do that, we need to assign values to all of the variables. There's a notation for this. If for instance, $\phi(x_1, \ldots, x_n)$ is an atomic formula that uses n variables, $x_1, \ldots, x_n$, and $a_1, \ldots, a_n \in M$, then we can evaluate $\phi(x_1, \ldots, x_n)$ at $a_1, \ldots, a_n$. If it turns out to be true, then we write $\mathcal{M} \models \phi(a_1, \ldots, a_n)$, otherwise, we write $\mathcal{M} \not\models \phi(a_1, \ldots, a_n)$. (The turnstile $\models$ is pronounced "models".) Sometimes, we'll skip the variables and just say $\mathcal{M} \models \phi$, this means that $\mathcal{M} \models \phi(a_1, \ldots, a_n)$ works for any choice of $a_1, \ldots, a_n$.

But I promised logic symbols earlier, so let's bring in those too:

Definition 5. The set of all $\mathcal{L}$-formulas is the smallest set such that:

- Atomic formulas are formulas.
- If ϕ is a formula, then $\neg\phi$ is a formula ($\neg$ stands for "not").
- If ϕ, ψ are formulas, then $\phi \vee \psi$, $\phi \wedge \psi$ and $\phi \to \psi$ are formulas ($\vee$ stands for "or"; $\wedge$ stands for "and"; $\to$ stands for "implies" or "if/then").
- If x is a variable and ϕ is a formula, then $\exists x\, \phi$ and $\forall x\, \phi$ are formulas ($\exists$ and $\forall$ stand for "there $\exists$xists" and 'for $\forall$ll').

The interpretation of a formula in a given structure will again be a boolean-valued function, or a subset of M^k for some k. (These subsets are called the *definable* sets). But how can we quickly look through and find out what k is?

In a formula such as $\forall x, y \leq x * y$, we label a variable which under a quantifier, such as x here, a *bound* variable, and the other variables, in this case y, *free*. The free variables should be thought of as the actual inputs to the formula, while the bound variables are just things we use to calculate whether the formula is true. When we use (usually Greek letter names) to refer to formulas, we'll put the free variables in parentheses, like this: $\phi(x, y)$ will be a function where x and y are the free variables.

For example, I might refer to $\forall x, y \leq x * y$ as $\psi(y)$, because y is the only free variable. If I then want to interpret it in a particular structure $\mathcal{M}$, I need to first pick a value for y, and then I can calculate a truth value, although calculating it will involve thinking about possible values for x. If $a \in M$, and when I set y to be a, this turns out to be true, I say $\mathcal{M} \models \psi(a)$, otherwise, I say $\mathcal{M} \not\models \psi(a)$.

To interpret the logical symbols, we use the following rules, for $a_1, \ldots, a_k \in M$ where k is the number of free variables:

- We know how to interpret atomic formulas.
- We say that $\mathcal{M} \models \neg\phi(a_1, \ldots, a_k)$ when $\mathcal{M} \not\models \phi(a_1, \ldots, a_k)$.
- We say that $\mathcal{M} \models \phi \vee \psi(a_1, \ldots, a_k)$ when either $\mathcal{M} \models \phi(a_1, \ldots, a_j)$ or $\mathcal{M} \models \psi(a_1, \ldots, a_k)$.

- We say that $\mathcal{M} \models \phi \wedge \psi(a_1, \ldots, a_k)$ when $\mathcal{M} \models \phi(a_1, \ldots, a_j)$ and $\mathcal{M} \models \psi(a_1, \ldots, a_k)$.
- We say that $\mathcal{M} \models \phi \to \psi(a_1, \ldots, a_k)$ when $\mathcal{M} \models \psi(a_1, \ldots, a_k) \vee \neg\phi(a_1, \ldots, a_j)$.
- We can just say that $\mathcal{M} \models \phi$ when $\mathcal{M} \models \phi(a_1, \ldots, a_k)$ for every choice of $a_1, \ldots, a_k$.

A formula with no free variables is a *sentence*, and these are what our axioms and theorems will be. In particular, because a sentence ϕ has no free variables, we can ask whether $\mathcal{M} \models \phi$ without specifying any more information.

1.5 Theories

Ok, finally we can capture the idea of an axiom system. After all, no matter how well we choose our language, a graph is more than a structure in the language of graph theory, a group is more than a structure in the language of group theory.

Definition 6. An $\mathcal{L}$-*theory* is a set of $\mathcal{L}$-sentences.

