Phase space analysis with exponential weights and non-selfadjoint spectral problems

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1. FBI-transforms, wavefront sets and applications

Microlocal analysis started in the 60ies,

> Kohn-Nirenberg, Hörmander, Maslov, Egorov in the setting of distributions mod. smooth functions,
>
> Sato-Kawai-Kashiwara [Sa70, SaKaKa71] in the setting of hyperfunctions mod. analytic functions.

Study of singularities of solutions of linear PDE, applications to spectral theory and other branches of analysis. Important tools: Pseudodifferential operators and Fourier integral operators.

The wave front set, $\mathrm{WF}(u)$ (Sato[Sa70], Hörmander [Ho71b]) is a central notion. It refines the one of singular support, $\operatorname{sing\,supp}(u)$. Let X be an open subset of $\mathbf{R}^n$ or a smooth manifold. Let $T^*X \simeq X_x \times \mathbf{R}^n_\xi$ be the cotangent bundle, write $0 = \{(x,\xi) \in T^*X;\, \xi = 0\}$. If $u \in \mathcal{D}'(X)$, then $WF(u)$ is a closed conic subset of $T^*X \setminus 0$ such that $\pi_x(WF(u)) = \operatorname{sing\,supp}(u)$, where $\pi_x : T^*X \to X$ is the natural projection.

To illustrate, consider a solution $u \in \mathcal{D}'(\mathbf{R}^{n+1}_{t,x})$ of $Pu = 0$ where $P = -D_t^2 + \sum_1^n D_{x_j}^2$ is the wave operator. If $(0,0) \in \operatorname{sing\,supp}(u)$, we know that $\operatorname{sing\,supp}(u)$ contains some union of light rays passing through $(0,0)$ of the form $x = t\omega$, $t \in \mathbf{R}$, where $\omega \in S^{n-1}$. How can we determine this union? $WF(u)$ is a closed conic subset of $T^*\mathbf{R}^{n+1} \setminus 0$, with $\pi_{t,x}(WF(u)) = \operatorname{sing\,supp}(u)$. When $Pu \in C^\infty$, we know that $WF(u) \subset p^{-1}(0)$, where $p = -\tau^2 + \xi^2$ is the principal symbol of P and we have a fundamental theorem on propagation of singularities (Hörmander, Sato-Kawai-Kashiwara) which tells us that $WF(u)$ is a union of maximally extended integral curves of $H_p = p'_\tau \partial_t + p'_\xi \cdot \partial_x$ in $p^{-1}(0)$. Thus $WF(u) \cap T^*_{(0,0)}\mathbf{R}^{n+1} \setminus \{(0,0)\}$ determines the lightrays through $(0,0)$ that are contained in $\operatorname{sing\,supp}(u)$.

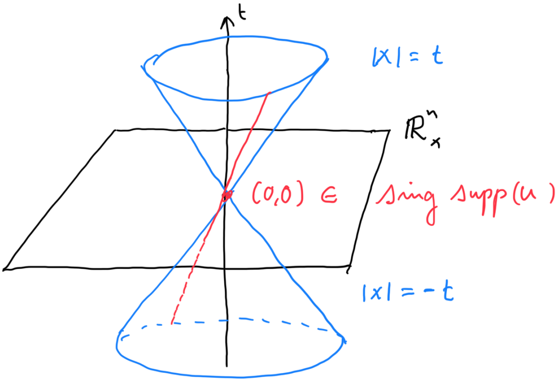


Figure: Wavecone

Our approach in the analytic framework is based on methods and ideas for Fourier integral operators with complex phase. Originally, it was used to study propagation of singularities of solutions to linear PDE (microlocal analysis), then it was globalized and applied to spectral problems (phase space analysis).

Plan of the lectures:

1. a) Analytic wavefront sets, local FBI-transforms and exponentially weighted spaces of holomorphic functions.
b) Propagation of singularities, eigenvalues of non-self-adjoint operators and resonances in the semi-classical limit (finally not included here) – a survey.

2. Eigenvalues of elliptic non-self-adjoint operators:
a) The analytic case, using semi-global weighted spaces.
b) The case of random perturbations (general Weyl law). (Finally not included here, see [Sj19].)

3. Resonances:
Global weighted spaces.
The role of trapped classical trajectories.
Potential well in an island for a semi-classical Schrödinger operator, shape resonances and higher levels.

1a. Analytic wavefront sets and FBI transforms

Let

$$\mathcal{F}u(\xi) = \int e^{-ix\cdot\xi} u(x)dx, \quad \xi \in \mathbf{R}^n$$

denote the Fourier transform of a distribution $u \in \mathcal{S}'(\mathbf{R}^n)$. The most direct definition of the usual wavefront set ([Ho71b]) is undoubtedly:

Definition (1.1)

Let $u \in \mathcal{D}'(X)$ where $X \subset \mathbf{R}^n$ is open. Let $(x_0, \xi_0) \in T^*X \setminus 0$. We say that $(x_0, \xi_0) \notin WF(u)$ iff $\exists\ \chi \in C_0^\infty(X)$ with $\chi(x_0) \neq 0$ and a conic neighborhood $V \in \mathbf{R}^n \setminus 0$ such that with $\langle\xi\rangle = (1+\xi^2)^{1/2}$,

$$\mathcal{F}(\chi u)(\xi) = \mathcal{O}_N(\langle\xi\rangle^{-N}) \text{ in } V \text{ for every } N \geq 0. \tag{1}$$

$\mathrm{WF}(u)$ *is a closed conic subset of $T^*X \setminus 0$.*

We have $\pi_x(WF(u)) = \operatorname{sing\,supp}(u)$ if $\pi_x : T^*X \setminus 0 \to X$ is the natural projection.

The definition of Sato uses the representation of hyperfunctions as sums of boundary values of holomorphic functions. Somewhat later Hörmander [Ho71c] defined the analytic wavefront set by modifying (1) in two ways:

- Replace the rapid decay by exponential decay.
- Since cutoffs are not analytic, use special sequences of cutoffs, that depend on $|\xi|$, introduced by Ehrenpreis [Ehr60], Mandelbrojt [Ma42, Ma52], ...

A third approach is to work with Fourier transforms with Gaussians. Many different names: FBI, Bargmann-Segal, Gabor, wavepacket .... transforms. In the context of analytic microlocal analysis they were introduced and used by D. Iagolnitzer, H. Stapp [IaSt69], J. Bros, Iagolnitzer [BrIa75]. This is the method we follow here. See [Sj82, Ma02a].

Let $\chi \in C_0^\infty(\mathbf{R}^n)$ be $=1$ near 0. We say that $(x_0, \xi_0) \notin WF_{\mathrm{a}}(u)$ if

$$Tu(x,\xi) := \int e^{i(x-y)\cdot\xi - |\xi|(x-y)^2} \chi(x-y)u(y)dy \qquad (2)$$

is $\mathcal{O}(e^{-|\xi|/C})$ in a conic neighborhood of (x_0, ξ_0).

Weighted spaces and symbols. Let $\Omega \subset \mathbf{C}^n$ be open, $\Phi \in C(\Omega; \mathbf{R})$. By definition, the function $u = u(z; h)$ on $\Omega \times]0, h_0[$ belongs to $H_\Phi^{\mathrm{loc}}(\Omega)$ if

- $u(\cdot; h) \in \mathrm{Hol}(\Omega)$, for all h, where $\mathrm{Hol}\,(\Omega)$ denotes the space of holomorphic functions on Ω.
- $\forall K \Subset \Omega$, $\varepsilon > 0$, $\exists C > 0$ such that $|u(z; h)| \leq Ce^{(\Phi(z)+\varepsilon)/h}$, $z \in K$.

Put

$$H_\Phi(\Omega) = \mathrm{Hol}\,(\Omega) \cap L^2(\Omega; e^{-2\Phi/h} L(dx)),\ L(dx) = \text{Lebesgue measure.}$$

When $u \in H_0^{\mathrm{loc}}(\Omega)$, we say that u is an analytic symbol. When $u = \mathcal{O}(h^{-m})$ locally uniformly on Ω, we say that u is of finite order $m \in \mathbf{R}$.

Equivalence: $u \sim v$, for $u, v \in H_\Phi^{\mathrm{loc}}(\Omega)$, means that there exists $C^0(\Omega) \ni \widetilde{\Phi} < \Phi$, such that $u - v \in H_{\widetilde{\Phi}}^{\mathrm{loc}}(\Omega)$.

By $H_{\Phi, x_0} :=$ the space of germs of functions in $H_\Phi^{\mathrm{loc}}(\Omega)$ at $x_0 \in \Omega$. We have a corresponding equivalence relation.

Classical analytic symbols (Boutet de Monvel, Krée [BoKr67]). Let $a_k \in \mathrm{Hol}\,(\Omega)$, $k = 0, 1, \dots$ and assume that for every $\widetilde{\Omega} \Subset \Omega$, $\exists C = C_{\widetilde{\Omega}} > 0$ such that

$$|a_k(z)| \le C^{k+1} k^k, \;\; z \in \widetilde{\Omega}. \tag{3}$$

$a = \sum_0^\infty a_k(z) h^k$ is called a formal classical analytic symbol.