If T is a $\mathcal{L}$-*theory*, and $\mathcal{M}$ is a $\mathcal{L}$-structure, then we say that $\mathcal{M} \models T$ when $\mathcal{M} \models \phi$ for EVERY $\phi \in T$. When this happens, we also say that $\mathcal{M}$ is a *model* of T. (Finally, a justification of the word "model" in "Model Theory"!)

Each of the classes of objects we talked about at the beginning of class can, if we're careful, be defined as "the models of T" for some well-chosen language $\mathcal{L}$ and some well-chosen $\mathcal{L}$-theory T. There is, for instance, a theory of graphs and a theory of groups. In the homework, we will look at how to write down theories for a few different kinds of objects.

There's one particular construction of theories I'd like to go over. If $\mathcal{M}$ is a structure, then $\mathrm{Th}(\mathcal{M})$, the "(complete) theory of $\mathcal{M}$," is the set of all sentences ϕ such that $\mathcal{M} \models \phi$.

Monday Problems

1.6 Arithmetic/Number Theory

Let $\mathcal{L} = \{0, 1, +, *, \leq\}$. Remember how we interpreted $\mathbb{N}$ as an $\mathcal{L}$-structure.

Problem 1. A subset $A \subseteq \mathbb{N}^k$ is called *definable* when there is some $\mathcal{L}$-formula $\phi(x_1, \ldots, x_k)$ (with k free variables) such that for all $a_1, \ldots, a_k \in \mathbb{N}$, $\mathbb{N} \models \phi(a_1, \ldots, a_k)$ if and only if $(a_1, \ldots, a_k) \in A$. Prove that the following sets are definable:

- The set of even numbers
- The set of pairs (x, y) such that $x|y$
- The set of prime numbers
- The set of twin primes

(Note: $\mathbb{N}^1$ is basically the same thing as $\mathbb{N}$.)

1.7 Posets

Problem 2. Explain how a poset can be thought of as a structure in the language $\mathcal{L} = \{\leq\}$ with one binary relation. Find a sentence ϕ such that for any $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models \phi$ if and only if $\mathcal{M}$ is a poset. Does such a sentence exist for linear orderings?

1.8 Graphs

Let $\mathcal{L} = \{E\}$ be the language with one binary relation.

Problem 3. Explain how a graph can be thought of as an $\mathcal{L}$-structure. Find a sentence ϕ such that for any $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models \phi$ if and only if $\mathcal{M}$ is a (simple, undirected) graph.

Problem 4. Find a theory T such that for any $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models T$ if and only if $\mathcal{M}$ is a bipartite (2-colorable) graph.

Problem 5. Show that there is a theory T such that for any *finite* $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models T$ if and only if $\mathcal{M}$ is a planar graph. (You may find it useful to look up Kuratowski's Theorem, but it is not necessary.)

1.9 Linear Algebra

Problem 6. What first-order language would you use to describe vector spaces?

In the language you chose, what do terms look like?

1.10 Rings and Fields

Let $\mathcal{L} = \{0, 1, +, *\}$.

Problem 7. Interpret $\mathbb{N}, \mathbb{R}$ as $\mathcal{L}$-structures in this language using the usual definitions of $0, 1, +, *$. Show that for each of these structures, there is a formula $\phi(x, y)$ such that if $\mathcal{M}$ is the structure, then for all $a, b \in M$, $a \leq b$ if and only if $\mathcal{M} \models \phi(x, y)$. Another way of saying this is that "$\leq$ is definable in $\mathcal{M}$."

Problem 8. Show that $\leq$ is definable in $\mathbb{Z}$ and $\mathbb{Q}$. (Hint: this needs a nontrivial theorem.)

Problem 9. A field K is *algebraically closed* if every nonconstant polynomial with coefficients in K has a zero in K. For example, $\mathbb{C}$ is algebraically closed.

Describe a theory T such that for any $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models T$ if and only if $\mathcal{M}$ is an algebraically closed field.

Problem 10. The *characteristic* of a field is either the smallest positive integer n such that in that field, $\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$, or 0 if there is no such number. (For instance, $\mathbb{R}$ has characteristic 0, while $\mathbb{Z}/5\mathbb{Z}$ has characteristic 5.)

Describe a theory T such that for any $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models T$ if and only if $\mathcal{M}$ is a field of characteristic 0.

2 Tuesday: Completeness and Compactness

Now that we have a formal model of an axiom system, it's time to look at what happens when we actually use these axioms to prove theorems. There are many definitions of what it means for a theory (again, a set of sentences, which we think of as axioms) to "prove" a theorem, but all the *good* definitions that I'm aware of turn out to be equivalent, so we're only going to take a quick tour through proof theory on our way to how model theory really deals with proofs and implication. If you like thinking about these formal models of proof, then I can talk about them more in a Week 5 class on Gödel's Incompleteness Theorems.

What actually is a mathematical proof? In most contexts, the answer involves convincing text, written mostly in a natural, human language, but let's go fully abstract and symbolic here. How can we write a proof totally formally in first-order logic?

There are, as I said, a few good answers, but ultimately, a formal proof is a list of first-order formulas, where each formula is derived somehow from the allowed axioms and the sentences before it.

Definition 7. Let ϕ be a formula, and let T be a theory. Then a *formal proof* of ϕ from T is a list $\chi_1, \ldots, \chi_n$ of formulas, such that for each $1 \le i \le n$, χ_i satisfies one of the following conditions:

- $\chi_i \in T$ (you're allowed to assume anything from T)
- χ_i is a logical axiom (these are some formulas that are true in every $\mathcal{L}$-structure)
- There are some $j, k < i$ such that χ_k is $\chi_j \to \chi_i$. (This rule is called *modus ponens*.)
- There is some $j < i$ and some variable x such that $\chi_i = \forall x, \chi_j$.

and $\phi \in \{\chi_1, \ldots, \chi_n\}$.

Definition 8. If ϕ is a formula, and T a theory, we say that "T proves ϕ" ($T \vdash \phi$) when there is a formal proof of ϕ from T.

This definition captures the idea of actual mathematical proofs, and what it means for a statement to follow from a set of axioms. But these proofs aren't very useful if they don't actually tell us whether a statement is true for actual models! Luckily, these proofs are useful in just that way. Any statement that follows from the axioms is actually true in every model.

Theorem 2.1 (Soundness)**.** *If $T \vdash \phi$, then for every model $\mathcal{M} \models T$, then $\mathcal{M} \models \phi$.*

Proof. If $T \vdash \phi$, then we can write down a proof for ϕ, call it $\chi_1, \ldots, \chi_n$, and $\phi = \chi_j$ for some j. Fix a model $\mathcal{M}$, and we can show by induction and casework that $\mathcal{M} \models \chi_i$ for each i. That's clearly true if $\chi_i \in T$, or if χ_i is a logical axiom (ok, we actually should check that the logical axioms are true in every $\mathcal{L}$-structure, but they are.)

Then we have two inductive steps. If $\mathcal{M} \models \chi_j$ and $\mathcal{M} \models \chi_j \to \chi_i$, then $\mathcal{M} \models \chi_i$ (Exercise if you don't believe me!)

Lastly, if $\mathcal{M} \models \phi$, then $\mathcal{M} \models \forall x, \phi$. This is basically just how we defined $\mathcal{M} \models \phi$ when ϕ has free variables. □

Now that we've established that this idea of proof actually describes real behavior of models, let's look at some properties that theories can have, defined in terms of $\vdash$:

Definition 9.
- A theory T is *inconsistent* if there is some sentence ϕ such that $T \vdash \phi$ and $T \vdash \neg\phi$. (Otherwise it's consistent.)
- A theory T is *complete* if it is consistent and for every ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$ (but necessarily not both.)

Inconsistent theories are bad - their consequences are contradictory, and once we get that one contradiction, we can derive whatever we want from it. That is, if T is inconsistent, then for every ϕ, $T \vdash \phi$. (To see this, it's enough to check that $\{\phi, \neg\phi\} \vdash \psi$.) Also, no model actually models contradictory sentences, so by Soundness, we see that T cannot have any models at all.

Meanwhile, complete theories are great! They tell us everything we might possibly want to know about their models - well, everything that first-order sentences describe. For an example of a complete theory, pick your favorite structure $\mathcal{M}$. We define *the theory of* $\mathcal{M}$, $\mathrm{Th}(\mathcal{M})$, to be the set of all ϕ such that $\mathcal{M} \models \phi$. This will be a complete theory, and in general, if T is a complete theory, then for each model $\mathcal{M} \models T$, $\mathrm{Th}(\mathcal{M})$ will be exactly the *consequences* of T. That is, $\{\phi : T \vdash \phi\}$. (Soundness ensures that $\mathcal{M} \models \phi$ for every such ϕ, and any more would give a contradiction.)