We have a realization of a on $\widetilde{\Omega}$ by

$$a_{\widetilde{\Omega}}(z; h) = \sum_{0 \le k \le (eC_{\widetilde{\Omega}} h)^{-1}} a_k(z) h^k \in H_\Phi^{loc}(\widetilde{\Omega}).$$

If $\widehat{\Omega}$ is another relatively compact subset of Ω, then $a_{\widehat{\Omega}}$ and $a_{\widetilde{\Omega}}$ are equivalent on $\widetilde{\Omega} \cap \widehat{\Omega}$.

FBI-transforms. Let $\phi \in \mathrm{Hol}\,(\mathrm{neigh}\,((x_0, y_0), \mathbf{C}^{2n}))$, $y_0 \in \mathbf{R}^n$ and assume that

$$\begin{gathered} \phi'_y(x_0, y_0) = -\eta_0 \in \mathbf{R}^n, \;\; \Im \phi''_{yy}(x_0, y_0) > 0, \\ \det \phi''_{xy}(x_0, y_0) \neq 0. \end{gathered} \tag{4}$$

Let $a(x, y; h)$ be a classical analytic symbol of order 0, defined near (x_0, y_0), elliptic in the sense that $a_0(x_0, y_0) \neq 0$ and let $\chi \in C_0^\infty(\mathrm{neigh}\,(y_0, \mathbf{R}^n))$ be equal to one near y_0. If $u \in \mathcal{D}'(\mathbf{R}^n)$ (or just defined in a neighborhood of the support of χ), we put

$$Tu(x; h) = \int e^{i\phi(x,y)/h} a(x, y; h)\chi(y)u(y)dy, \; x \in \mathrm{neigh}\,(x_0, \mathbf{C}^n). \qquad (5)$$

Proposition

$Tu \in H^{\mathrm{loc}}_{\Phi_0}(\mathrm{neigh}\,(x_0))$, where

$$\Phi_0 = \sup_{y \in \mathrm{neigh}\,(y_0, \mathbf{R}^n)} -\Im\phi(x, y) \in C^\infty(\mathrm{neigh}\,(x_0, \mathbf{C}^n); \mathbf{R}) \textit{ is real-analytic.}$$

Example A Bargmann transform with $\phi(x, y) = i(x - y)^2/2$. Then $\Phi_0(x) = (\Im x)^2/2$ and the exponential factor in (5) becomes

$$e^{\frac{i}{2h}(x-y)^2} = e^{\Phi_0(x)/h} e^{\frac{i}{h}(\Re x - y)\cdot(-\Im x) - \frac{1}{2h}(\Re x - y)^2}.$$

cf. (2).

T is a Fourier integral operator with associated complex canonical transformation:

$$\kappa_T : \operatorname{neigh}((y_0, \eta_0), \mathbf{C}^{2n}) \ni (y, -\partial_y \phi(x,y)) \mapsto \\ (x, \partial_x(\phi(x,y)) \in \operatorname{neigh}((x_0, \xi_0), \mathbf{C}^{2n}), \ \xi_0 = \partial\phi(x_0, y_0).$$

Let

$$\Lambda_{\Phi_0} = \{(x, \frac{2}{i}\partial_x \Phi_0(x)); \ x \in \operatorname{neigh}(x_0, \mathbf{C}^n)\}.$$

Proposition

We have $\Lambda_{\Phi_0} = \kappa_T(\mathbf{R}^{2n})$. Further, Φ_0 is strictly pluri-subharmonic.

Assume that $\eta_0 \neq 0$. For $x \in \mathrm{neigh}\,(x_0)$, write

$$(y(x), \eta(x)) = \kappa_T^{-1}(x, (2/i)\partial_x \Phi_0(x)) \in T^*\mathbf{R}^n \setminus 0.$$

$y(x)$ is the local real maximum of $-\Im\phi(x, \cdot)$.

Definition

Let u be a distribution defined near y_0, independent of h. We say that $(y(x), \eta(x)) \notin \mathrm{WF}_a(u)$ if $Tu \sim 0$ in $H_{\Phi_0,x}$.

This leads to the definition of a closed conic subset $\mathrm{WF}_a(u) \subset T^*X \setminus 0$ when $u \in \mathcal{D}'(X)$, $X \subset \mathbf{R}^n$ open. We have

$$\pi_x\left(\mathrm{WF}_a(u)\right) = \mathrm{sing\,supp}_a(u), \text{ the analytic singular support of } u.$$

1b. Propagation of singularities, eigenvalues of non-self-adjoint operators and resonances – a survey

Let P be a differential operator with analytic coefficients on an open set $X \subset \mathbf{R}^n$. Let p be the principal symbol. The following theorem is due to N. Hanges [Ha81]. It improves the classical results of L. Hörmander [Ho71c] and Sato, Kawai and Kashiwara [SaKaKa71] in that it only requires one real bicharacteristic strip. See also [HaSj82].

Theorem

Assume that $H_p = p'_\xi \cdot \partial_x - p'_x \cdot \partial_\xi$ has a real integral curve $\gamma : [a, b] \to p^{-1}(0) \cap T^*X \setminus 0$, $a < b$. If $u \in \mathcal{D}'(X)$, $\mathrm{WF}_a(Pu) \cap \gamma([a, b]) = \emptyset$, then $\gamma([a, b])$ is either contained in, or disjoint from $\mathrm{WF}_a(u)$.

There are many results on propagation of singularities, especially for boundary value problems, e.g. by J. Ralston [Ra76] (Gaussian beams) and G. Eskin [Es85] (propagation and fundamental solutions in the interior).

We have seen that T is a Fourier integral operator with associated canonical transformation κ_T with $\kappa_T(T^*X) = \Lambda_{\Phi_0}$. We have a "Egorov theorem". Let $(y_0, \eta_0) \in T^*\mathbf{R}^n \setminus 0$ be a point where $p(y_0, \eta_0) = 0$, $(x_0, \xi_0) = \kappa_T(y_0, \eta_0)$. Then there exists a semi-classical pseudodifferential operator $Q(x, hD_x; h)$ with classical analytic symbol $Q(x, \xi; h) \sim q(x, \xi) + hq_1 + \dots$ such that

$$QTu \sim Th^m Pu \tag{6}$$

for every fixed $u \in \mathcal{D}'(\mathrm{neigh}\,(0, \mathbf{R}^n))$. We have

$$q \circ \kappa_T = p.$$

Q acts in $H^{\mathrm{loc}}_{\Phi_0}$ and also in H^{loc}_{Φ} when Φ is close to Φ_0 in C^2 (it is often very useful to replace Φ_0 by a deformation Φ). One proof ([HiSj18]) of Hanges' theorem is based on the possibility of choosing T so that Q in (6) is equal to hD_{x_n}.

Non-self-adjoint operators. Appear naturally in many contexts; fluid dynamics, Kramers-Fokker-Planck, damped wave equations,....

Difficulty: Spectral instability, often no useful spectral resolution.

Advantage: Often possible to study individual eigenvalues not only in 1D (as in the self-adjoint case) but also in 2D. This is a kind of complete integrability, related to the absence of small denominators. (Cherry's theorem).

With Michael Hitrik we have written a series of papers about analytic semi-classical non-self-adjoint operators in 2D of the form

$$P_\epsilon = P_0 + i\epsilon Q + \mathcal{O}(\epsilon^2)$$

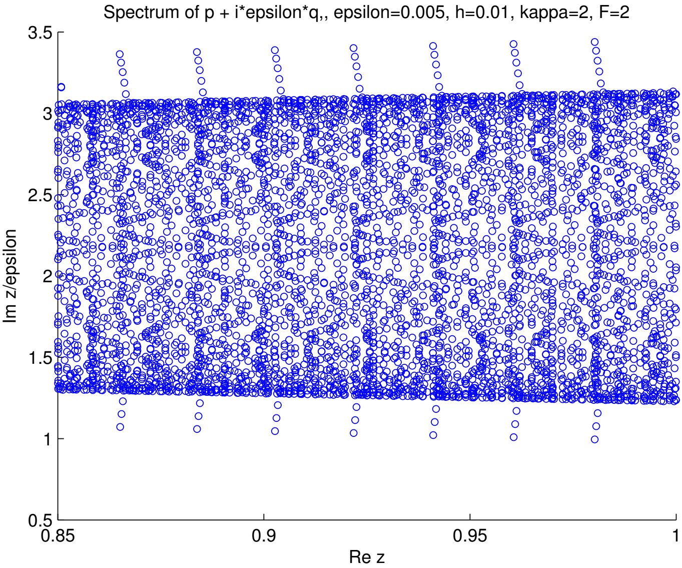
where P_0 is self-adjoint with leading (real) symbol p completely integrable.

Then the energy surface $p^{-1}(0)$ (fixing the real energy to 0) is decomposed in H_p-flow invariant sets Λ which "most of the time" are torii (Arnold-Mineur-Liouville theorem). Each torus Λ has a rotation number that may be rational, but most of the time is irrational and even Diophantine.

The spectrum near 0 is contained in a band of width $\asymp \epsilon$, parallel to the real axis. We have a Weyl law for the distribution of the real parts of the eigenvalues (Markus–Matseev [MaMa79]).

[HiSjVu07]: Diophantine torii generate distorted lattices of eigenvalues. Tools: semi-global FBI transforms and related weighted spaces of holomorphic functions.

Numerical simulations for an operator on the two-torus:
We get a "centipede; mille-pattes" whose body fits with the range of torus averages. The legs were more mysterious.



Hitrik-Sj [HiSj18]: The legs are generated by rational torii and the eigenvalues in the legs are obtained by the secular method, cf. [LiLi92].

2. Eigenvalues of elliptic non-self-adjoint differential operators

Non self-adjoint operators appear naturally in a number of areas:

- General linear PDE: solvability theory (non-normal operators, H. Lewy [Le57], L. Hörmander [Ho60a, Ho60b], ...).
- Mathematical physics: Damped wave equation, (Kramers-)Fokker-Planck operator, scattering poles.
- Fluid dynamics: Linearizations around special stationary flows.

An important difference with the self-adjoint case is that the resolvent may be large far away from the spectrum $\sigma(P)$ of the closed operator P:

$$\|(z-P)^{-1}\| \gg \frac{1}{\mathrm{dist}\,(z,\sigma(P))},$$

which implies spectral instability: a small perturbation of P may move the eigenvalues a lot.

We will discuss non-self-adjoint 2-dimensional problems, allowing very detailed results about individual eigenvalues. (A series of papers with Michael Hitrik, see [HiSj18] and further references there.)

2.1. WKB-method and quasi-modes.

Let $P(x, hD_x; h) = p(x, hD) + hp_1(x, hD) + ..$ be a semi-classical (pseudo-)differential operator on a smooth manifold X, $(x_0, \xi_0) \in T^*X$, $p(x_0, \xi_0) = z_0 \in \mathbf{C}$ and assume that $\frac{1}{i}\{p, \overline{p}\}(x_0, \xi_0) > 0$. Then we can construct a quasimode of the form $u(x; h) = e^{\frac{i}{h}\phi(x)}a(x; h)$, solving

$$(P - z_0)(u(x; h)) = \mathcal{O}(h^\infty) \text{ i.e. } \mathcal{O}_N(h^N),\ \forall N > 0,$$

normalized in L^2 and exponentially small away from x_0. Hörmander [Ho60a, Ho60b] in a different context, E.B. Davies [Da99], M. Zworski [Zw01], N. Dencker–Sj–Zworski [DeSjZw04], K. Pravda-Starov [Pr06]. This implies that $\|(P - z_0)^{-1}\|$ is very large when the resolvent exists, and spectral instability near z_0. (It does not imply that z_0 is close to the spectrum.)

Example: Davies' operator: $(hD_x)^2 + ix^2$ on $\mathbf{R}$.

2.2. Semi-global FBI-transforms

When $X = \mathbf{R}^n$, we can take a Bargmann transform as in the example above. Let now X be a compact real-analytic Riemannian manifold of dimension n. Let $d(x, y)$ be the distance We shall define an FBI-transform as above but with a global choice of phase (cf. [Bo78, GoLeSt96, Sj96, Zw99, HiSj04]). Let $\widetilde{X}$ be a complex neighborhood of X. The function $d(x, y)^2$ is analytic near the diagonal in $X \times X$ and extends holomorphically to a neighborhood of the diagonal in $\widetilde{X} \times \widetilde{X}$, if $\widetilde{X}$ is close enough to X. Put

$$\phi(x, y)^2 = i\lambda d(x, y)^2,$$

where $\lambda > 0$ is constant, large enough, depending on the size of the bounded region in T^*X that we want to cover.

For $x \in \widetilde{X}$, $|\Im x| < 1/C$, put

$$Tu(x;h) = h^{-\frac{3n}{4}} \int e^{\frac{i}{h}\phi(x,y)} a(x,y;h)\chi(x,y)u(y)dy, \ \ u \in \mathcal{D}'(X),$$

where χ is a suitable smooth cut-off function, equal to 1 near $\mathrm{diag}\,(X \times X)$ and a is an elliptic classical analytic symbol. We have the following facts:

As before we can introduce the function $\Phi_0(x) = \sup_{y \in X} -\Im\phi(x,y) = -\Im\phi(x,y(x))$, $x \in \widetilde{X}$, $|\Im x| < 1/C$. It is strictly pluri-subharmonic and of the order of magnitude $\sim |\Im x|^2$. $\Lambda_{\Phi_0} := \{(x, \frac{2}{i}\partial\Phi_0) \in T^*\widetilde{X}\}$ is given by $\Lambda_{\Phi_0} = \kappa_T(T^*X)$, where κ_T is defined as before, now with a domain containing an arbitrarily large set of the form $\{(y,\eta) \in T^*X;\ |\eta| \leq \mathcal{O}(1)\}$.

Deformations of real phase space and averaging. Let Λ_t, $t \in \mathrm{neigh}\,(0, \mathbf{R})$ be a smooth family of IR-manifolds in the complexified cotangent space with $\Lambda_0 = T^*X$. Here "IR" means that the restriction of the symplectic form is real and nondegenerate. For every choice of real analytic coordinates, Λ_t is of the form $\{\rho + itH_{G_t}(\rho);\ \rho \in T^*X\}$. G_0 is independent of the choice of local coordinates. Applying κ_T, we get

$$\kappa_T(\Lambda_t) = \Lambda_{\Phi_t}, \quad \partial_t \Phi_0 \circ \kappa_T = G_0.$$

Also

$$p_{|_{\Lambda_t}} \simeq p(\rho + itH_{G_t}(\rho)) = p(\rho) - itH_p(G_0) + \mathcal{O}(t^2).$$

(When G_t is analytic we can replace $\rho + itH_{G_0}$ by $\exp(itH_{G_0})(\rho)$.
Let $p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2)$ be a small perturbation of a real Hamiltonian p, $|\epsilon| \ll 1$. Let G be real and analytic, $\Lambda_{\epsilon G} = \exp(i\epsilon H_G)(T^*X)$.

Then

$$p_{\epsilon|_{\Lambda_\epsilon}} \simeq p(\rho + i\epsilon H_G(\rho)) = p_\epsilon(\rho) + i\epsilon(q - H_p(G)) + \mathcal{O}(\epsilon^2)$$

We can take G with $H_p(G) = q - \langle q \rangle_T$,

$$\langle q \rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} q \circ \exp(tH_p)dt,$$

and we get

$$p_{|_{\Lambda_{\epsilon G}}} = p(\rho) + i\epsilon\langle q \rangle_T + \mathcal{O}_T(\epsilon^2).$$

Particularly efficient when the H_p-flow is periodic. (A. Weinstein [We77], Y. Colin de Verdière [Co77], A. Grigis [Gr91] and in the present context Hitrik-Sj [HiSj04, HiSj08].)

2.3. The analytic case in 1D

For self-adjoint (pseudo-)differential operators in dimension 1, we often have a Bohr-Sommerfeld rule to determine the asymptotic behaviour of the eigenvalues. (B. Helffer – D. Robert [HeRo82] in this degree of generality). In the non-self-adjoint case we get the same results for small perturbations $P_\epsilon = P + i\epsilon Q$ if the coefficients of P_0 and Q *are analytic*. Averaging: Assume that $p^{-1}(0)$ is a simple closed curve on which $dp \neq 0$. Let $\langle q\rangle_E$ be the average of q on $p^{-1}(E)$ and view $\langle q\rangle$ as a function on phase space: $\langle q\rangle(\rho) = \langle q\rangle_{p(\rho)}$. Then

$$p_{|_{\Lambda_{\epsilon G}}} = p(\rho) + i\epsilon\langle q\rangle_{p(\rho)}.$$

We conclude in principle that the eigenvalues of P_ϵ in a fixed neighborhood of 0 are situated in a $\mathcal{O}(\epsilon^2)$-neighborhood of the curve $\{E + i\epsilon\langle q\rangle_E;\ E \in \mathrm{neigh}\,(0, \mathbf{R})\}$.

2.4. The analytic case in 2D

A. Melin–Sj [MeSj02, MeSj03], M. Hitrik–Sj, Hitrik–Sj–S. Vũ Ngọc: For analytic non-self-adjoint operators in dimension 2 one can often determine individual eigenvalues by means of Bohr-Sommerfeld rules in the complex domain. Especially, small perturbations of self-adjoint operators (cf. the damped wave equation) have been studied. We discuss one such result [HiSjVu07].