Now let's introduce a theorem that's way more powerful than it looks at first.

Theorem 2.2. *A theory T is consistent if and only if every finite subtheory $T_0 \subseteq T$ is consistent.*

Proof. First - does anyone have any suggestions for how to prove this?

It is easier to prove the contrapositive - if a subtheory $T_0 \subseteq T$ is inconsistent, then T is definitely inconsistent, because $T_0 \vdash \phi$ implies $T \vdash \phi$.

Why though do we know that if T is inconsistent, then there is some *finite* T_0 that is also inconsistent? The fundamental idea is that *proofs are finite*. If $T \vdash \phi$ and $T \vdash \neg\phi$, then we can write down a finite list T_0 of just the axioms we need in those two proofs. □

This is useful enough for describing how $\vdash$ works, but when we combine it with the following theorem, it becomes massively useful for describing models.

Theorem 2.3 (Completeness)**.** *A theory T is consistent if and only if it has a model.*

One direction of this is just Soundness, but the other direction is much deeper. Essentially this tells us that the only thing stopping a theory from having a model is if it actually has a contradiction. The proof is too technical for me to want to lecture it, but we'll see another proof later this week that gets the basic idea across. If you're interested, I can have some optional homework problems that walk you through the completeness proof.

The completeness theorem can be phrased in another remarkable way: that $T \vdash \phi$ if and only if $T \models \phi$, where $T \models \phi$ means that if $\mathcal{M} \models T$, then $\mathcal{M} \models \phi$. To prove this version, observe that $T \vdash \phi$ if and only if $T \cup \{\neg\phi\}$ is inconsistent, and $T \models \phi$ if and only if there is no model of $T \cup \{\neg\phi\}$.

This brings us to perhaps my favorite theorem ever:

Theorem 2.4 (Compactness)**.** *A theory T has a model if and only if each finite subtheory T_0 has a model.*

This follows pretty quickly from what we've said earlier about the finiteness of proofs and the completeness theorem. There's also an awesome proof using ultrafilters - keep your eyes peeled for a Week 5 class proposal on ultraproducts. A word of warning about the proof - the completeness and compactness theorems both need a fair amount of Choice to prove (specifically, a magic wand called the ultrafilter lemma).

Compactness can do several tricks, but right now I'd like to show off its usefulness for constructions. If you want to build a strange structure, it can be useful to design a custom theory (often in a new language with extra constants) and then show that each of its finite subtheories has a model. Then you get a model of your custom theory, which hopefully has the properties you want.

Tuesday Problems

Problem 11. Show that if a theory T has arbitrarily large finite models, it has an infinite model.

Problem 12. Let T and T' be $\mathcal{L}$-theories such that every $\mathcal{L}$-structure models either T or T', but not both. Show that there is a formula ϕ such that $\mathcal{M} \models \phi$ implies $\mathcal{M} \models T$.

3 Groups

Problem 13. A group G is called a *torsion* group when for every $g \in G$, there is some natural number n such that $g^n = e$. Show that there is no theory T in the language $\mathcal{L} = \{e, *, ^{-1}\}$ such that for every $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models T$ if and only if $\mathcal{M}$ is a torsion group.

4 Graphs

Let $\mathcal{L} = \{E\}$ consist of single binary relation, representing the edge relation on graphs.

Problem 14. Show that there is no single $\mathcal{L}$-formula ϕ such that for any $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models \phi$ if and only if $\mathcal{M}$ is a bipartite graph.

Problem 15. Show that there is no $\mathcal{L}$-theory T such that for any $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models \phi$ if and only if $\mathcal{M}$ is a connected graph.

Problem 16. Let $k \in \mathbb{N}$. Show that a graph is k-colorable if and only if all of its finite subgraphs are k-colorable.

5 Fields

Problem 17. Let $\mathcal{L} = \{0, 1, +, *\}$ and let ϕ be an $\mathcal{L}$-sentence. Show that if there are fields of arbitrarily high characteristic that model ϕ, then there is a field of characteristic 0 that models ϕ.

6 Wednesday: Elementary Inclusion

Today, we'll begin with a definition.