Let

$$P_\epsilon(x, hD; h) = \sum_{|\alpha|\le m} a_\alpha(x, \epsilon; h)(hD_x)^\alpha$$

be a semi-classical differential operator of order m on a compact analytic surface X (or on $\mathbf{R}^2$), where

- a_α is smooth in $\epsilon \in \mathrm{neigh}\,(0, \mathbf{R})$, holomorphic in x,
- $a_\alpha(x, \epsilon; h) = a_\alpha(x, \epsilon) + \mathcal{O}(h)$,
- P_ϵ is elliptic in the classical sense: $\left|\sum_{|\alpha|=m} a_\alpha(x, 0; 0)\xi^\alpha\right| \asymp |\xi|^m$,
- $P_{\epsilon=0} = P(x, hD; h)$ is self-adjoint in $L^2(X; dx)$.

The leading symbol $p_\epsilon(x,\xi) = P_\epsilon(x,\xi;0)$ is of the form $p(x,\xi) + i\epsilon q(x,\xi) + \mathcal{O}(\epsilon^2)$ where p is real and we assume q real for simplicity.

Assume that p is completely integrable, (there exists a non-trivial analytic function which Poisson commutes with p) and that

$$p^{-1}(0) \text{ is connected and } dp \neq 0 \text{ on that set.} \tag{7}$$

Example

$P_\epsilon = -h^2\Delta + i\epsilon V(x)$ *on a surface of revolution.*

Then we have a decomposition

$$p^{-1}(0) \cap T^*X = \bigsqcup_{\Lambda \in J} \Lambda, \tag{8}$$

where Λ are compact connected sets, invariant under the H_p flow. Here $H_p = p'_\xi \cdot \partial_x - p'_\xi \cdot \partial_\xi$. Typically, Λ are Lagrangian tori forming 1 parameter families: the regular part. (Arnold-Mineur-Liouville theorem). There can also be degenerations: $\Lambda \in S$.

Each torus $\Lambda \in J \setminus S$ has a rotation number $\omega(\Lambda) = [a_1 : a_2] \in \mathbf{RP}^1$ depending analytically on Λ.
We say that $\Lambda \in J \setminus S$ is respectively rational, irrational, diophantine if a_1/a_2 has the corresponding property. Diophantine means that there exist $\alpha > 0$, $d > 0$ such that

$$|(a_1, a_2) \cdot k| \geq \frac{\alpha}{|k|^{1+d}},\ 0 \neq k \in \mathbf{Z}^2. \tag{9}$$

We introduce

$$\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp(tH_p) dt,\ \ T > 0, \tag{10}$$

and consider the compact intervals

$$Q_\infty(\Lambda) := [\lim_{T\to\infty} \inf_{\Lambda} \langle q \rangle_T, \lim_{T\to\infty} \sup_{\Lambda} \langle q \rangle_T]. \tag{11}$$

Then, when $\epsilon, \delta \to 0$,

$$\begin{aligned} &\{z \in \sigma(P_\epsilon);\ |\Re z| \leq \delta\} \subset \\ &[-\delta, \delta] + i\epsilon[\inf_{\Lambda\in J} \inf Q_\infty(\Lambda) - o(1), \sup_{\Lambda\in J} \sup Q_\infty(\Lambda) + o(1)], \end{aligned} \tag{12}$$

For each torus $\Lambda \in J \setminus S$, we let $\langle\langle q \rangle\rangle(\Lambda)$ be the average of $q_{|_\Lambda}$ When Λ is irrational then $Q_\infty(\Lambda) = \{\langle\langle q \rangle\rangle(\Lambda)\}$.
Let $F_0 \in \cup_{\Lambda \in J} Q_\infty(\Lambda)$ and assume that there exists a Diophantine torus Λ_d (or finitely many), such that

$$\langle\langle q \rangle\rangle(\Lambda_d) = F_0, \quad d_\Lambda \langle\langle q \rangle\rangle(\Lambda_d) \neq 0 \neq d_\Lambda \omega(\Lambda_d). \tag{13}$$

Using averaging, in particular complex Birkhoff normal forms, we obtained:

Theorem ([HiSjVu07])

Assume also that F_0 *does not belong to* $Q_\infty(\Lambda)$ *for any other* $\Lambda \in J$. *Let* $0 < \delta < K < \infty$. *Then* $\exists C > 0$ *such that for* $h > 0$ *small enough, and* $h^K \leq \epsilon \leq h^\delta$, *the eigenvalues of* P_ϵ *in the rectangle* $|\Re z| < h^\delta / C$, $|\Im z - \epsilon F_0| < \epsilon h^\delta / C$ *form a distorted lattice, given by a complex Bohr-Sommerfeld condition, with horizontal spacing* $\asymp h$ *and vertical spacing* $\asymp \epsilon h$.

2.5. Numerical illustrations in 2D

See [HiSj18]. Numerically easy situation: $X = \mathbf{T}^2$,

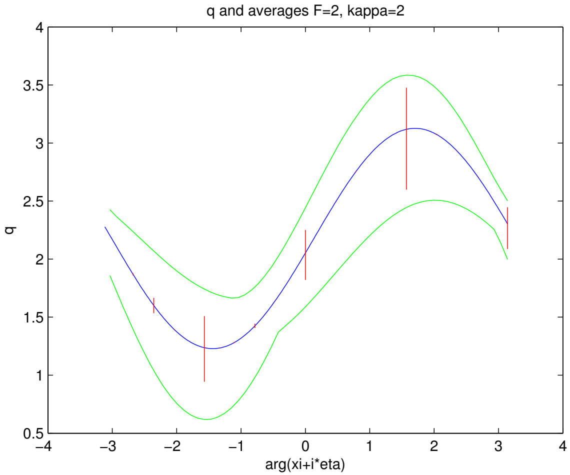
$$P_\epsilon = -h^2\Delta_{x,y} + i\epsilon(q_0(x,y) + q_1(x,y)hD_x + q_2(x,y)hD_y)$$

where q_j are real trigonometric polynomials of degree F.
Symbol:

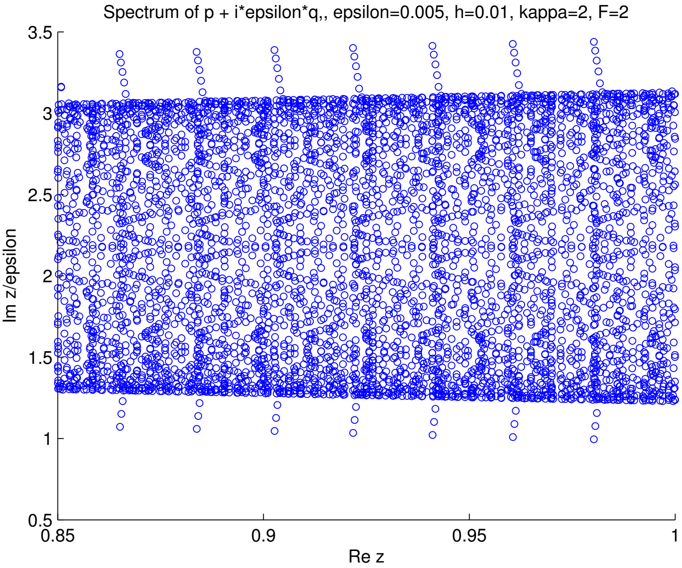
$$\xi^2 + \eta^2 + i\epsilon \underbrace{(q_0(x,y) + q_1(x,y)\xi + q_2(x,y)\eta)}_{=:q(x,y,\xi,\eta)}.$$

We look at the eigenvalues z with $0.85 \le \Re z \le 1$. The energy surface $\xi^2 + \eta^2 = 1$ is foliated into invariant tori, $\xi = \mathrm{const}$, $\eta = \mathrm{const}$, that we parametrize by $\arg(\xi + i\eta)$:

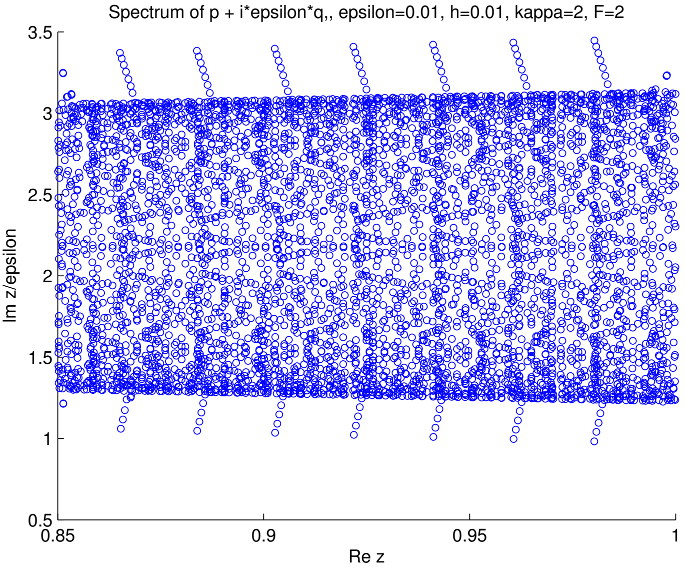
- The torus average of q,
- The torus max and min of q
- $Q_\infty(\Lambda)$ for each relevant rational torus.

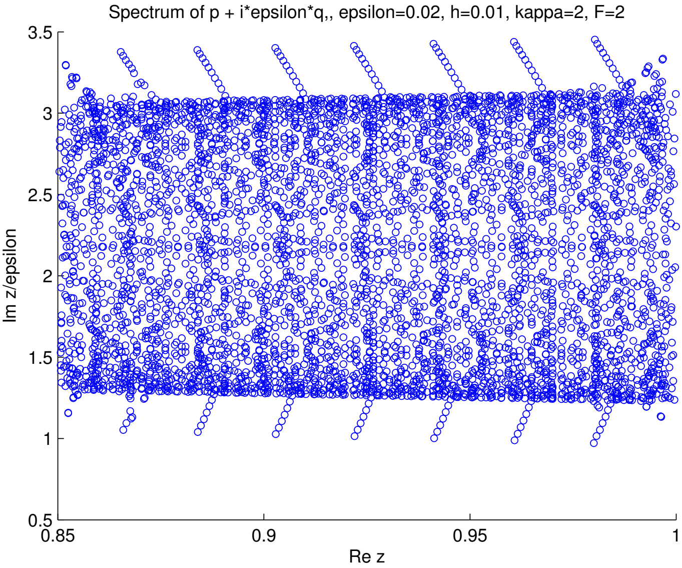


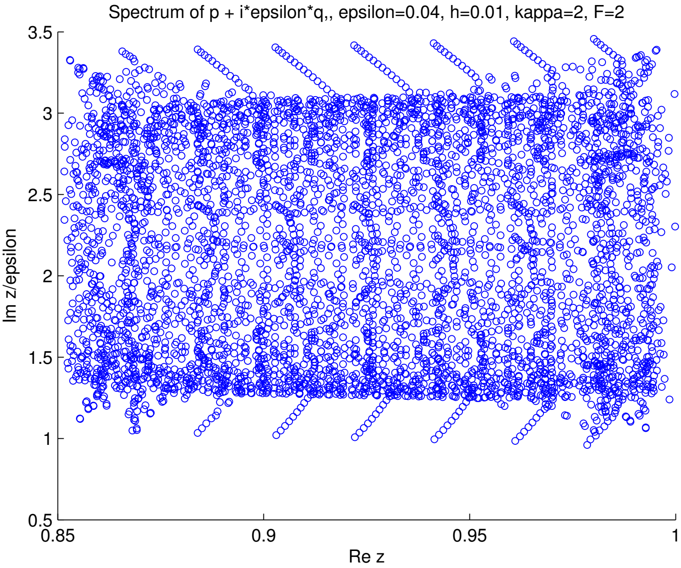
Below we plot the spectrum for the same fixed q, with $h = 0.01$, and ϵ successively doubling from $h/2$ to $16h$. We get a "centipede; mille pattes" whose body fits with the range of torus averages. The legs were more mysterious. Influence of rational tori? Relation with the operator $(hD)^2 + i\cos x$??

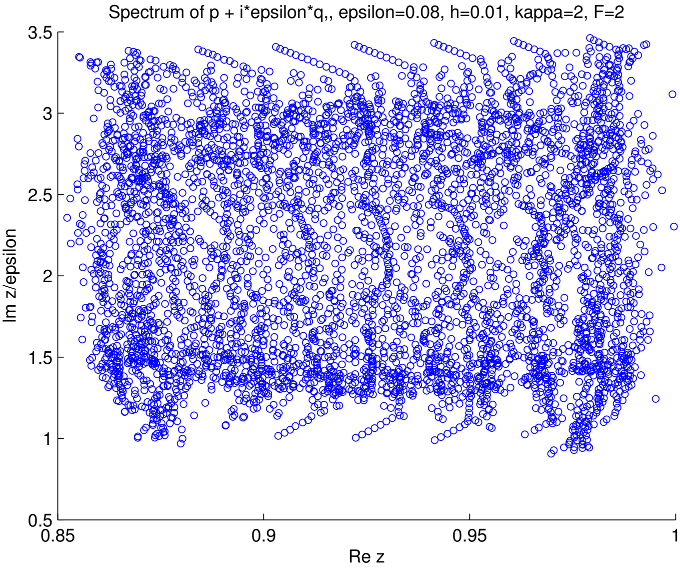


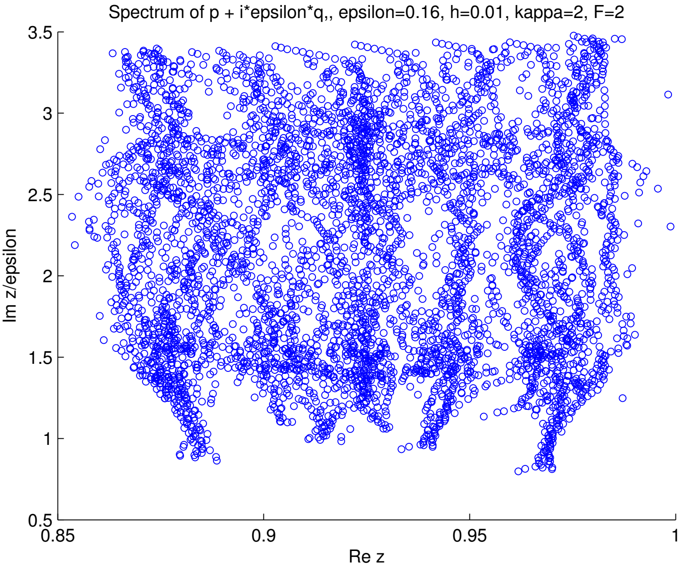
Spectrum of p + i*epsilon*q,, epsilon=0.005, h=0.01, kappa=2, F=2
3.5
3
2.5
2
1.5
1
0.5
Im z/epsilon
0.85
0.9
0.95
1
Re z


Spectrum of p + i*epsilon*q,, epsilon=0.01, h=0.01, kappa=2, F=2
3.5
3
2.5
2
1.5
1
0.5
Im z/epsilon
0.85
0.9
0.95
1
Re z


Spectrum of p + i*epsilon*q,, epsilon=0.02, h=0.01, kappa=2, F=2
Im z/epsilon
Re z
3.5
3
2.5
2
1.5
1
0.5
0.85
0.9
0.95
1


Spectrum of p + i*epsilon*q,, epsilon=0.04, h=0.01, kappa=2, F=2
3.5
3
2.5
2
1.5
1
0.5
Im z/epsilon
0.85
0.9
0.95
1
Re z


Spectrum of p + i*epsilon*q,, epsilon=0.08, h=0.01, kappa=2, F=2
3.5
3
2.5
2
1.5
1
0.5
Im z/epsilon
0.85
0.9
0.95
1
Re z


Spectrum of p + i*epsilon*q,, epsilon=0.16, h=0.01, kappa=2, F=2
3.5
3
2.5
2
1.5
1
0.5
Im z/epsilon
0.85
0.9
0.95
1
Re z

2.6. The centipede (le millepatte)

Let P_ϵ satisfy the general conditions above. We consider the decomposition $p^{-1}(0) = \bigsqcup_{\Lambda \in J} \Lambda$ in (8).
Recall that $Q_\infty(\Lambda)$, $\Lambda \in J \setminus S$ is reduced to the point $\langle\langle q \rangle\rangle_\Lambda$ when $\Lambda \in J \setminus S$ is irrational and is an interval containing $\langle\langle q \rangle\rangle_\Lambda$ when Λ is rational.

Let $\Lambda_0 \in J \setminus S$ be a rational torus and assume that $(d_\Lambda \omega)(\Lambda_0) \neq 0,$

$$\inf Q_\infty(\Lambda_0) < \inf_{\Lambda \in J \setminus \{\Lambda_0\}} \inf Q_\infty(\Lambda)$$

or the reversed inequality with all "inf" replaced by "sup".
Choose action-angle coordinates (x, ξ) near Λ_0 so that Λ_0 is given by $\xi = 0$ in $T^*\mathbf{T}^2$, $p = p(\xi)$, and

$$\partial_{\xi_2} p(0) > 0,\ \partial_{\xi_1} p(0) = 0,\ \partial_{\xi_1}^2 p(0) \neq 0,$$

where we keep the assumption from (13), that the derivative of the rotation number is $\neq 0$. Then

$$\partial_{\xi_1} p(\xi) = 0 \Leftrightarrow \xi_1 = f(\xi_2),$$

where f is analytic and the tori $\Lambda_E \subset p^{-1}(E)$, given by $p(f(\xi_2), \xi_2) = E$ are rational with the same rotation number as Λ_0.