Definition 10. Let $\mathcal{M}$ and $\mathcal{N}$ be structures. We say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$, or $\mathcal{N}$ is an *extension* of $\mathcal{M}$ if

- The universe of $\mathcal{M}$ is a subset of the universe of $\mathcal{N}$: $M \subseteq N$
- For every constant symbol c, $c^{\mathcal{M}} = c^{\mathcal{N}}$
- For every function symbol f, and each $m_1, \ldots, m_{n_f} \in M$, $f^{\mathcal{M}}(m_1, \ldots, m_{n_f}) = f^{\mathcal{N}}(m_1, \ldots, m_{n_f})$.
- For every relation symbol r, and each $m_1, \ldots, m_{n_r} \in M$, $r^{\mathcal{M}}(m_1, \ldots, m_{n_r})$ is true if and only if $r^{\mathcal{N}}(m_1, \ldots, m_{n_r})$.

Each subset of the universe of a structure $\mathcal{M}$ can be made a substructure in a unique way, if it contains all of the constants and is closed under the application of the function symbols.

For some examples in the language $\{0, 1, +, *, \leq\}$, we have $\mathbb{N}$ is a substructure of $\mathbb{Z}$ is a substructure of $\mathbb{Q}$ is a substructure of $\mathbb{R}$.

Substructures in general can be very different from their extensions. For instance, $\mathbb{N} \models \forall x, 0 \leq x$, while the rest of our examples do not, and $\mathbb{R} \models \exists x, x * x = 1 + 1$, while the rest do not. To exclude these situations, let's introduce a stronger definition:

Definition 11. We say that $\mathcal{M}$ is an *elementary* substructure of $\mathcal{N}$ and $\mathcal{N}$ an *elementary* extension of $\mathcal{M}$ when $\mathcal{M}$ is a substructure of $\mathcal{N}$ such that for all formulas $\phi(x_1, \ldots, x_k)$ and all $m_1, \ldots, m_k$, $\mathcal{M} \models \phi(m_1, \ldots, m_k)$ if and only if $\mathcal{N} \models \phi(m_1, \ldots, m_k)$.

This gets the notation $\mathcal{M} \preceq \mathcal{N}$.

If $\mathcal{M} \preceq \mathcal{N}$, then among other things, $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. Unlike the cases of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, which behave fundamentally differently, a structure and its elementary extensions are extremely similar, so a common proof technique is actually to prove something in a structure by first proving it in an elementary extension where it's easier to pull off some trick.

I can talk more about specific examples on Friday, but I'd like to briefly introduce you to some elementary extensions. If you strip away the arithmetic, and just look at the ordering (that is, you use the language $\leq$), then $\mathbb{Q} \preceq \mathbb{R}$. These are both models of the theory of *dense linear orders without* *endpoints*, which I may say more about later. For now, I'll just tell you that it's a complete theory with a nice finite axiomatization, and if you want to check that this is an elementary extension, you can figure out the rest of the details.

Another example, which we will meet again on Friday, is the elementary extension of $\mathbb{R}$ in the language $\{0, 1, +, *, \leq\}$ known as the *hyperreals*. This structure has infinitesimal and infinite elements in it, which we can use to get easier proofs of calculus facts, with slide back down to $\mathbb{R}$, as if by magic.[1]

There is a quick test for determining if a substructure is elementary:

Theorem 6.1 (Tarksi-Vaught Test). *Let $\mathcal{M}$ be a substructure of $\mathcal{N}$. We have $\mathcal{M} \preceq \mathcal{N}$ if and only if for every formula $\phi(x, y_1, \ldots, y_k)$ and $m_1, \ldots, m_k \in M$, then if there exists $n \in N$ such that $\mathcal{N} \models \phi(n, m_1, \ldots, m_k)$, then there also exists $m \in M$ such that $\mathcal{N} \models \phi(n, m_1, \ldots, m_k)$.*

The proof of this is in the homework! It starts by showing that if a sentence has no quantifiers, then it's always true in $\mathcal{M}$ if and only if it's true in $\mathcal{N}$. Then if we rewrite all the $\forall$s using $\exists$ and $\neg$, we can use the quantifier-free formulas as a base case, and induct on the number of $\exists$s. The inductive step has two directions - one direction is easy (if something exists in M, then it exists in N), and the other direction is given by the Tarski-Vaught Test.

6.1 The Löwenheim-Skolem Theorems

We now have the tools to state two remarkable theorems.