Define

$$\langle q\rangle_2(x_1,\xi)=\frac{1}{2\pi}\int_0^{2\pi} q(x,\xi)dx_2,$$

and assume that for $\xi\in\mathrm{neigh}\,(0,\mathbf{R}^2)$, $\mathbf{T}\ni x_1\mapsto\langle q\rangle_2(x_1,\xi)$ has a unique minimum $x_1(\xi)$ which is nondegenerate. Observe that $\langle q\rangle_2(x_1(0),0)=\inf Q_\infty(\Lambda_0)$. We finally assume (for simplicity) that the subprincipal symbol of P vanishes. Let $x_1(\xi_2)=x_1(f(\xi_2),\xi_2)$. Let $\delta\in]1/18,1/9[$ be fixed,

$$h^{1/(1-\delta)}\ll\epsilon\ll h^{6/(5+12\delta)}. \tag{14}$$

Put

$$\widetilde{h}=\frac{h}{\sqrt{\epsilon}}\ll 1.$$

Theorem ([HiSj18])

$\exists C_1 > 0$ *such that* $\forall C_0 > 0$, *we have the following description of the eigenvalues of* P_ϵ *in the region*

$$\{z \in \mathbf{C};\ |\Re z| < \frac{1}{C_1},\ \Im z \leq \epsilon(\inf Q_\infty(\Re z) + C_0\widetilde{h})\}, :$$

For $h > 0$ *small enough, the eigenvalues are simple and given by*

$$\begin{aligned} \lambda_{j,k} =& p(f(\xi_2(j)), \xi_2(j)) + i\epsilon\langle q\rangle_2(x_1(\xi_2(j)), f(\xi_2(j)), \xi_2(j)) \\ &+ \epsilon\widetilde{h}(\lambda^0_{j,k} + \lambda^1_{j,k}\widetilde{h} + \lambda^2_{j,k}\widetilde{h}^2 + \dots), \end{aligned} \tag{15}$$

with $j \in \mathbf{Z}$, $\xi_2(j) = h(j - \theta_2) \in \mathrm{neigh}\,(0, \mathbf{R})$, $\mathbf{N} \ni k \leq \mathcal{O}(1)$, *where* $\lambda^\nu_{j,k} = \lambda^\nu_k(\xi_2(j), \sqrt{\epsilon})$ *is a smooth function of* $\xi_2(j) \in \mathrm{neigh}\,(0, \mathbf{R})$ *and* $\sqrt{\epsilon} \in \mathrm{neigh}\,(0, \overline{\mathbf{R}_+})$. *Here,* $\theta_2 = k_0(\alpha_2)/4 + S_2/2\pi h$, *where* $k_0(\alpha_2) =$ *Maslov index,* $S_2 =$ *classical action, of the natural cycle in* Λ_0.

Here $\lambda_k^0(\xi_2, 0)$ are eigenvalues of a complex harmonic oscillator,

$$\lambda_k^0(\xi_2, 0) =$$
$$e^{i\pi/4}(\partial_{\xi_1}^2 p(f(\xi_2), \xi_2))^{\frac{1}{2}}(\partial_{x_1}^2 \langle q \rangle_2(x_1(\xi_2), f(\xi_2), \xi_2))^{\frac{1}{2}} \left(k + \frac{1}{2}\right). \tag{16}$$

2.7. About the proof

We can make ϵ-deformations of T^*X as already explained.
The deformed spaces are closely related to normal forms that are also obtained by the method of averaging. In the case of a Diophantine torus Λ_0, the normal form is simply:

$$\Lambda_0 = \{\xi = 0\} \text{ in } T^*\mathbf{R}^2, \quad P_\epsilon = \underbrace{P_\epsilon(\xi; h)}_{\text{independent of } x} + \mathcal{O}((\epsilon, h, \xi)^\infty).$$

The reason for that is that we can solve

$$H_p G = q - \underbrace{\langle\langle q \rangle\rangle_\Lambda}_{\text{torus average}}, \ \Lambda \in J$$

to infinte order at $\Lambda = \Lambda_0$. (Small divisors.)

When Λ_0 is rational, there are zero divisors and in the action angle coordinates, we can only solve

$$H_p G = q - \langle q \rangle_2(x_1, \xi), \text{ for } \xi_1 = f(\xi_2),$$

i.e. we can only eliminate the x_2 variable. This amounts to the so called secular method [LiLi92].

Normal form for P_ϵ: After conjugation with an elliptic Fourier integral operator with complex phase we get microlocally near $\Lambda_0 = \{\xi = 0\}$ the operator $\widehat{P}_\epsilon$ such that

- The symbol is independent of x_2 up to $\mathcal{O}(\epsilon^{N+1} + (\xi_1 - f(\xi_2))^N + h^\infty)$,
- The subprincipal symbol is $\mathcal{O}(\epsilon)$,
- Up to $\mathcal{O}(\epsilon^2)$ the leading symbol is

$$p(f(\xi_2), \xi_2) + g(\xi)(\xi_1 - f(\xi_2))^2 + i\epsilon\langle q \rangle_2(x_1, \xi).$$

In order to treat this operator, use Fourier series in x_2 and get the family of operators.

$$p(f(jh), jh) + g(hD_{x_1}, \xi_2)(hD_{x_1} - f(jh))^2 + i\epsilon\langle q\rangle_2(x_1, \xi_1, jh). \tag{17}$$

To be able to absorb the errors we need a good control over the resolvent of these operators when the spectral parameter is not too high in the upper half-plane. To understand (17), we can consider the model case

$$(hD_{x_1})^2 + i\epsilon x_1^2 = \epsilon((\widetilde{h}D_{x_1})^2 + ix_1^2) \text{ on } \mathbf{R},$$

whose spectrum is given by the simple eigenvalues $\epsilon e^{i\pi/4}(2k+1)\widetilde{h}$, $k \in \mathbf{N}$.

3. Resonances

Resonances, or scattering poles is a vast subject. Here I will concentrate on the semi-classical Schrödinger operator and apply the FBI approach as it was developed in [HeSj86]. See [DyZw19], [Sj02] for other monographs. Let

$$P = -h^2\Delta + V(x),\ x \in \mathbf{R}^n, \tag{18}$$

where V is smooth, real and has a holomorphic extension (also denoted by V) to a truncated sector

$$\Gamma = \{x \in \mathbf{C}^n;\ |\Re x| > C,\ |\Im x| < |Rex|/C\},\ \text{for some } C > 0, \tag{19}$$

and

$$V(x) \to 0,\ x \to \infty \text{ in } \Gamma. \tag{20}$$

Using exterior complex distorsions (B. Simon [Si78], W. Hunziker [Hu86] and [SjZw91]) one can show that $(P - z)^{-1} : L^2 \to H^2$ extends meromorphically as a map $L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$ from the open upper half-plane to a sector

$$\{z \in \mathbf{C};\ -1/\widetilde{C} < \arg z \leq 0\} \tag{21}$$

The poles are called scattering poles or resonances. No smallness for h is required so far; for the study near ∞ there is an effective Planck's constant $\widetilde{h} = h/\langle x\rangle$ which tends to 0 when $x \to \infty$.

3.1 Global weighted spaces

Let $R(x) = \langle x \rangle$, $r(x) = 1$, $\widetilde{r}(x, \xi) = (r(x)^2 + \xi^2)^{1/2}$. R indicates the natural scale in x-space and $\widetilde{r}$ that in the ξ-directions.
If $a \in C^\infty(\mathbf{R}^{2n})$ and $m > 0$ is smooth, we write $a \in S(m)$, if for all $\alpha, \beta \in \mathbf{N}^n$,

$$\partial_x^\alpha \partial_\xi^\beta a(x, \xi) = \mathcal{O}(1) m(x, \xi) R(x)^{-|\alpha|} \widetilde{r}(x, \xi)^{-|\beta|}. \tag{22}$$

We require m to be an order function in the sense that $m \in S(m)$.
r, R, $\widetilde{r}$ are order functions.
Let $G \in S(\widetilde{r}R)$ be real-valued. Consider the manifold

$$\Lambda_G = \{(x, \xi) \in \mathbf{C}^{2n}; \Im(x, \xi) = H_G(\Re(x, \xi))\}. \tag{23}$$

We have a corresponding "exponent"

$$H = -\Re\xi \cdot \Im x + G(\Re(x, \xi)) = G(\Re(x, \xi)) - \Re\xi \cdot G'_\xi(\Re(x, \xi)). \tag{24}$$

(23) gives a parametrization $\mathbf{R}^{2n} \ni \rho \mapsto \rho + iH_G(\rho)$ of Λ_G allowing to define symbol spaces $S(m) = S(m, \Lambda_G)$ of functions on Λ_G.

Let $\lambda = \lambda(\alpha) \in S(\tilde{r}R^{-1}, \Lambda_G)$ be positive, elliptic (in the sense that λ is non-vanishing and $1/\lambda \in S((\tilde{r}R^{-1})^{-1}, \Lambda_G))$ and put

$$\phi(\alpha, y) = (\alpha_x - y)\alpha_\xi + i\lambda(\alpha)(\alpha_x - y)^2/2, \ \alpha = (\alpha_x, \alpha_\xi) \in \Lambda_G, \ y \in \mathbf{C}^n. \tag{25}$$

The amplitude will be a $\mathbf{C}^{n+1}$-valued smooth function $\mathbf{t}(\alpha, y; h)$ on $\Lambda_G \times \mathbf{C}^n_y$ which is affine linear in y. Restrict the attention to a region

$$|y - \alpha_x| < \mathcal{O}(1)R(\alpha_x), \tag{26}$$

and assume that $\mathbf{t} \in h^{-3n/4}S(\tilde{r}^{n/4}R^{-n/4})$ and that $\mathbf{t}, \partial_{y_1}\mathbf{t}, ..., \partial_{y_n}\mathbf{t}$ are maximally linearly independent in the natural sense.
Let $\chi \in C_0^\infty(B(0, 1/C))$ be equal to one in $B(0, 1/(2C))$, where $C > 0$ is large enough. We define the FBI-transform $T : \mathcal{D}'(\mathbf{R}^n) \to C^\infty(\Lambda_G; \mathbf{C}^{n+1})$ by

$$Tu(\alpha; h) = \int_{\mathbf{R}^n} e^{\frac{i}{h}\phi(\alpha,y)}\mathbf{t}(\alpha, y; h)\chi_\alpha(y)u(y)dy, \tag{27}$$

where $\chi_\alpha(y) = \chi((y - \Re\alpha_x)/R(\Re\alpha_x))$.

We also assume:

$$\begin{aligned}&\exists\, g_0 = g_0(x) \in S(rR), \text{ such that } G(x,\xi) - g_0(x)\\ &\text{has its support in a region where } |\xi| \leq \mathcal{O}(r(x))\\ &\text{and } G(x,\xi) - g_0(x) \text{ is sufficiently small in } S(rR).\end{aligned} \tag{28}$$

Definition

$H(\Lambda_G, m)$ is the completion of $C_0^\infty(\mathbf{R}^n)$ for the norm

$$\|u\|_{H(\Lambda_G,m)} = \|Tu\|_{L^2(\Lambda_G, m^2 e^{-2H/h} d\alpha)}. \tag{29}$$

Cf. the recent works [GaZw19a], [GaZw19b], [BoJe20]!

Let $p = \xi^2 + V(x)$ be the symbol of P. We can view $P : H(\Lambda_{tG}, \widetilde{r}^2) \to H(\Lambda_{tG})$ for $0 < t \ll 1$ as a pseudodifferential operator with leading symbol

$$p_{|_{\Lambda_{tG}}} \simeq p(\rho + itH_G(\rho)) = p(\rho) - itH_p G + \mathcal{O}(t^2).$$

Let $G \in S(\widetilde{r}R)$ be an escape function in the sense that for a given energy level $E_0 > 0$, we have

$$H_p G \asymp 1 \text{ on } p^{-1}(E_0) \cap \{(x, \xi) \in \mathbf{R}^{2n}; |x| \gg 1\}.$$

(The standard choice is $G(x, \xi) = x \cdot \xi$, truncated in the region, $|\xi| \gg 1$. The term escape function was used by Morawetz, Ralston and Strauss [MoRaSt77].) This implies that if we fix $t > 0$ small enough, then $p_{|_{\Lambda_{tG}}}(\rho) \notin \mathrm{neigh}\,(E_0, \mathbf{C})$ away from a bounded set in phase space. By Fredholm theory, it follows that P has purely discrete spectrum in $\mathrm{neigh}\,(E_0, \mathbf{C})$. The eigenvalues are precisely the resonances.

3.2 The role of trapped trajectories

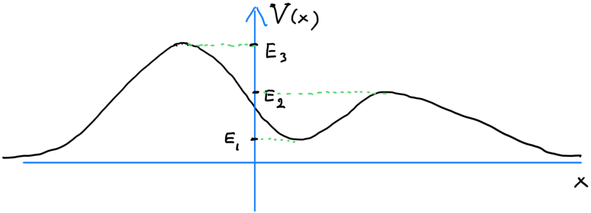
For $E > 0$, let

$$\Gamma_{\pm}(E) = \{\rho \in p^{-1}(E);\, \exp(tH_p)(\rho) \not\to \infty,\ t \to \mp\infty\}.$$

One can show that

$$\rho \in \Gamma_{\pm}(E) \Leftrightarrow |\exp(tH_p)(\rho)| \leq C(\rho),\ \mp t \geq 0.$$

The set $K(E) = \Gamma_+(E) \cap \Gamma_-(E)$ is compact; the set of trapped trajectories.



$$K(E) = \begin{cases} \emptyset , & E \in]0,E_1[\cup]E_2,E_3[\cup]E_3,+\infty[, \\ \text{a point} , & E \in \{E_1 , E_3\}, \\ \text{a closed trajectory}, & E \in]E_1,E_2]. \end{cases}$$

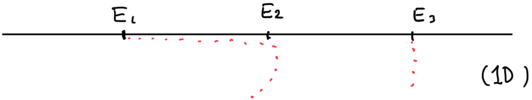


Figure: K(E)

- If V is analytic and $K(E) = \emptyset$ for some $E > 0$, then $\exists$ neighborhood $W \subset \mathbf{C}$ of E, independent of h such that P has no resonances in W when h is small enough (implicit in [HeSj86]: $\exists$ an escape function such that $H_pG > 0$ on **all** of $p^{-1}(E)$).
- Without the analyticity assumption we still have a W as above such that for every fixed $C > 0$ there are no resonances in $\{z \in W; -Ch\ln(1/h)\}$ for h small enough (A. Martinez [Ma02b]).
- For every $E > 0$ there exists W as above such that the number of resonances in W is $\leq \mathcal{O}(h^{-n})$. Such results in the context of obstacle scattering go back to R. Melrose [Me84]
- We have dynamical upper bounds: When the classical dynamical system is hyperbolic, there are upper bounds on the $\#$ of resonances in rectangles $]-a,a[+i]-\delta,0]$ that depend on the Minkowski dimension of the trapped set. See [Sj86, Sj90] as well as later results for hyperbolic surfaces by Zworski [Zw99] and others.
- for certain levels (that are analytic singularites of an E-dependent phase space volume), the number of resonances in any neighborhood W of E is $\geq h^{-n}/C$. Follows from trace formulae. See [Sj01].

3.3 Potential well in an island, shape resonances and higher levels

We are mainly interested in the resonances near a limiting level $E_0 > 0$ that we reduce to 0 by substracting E_0 from the potential in our Schrödinger operator. Let $n \geq 2$ and let $V \in C^\infty(\mathbf{R}^n; \mathbf{R})$ denote the modified potential so that

$$\begin{aligned}&V \text{ has a holomorphic extension to a truncated sector}\\ &\{x \in \mathbf{C}^n;\ |\Re x| > C,\ |\Im x| < |\Re x|/C\},\end{aligned} \tag{30}$$

$$V(x) \to -E_0,\ |x| \to \infty, \quad E_0 > 0. \tag{31}$$

Assume that for some $E > -E_0$,

$$V^{-1}(]-\infty, E[) = U_E \cup S_E, \tag{32}$$

with U_E, S_E open connected, U_E bounded, $\overline{U}_E \cap \overline{S}_E = \emptyset$. Assume also that there are no trapped trajectories in $p^{-1}(E) \cap \pi_x^{-1}(\overline{S}_E)$.

Let $P_{\rm int} = -h^2\Delta + V_{\rm int}$ be a reference operator obtained by increasing the potential ("filling the sea") near $\overline{S}_E$ so that the new potential $V_{\rm int}$ is equal to V in a neighborhood of $\overline{U}_E$ and $\geq E + 1/\mathcal{O}(1)$ outside. Then $P_{\rm int}$ has discrete real spectrum near E and in a neighborhood of E we can find a bijection b from the set of discrete eigenvalues onto the set of resonances of $P = -h^2\Delta + V$ in that neighborhood such that $b(\lambda) - \lambda = \mathcal{O}(e^{-1/(Ch)})$. See [HeSj86, CoDuKlSe87, FuLaMa11].

If we increase E, a "best case scenario" is that the assumptions above are fulfilled until we reach a critical level, say 0, and that,

$$\overline{U}_0 \cap \overline{S}_0 = \{x_0\}, \tag{33}$$

for some point $x_0 \in \mathbf{R}^n$, say $x_0 = 0$. We have $V(0) = 0$.