Theorem 6.2 (Löwenheim-Skolem). *Let $\mathcal{L}$ be a countable language, and let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure.*

- *There is a countable $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{N} \preceq \mathcal{M}$.*
- *For any cardinality $\kappa \geq |M|$, there is an $\mathcal{L}$-structure $\mathcal{N}$ with $|N| = \kappa$ such that $\mathcal{M} \preceq \mathcal{N}$.*

The first bulletpoint here is called *Downward* Löwenheim-Skolem, because the cardinality of the model goes down, and the second is called *Upward* Löwenheim-Skolem, because the cardinality of the model can go way, way up. As with nearly all interesting theorems about cardinality, it has some bizarre consequences, including *Skolem's Paradox*, which I'll get to tomorrow!

6.2 Elementary Diagrams

For the rest of the day, let's discuss a tool for building elementary extensions, which you can use in the homework to prove Upward Löwenheim-Skolem and construct nonstandard models of arithmetic and analysis.

Given a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$, define the language $\mathcal{L}_\mathcal{M}$ by adding a new constant symbol c_m for every $m \in M$, getting $\mathcal{L}_\mathcal{M} := \mathcal{L} \cup \{c_m : m \in M\}$. Then we can interpret $\mathcal{M}$ as an $\mathcal{L}_\mathcal{M}$-structure by interpreting each c_m as m, let's call this structure $\mathcal{M}^*$.

Now let the *elementary diagram* of $\mathcal{M}$ be $\mathrm{Th}(\mathcal{M}^*)$. What kind of sentences are in this theory? In general, a $\mathcal{L}_\mathcal{M}$-formula is just an $\mathcal{L}$-formula where we've replaced some of the variables with elements of M. That means that the sentences will all take the form $\phi(c_{m_1}, \ldots, c_{m_n})$ where $\phi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula. So really, the elementary diagram of $\mathcal{M}$ is the set of all formulas $\phi(c_{m_1}, \ldots, c_{m_n})$ where ϕ is an $\mathcal{L}$-formula and $\mathcal{M} \models \phi(m_1, \ldots, m_n)$.

But we're doing *Model* Theory, so now that we've constructed a potentially interesting theory, we should find out what its models are. Firstly, they're $\mathcal{L}_\mathcal{M}$-structures. If $\mathcal{N}$ is a $\mathcal{L}_\mathcal{M}$-structure, then that just means it's an $\mathcal{L}$-structure with some extra information: the interpretation of the constants $\{c_m : m \in M\}$. Basically this amounts to a function $M \to N$, sending $m \mapsto c_m$. If $\mathcal{N}$

[1] The magic is ultrafilters.

turns out to model the elementary diagram of $\mathcal{M}$, then right away we know that as an $\mathcal{L}$-structure, it models $\mathrm{Th}(\mathcal{M})$, and we just need to understand what happens with the new constants. It turns out that the function $M \to N$ sending $m \mapsto c_m$ is an injection. (Exercise!) This means that the subset $\{c_m^{\mathcal{N}} : m \in M\}$ is in bijection with M. Let's consider the best case scenario, and assume that $M \subseteq N$, with $c_m^{\mathcal{N}} = m$. I claim this isn't really such a big assumption to make. In this case, I claim that actually, $\mathcal{M} \preceq \mathcal{N}$, as $\mathcal{L}$-structures. The exact details are left as a homework exercise.

Once you've proven this, it is easy to find elementary extensions for a given structure $\mathcal{M}$ - just find a model for its elementary diagram. We can add extra stipulations to this idea, and use it to find elementary extensions of a specified size, solving Upward Löwenheim-Skolem, and even create paradoxical-looking number systems out of good old $\mathbb{N}$ and $\mathbb{R}$.

Wednesday Problems

6.3 Checking the Tarski-Vaught Test

Let $\mathcal{M}$ be a substructure of $\mathcal{N}$.

Problem 18. Let $\phi(x_1, \ldots, x_k)$ be an atomic formula, and let $m_1, \ldots, m_k \in M$. Show that $\mathcal{M} \models \phi(m_1, \ldots, m_k)$ if and only if $\mathcal{N} \models \phi(m_1, \ldots, m_k)$.

Problem 19. Let $\phi(x_1, \ldots, x_k)$ be a quantifier-free formula, and let $m_1, \ldots, m_k \in M$. Show that $\mathcal{M} \models \phi(m_1, \ldots, m_k)$ if and only if $\mathcal{N} \models \phi(m_1, \ldots, m_k)$.

Problem 20. Finish the proof of the Tarski-Vaught Test:

Suppose that for every formula $\phi(x, y_1, \ldots, y_k)$ and all $m_1, \ldots, m_k \in M$, if there exists $n \in N$ such that $\mathcal{N} \models \phi(n, m_1, \ldots, m_k)$, then there also exists $m \in M$ such that $\mathcal{N} \models \phi(m, m_1, \ldots, m_k)$. Show that $\mathcal{M} \preceq \mathcal{N}$.

(Remember that you will want to swap out $\forall\phi$ with $\not\exists \not\phi$, and induct on the number of quantifiers.)

6.4 Upward Löwenheim-Skolem

Problem 21. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Show that if $\mathcal{N}$ is a model of the elementary diagram of $\mathcal{M}$, then there is an elementary extension of $\mathcal{M}$ with the same size as $\mathcal{N}$. (The universes have the same cardinality/are in bijection.)

Problem 22. Prove the following version of the Upward Löwenheim-Skolem theorem using compactness, elementary diagrams, and some extra symbols:

Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure, and let κ be a set. Show that there is an elementary extension $\mathcal{M} \preceq \mathcal{N}$ such that $|\kappa| \leq |N|$.

(To get to the version of ULS mentioned in the notes, you need to combine this with a slightly stronger version of DLS.)

6.5 More on Elementary Inclusion

Problem 23. Show that $\preceq$ satisfies the axioms of a partial order:

Reflexivity: Let $\mathcal{M}$ be a $\mathcal{L}$-structure, show that $\mathcal{M} \preceq \mathcal{M}$.

Antisymmetry: Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures such that $\mathcal{M} \preceq \mathcal{N}$ and $\mathcal{N} \preceq \mathcal{M}$, show that $\mathcal{M} = \mathcal{N}$.

Transitivity: Let $\mathcal{M}, \mathcal{N}, \mathcal{P}$ be $\mathcal{L}$-structures such that $\mathcal{M} \preceq \mathcal{N}$ and $\mathcal{N} \preceq \mathcal{P}$, show that $\mathcal{M} \preceq \mathcal{P}$.

Problem 24. Let $\mathcal{M}, \mathcal{N}, \mathcal{P}$ be $\mathcal{L}$-structures such that $\mathcal{M} \preceq \mathcal{P}$, $\mathcal{N} \preceq \mathcal{P}$, and $M \subseteq N$. Show that $\mathcal{M} \preceq \mathcal{N}$.

Problem 25. Prove the Tarksi-Vaught *Theorem*:

Let $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots$ be $\mathcal{L}$-structures such that for every $i < j$, $\mathcal{M}_i \preceq \mathcal{M}_j$. Show that there is a natural way to interpret $\bigcup_{i \in \mathbb{N}} M$ as an $\mathcal{L}$-structure $\mathcal{M}_\infty$, and that for all $i \in \mathbb{N}$, $\mathcal{M}_i \preceq \mathcal{M}_\infty$.

7 Thursday: Downward Löwenheim-Skolem

In class this day, I lectured through the proof of Downward Löwenheim-Skolem, but it was pretty dense. I cleaned up the proof and presented it as follows as the Thursday notes/homework, with certain steps left as exercises.

Here's the version of Downward Löwenheim-Skolem we're going to prove:

Theorem 7.1. *Let $\mathcal{L}$ be a countable language, and let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure. Then there exists a countable $\mathcal{N} \preceq \mathcal{N}$.*

In fact, to prove the entire Löwenheim-Skolem theorem, we'd need a stronger downward version, where we can basically shrink the model down to any specific size, but this is a good place to start.

7.1 Constructing a countable substructure

Our fundamental technique for this proof (and all variations on it) is to construct the smallest possible substructure of $\mathcal{M}$. It turns out that if we have a countable language, that smallest possible substructure is necessarily countable.

Problem 26. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $N = \{t^{\mathcal{M}} : t \text{ is a term with no variables}\}$. Prove that N is the universe of a substructure $\mathcal{N}$.

Problem 27. If $\mathcal{L}$ is countable, show that N is countable.

Problem 28 (Optional)**.** If $\mathcal{P}$ is another substructure of $\mathcal{M}$, show that $N \subseteq P$.

Problem 29 (Optional)**.** Let $\mathcal{L} = \{0, 1, +, *, \leq\}$. Show that if $\mathcal{M}$ is one of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, then $N = \mathbb{N}$.