Assume that

$$0 \text{ is a nondegenerate critical point for } V \text{ of signature } (n-1,1). \tag{34}$$

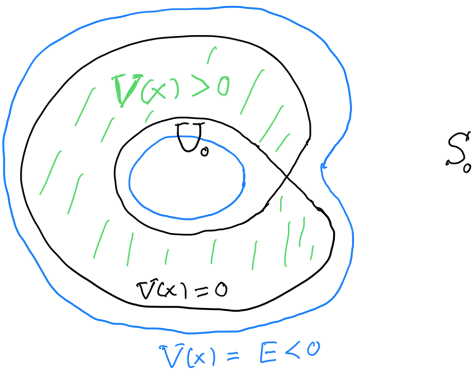


Figure: View from above

The point $(0,0) \in \mathbf{R}^{2n}$ is a stationary point and hence a trapped trajectory for the H_p flow, where $H_p = p'_\xi \cdot \partial_x - p'_\xi \cdot \partial_x$, $p(x,\xi) = \xi^2 + V(x)$.

Assume that

$$dV(x) \neq 0, \text{ when } x \in \partial U_0 \setminus \{0\}. \tag{35}$$

$$V \text{ is analytic in a neighborhood of } \overline{S}_0, \tag{36}$$

$$(0,0) \text{ is the only trapped trajectory in } p^{-1}(0)|_{\overline{S}_0}. \tag{37}$$

For $E \leq 0$, put

$$\omega(E) = \mathrm{vol}\left(p^{-1}(]-\infty, E])_{|U_0}\right). \tag{38}$$

Since $n \geq 2$, we check that $\omega \in C^1([-1/C, 0])$. Let ω also denote a C^1 extension to the interval $[-1/C, 1/C]$ so that $\omega(E)$ is well-defined up to a term $o(E)$ for $0 \leq E \leq 1/C$.

Theorem (Zerzeri–Sj 2020)

There exists a constant $t_0 > 0$ *and a constant* $0 < \delta_0 \ll 1$ *such that the following holds for every fixed* $C_0 > 0$*:*
For every $0 < \delta \leq \delta_0$*, there exists* $0 < \varepsilon(\delta) \ll 1$ *such that for every* $0 < \varepsilon \leq \varepsilon(\delta)$ *and* $0 < h \leq h(\varepsilon, \delta)$ *small enough:*

(A) *The number of resonances (of* P*) in* $]-C_0\varepsilon, \varepsilon[+i]-t_0\varepsilon, -\delta\varepsilon[$ *is* $\mathcal{O}_\delta(\varepsilon^n h^{-n})$*,*

(B) *For all* $a, b \in]-C_0\varepsilon, \varepsilon[$ *with* $a < b$*, the number of resonances in* $]a, b[+i]-\delta\varepsilon, 0]$ *is equal to* $(2\pi h)^{-n}(\omega(b) - \omega(a) + \mathcal{O}(\delta|\ln\delta|\varepsilon))$*, uniformly with respect to* a*,* b*,* h*.*

More precise results are known when $n = 1$. In this case, the function ω has a logarithmic singularity at 0. See [FuRa98], [BoFuRaZe14]

Proof, some ingrediants. The first thing is to find a nice escape function G, which vanishes over a neighborhood of $\overline{U_0}$ and in a $\sqrt{\epsilon}$-neighborhood of $(0,0)$

$$\partial_\rho^\alpha G = \mathcal{O}(1)(\varepsilon + \rho^2)^{1-|\alpha|/2}, \ \alpha \in \mathbf{N}^{2n}. \tag{39}$$

Define several reference operators P_ϵ, $P_\epsilon^{\mathrm{ext}}$, $P_\epsilon^{\mathrm{int}}$ by modifying the potential terms:

$$V_\epsilon = V + \epsilon\chi(\epsilon^{-1/2}(x, hD_x),$$

$$V_\epsilon^{\mathrm{ext}} = V_\epsilon + \text{"filling of the well"}$$

$$V_\epsilon^{\mathrm{int}} = V_\epsilon + \text{"filling of the sea"}$$

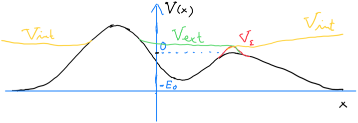


Figure: Reference potentials

Let

$$R =]-\mathcal{O}(\epsilon), \epsilon/\mathcal{O}(1)[+i]-\epsilon/\mathcal{O}(1), \mathcal{O}(\epsilon)[,\ R_\delta = \{z \in R;\ |\Im z| > \delta\epsilon\},$$

$$\begin{aligned} R_{\delta,\epsilon} = R_{\delta,A,B,\epsilon} = R_\delta \cup (A\epsilon +]-\delta\epsilon/4, \delta\epsilon/4[+ i[-\delta\epsilon, \delta\epsilon]) \\ \cup (B\epsilon +]-\delta\epsilon/4, \delta\epsilon/4[+ i[-\delta\epsilon, \delta\epsilon]). \end{aligned}$$

Then:
P_ϵ has no eigenvalues in R_δ,
$P_{\epsilon,A,B,\delta}$ (obtained from P_ϵ by creating two gaps of size $\epsilon\delta$ in the spectrum) has no eigenvalues in $R_{\epsilon,A,B,\delta}$.

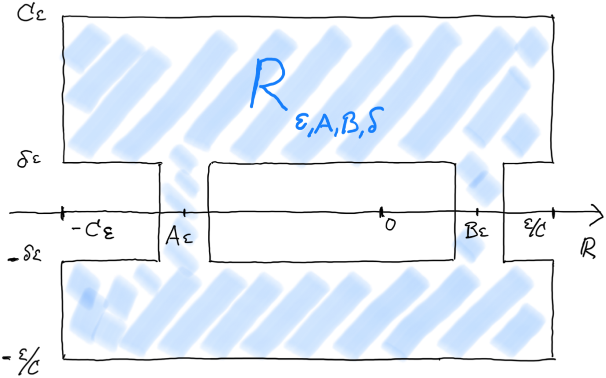


Figure: $R_{\epsilon,A,B,\delta}$

Relative determinants.

Recall (see e.g. [GoKr69]) that under suitable but very general assumptions,

$$\begin{aligned} |\det \mathcal{A}\mathcal{B}^{-1}| &= |\det\left(1+(\mathcal{A}-\mathcal{B})\mathcal{B}^{-1}\right)| \\ &\le \exp \|(\mathcal{A}-\mathcal{B})\mathcal{B}^{-1}\|_{\mathrm{tr}} \le \exp\left(\|\mathcal{A}-\mathcal{B}\|_{\mathrm{tr}}\|\mathcal{B}^{-1}\|\right). \end{aligned}$$

We have

$$\begin{aligned} \|P_\epsilon - P_\epsilon^{\mathrm{ext}}\|_{\mathrm{tr}} &= \mathcal{O}(h^{-n}) \\ \|P - P_\epsilon\|_{\mathrm{tr}} &= \mathcal{O}(\epsilon^{n+1}h^{-n}) \\ \|P_\epsilon - P_{\epsilon,\delta}\|_{\mathrm{tr}} &= \mathcal{O}(\epsilon\delta h^{-n}),\ \ P_{\epsilon,\delta} = P_{\epsilon,A,B,\delta}. \end{aligned}$$

Define

$$\begin{aligned} \mathcal{D}_P(z) &= \ln\left|\det(P-z)(P_\epsilon^{\mathrm{ext}}-z)^{-1}\right| \\ \mathcal{D}_{P_\epsilon}(z) &= \ln\left|\det(P_\epsilon-z)(P_\epsilon^{\mathrm{ext}}-z)^{-1}\right| \\ \mathcal{D}_{P_{\epsilon,\delta}}(z) &= \ln\left|\det(P_{\epsilon,\delta}-z)(P_\epsilon^{\mathrm{ext}}-z)^{-1}\right|. \end{aligned}$$

We have

$$\mathcal{D}_P - \mathcal{D}_{P_\epsilon} \begin{cases} \leq \mathcal{O}_\delta(1)\epsilon^n/h^n \text{ in } R_\delta, \\ \geq -\mathcal{O}_\delta(1)\epsilon^n/h^n \text{ in } R_\delta \cap \{\Re z \leq -\epsilon/\mathcal{O}(1)\}. \end{cases}$$

Similar estimates hold for $\mathcal{D}_P - \mathcal{D}_{P_{\epsilon,\delta}}$ in R_δ by $R_{\delta,\epsilon}$.
Standard arguments, including Jensen's formula, lead to:
Consider the holomorphic function $f(z) = \det((P-z)(P_\epsilon^{\mathrm{ext}} - z)^{-1})$ on R, whose zeros are the resonances of P. Then

$$|f(z)| \leq \exp\left(h^{-n}(\phi(z) + \mathcal{O}(\epsilon\delta))\right) \text{ in } R_{\delta,\epsilon},$$

where $\phi(z) = h^n \mathcal{D}_{P_{\epsilon,\delta}}$. We have

$$|f(z)| \geq \exp\left(h^{-n}(\phi(z) - \mathcal{O}(\epsilon\delta))\right) \text{ at plenty of points in } R_{\delta,\epsilon}.$$

We can then apply a result on counting of zeros of holomorphic functions with exponential growth: Theorem 1.1 in [Sj10] (or [Sj19, Theorem 12.1.1]).

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