7.2 Skolem Functions

In our proof of Downward Löwenheim-Skolem, we know that we can construct a countable substructure, but it's not necessarily elementary (see 29). We want to construct a countable substructure that *is* elementary, and to check that it's elementary, we'll use the Tarski-Vaught Test. What follows is an attempt to custom-build a substructure to satisfy the Tarski-Vaught Test.

Fundamentally what we want to do is to extend the language $\mathcal{L}$ to a language $\mathcal{L}^*$ by adding countably many function symbols, and extending $\mathcal{M}$ to a structure $\mathcal{M}^*$ by adding interpretations of those function symbols. Then we will find a countable substructure of $\mathcal{M}^*$, which we call $\mathcal{N}^*$. If we peel away the extra function symbols, we can also view $\mathcal{N}^*$ as an $\mathcal{L}$-substructure, $\mathcal{N}$. If we are careful with how we add the function symbols and their interpretations, we can ensure that $\mathcal{N}$ satisfies the requirements of the Tarski-Vaught Test, so $\mathcal{N} \preceq \mathcal{M}$, and we are done.

(The way I will construct this $\mathcal{L}^*$ is a bit more straightforward than how I did it in class today - but the basic idea is the same.)

The basic idea is that of *Skolem functions*. Let $\phi(x, y_1, \ldots, y_k)$ be an $\mathcal{L}$-formula. Then f_ϕ is a *Skolem function for ϕ in $\mathcal{M}$* if for all $m, m_1, \ldots, m_k \in M$ such that $\mathcal{M} \models \phi(m, m_1, \ldots, m_k)$, we have $\mathcal{M} \models \phi(f_\phi(m_1, \ldots, m_k), m_1, \ldots, m_k)$.

Problem 30 (Skolem functions exist)**.** Let $\phi(x, y_1, \ldots, y_k)$ be an $\mathcal{L}$-formula. Show that there exists a Skolem function f_ϕ for ϕ in $\mathcal{M}$. (Technically, this will use the axiom of choice. If you have questions about that, ask me.)

Define $\mathcal{L}^*$ to be the union $\mathcal{L} \cup \{f_\phi : \phi \text{ is an } \mathcal{L}\text{-formula}\}$. Define $\mathcal{M}^*$ by letting $f_\phi^{\mathcal{M}}$ be a Skolem function for ϕ in $\mathcal{M}$. (You know they exist by 30.)

Problem 31. Check that $\mathcal{L}^*$ is countable.

Problem 32. Let $\mathcal{N}^*$ be a countable substructure of $\mathcal{M}^*$, with universe N. Show that for every $n_1, \ldots, n_k \in N$, if there is $m \in M$ such that $\mathcal{M} \models \phi(m, n_1, \ldots, n_k)$, then $\mathcal{M} \models \phi(f_\varphi(n_1, \ldots, n_k), n_1, \ldots, n_k)$, and that $f_\varphi(n_1, \ldots, n_k) \in N$.

Conclude using the Tarski-Vaught Test that if $\mathcal{N}$ is the $\mathcal{L}$-substructure of $\mathcal{M}$ with universe N, then $\mathcal{N} \preceq \mathcal{M}$, concluding the proof of DLS.

8 Friday: Nonstandard Models of Arithmetic

On Friday, I cleared up a few loose ends from Thursday, and then we talked about nonstandard models of arithmetic. Nonstandard models of arithmetic are structures in the language $\{0, 1, +, *, \leq\}$ that satisfy the same theory Th($\mathbb{N}$) as the natural numbers, but are not isomorphic to $\mathbb{N}$. To understand just how bizarre these structures have to be, we first have to understand some things about the theory Th($\mathbb{N}$) (sometimes called *true arithmetic*), and all its models.

In lieu of detailed notes, this was more of a dialog, with me presenting the definition, and then asking the students to help fill in the picture. We ended up showing the following things, to various levels of rigor:

- Every nonstandard model is an elementary extension of $\mathbb{N}$.
- The copy of $\mathbb{N}$ forms an initial segment.
- A nonstandard model is not well-ordered.
- The infinite elements form "galaxies" - equivalence classes under the equivalence relation $x \sim y \iff \exists n \in \mathbb{Z} : x + n = y$.
- The order type of the infinite galaxies is a dense linear order.
- If I teach this again and get around to using DLOs as an example, I could characterize the order type of countable nonstandard models.