LOCAL AND GLOBAL WELL-POSEDNESS OF WAVE MAPS ON $\mathbf{R}^{1+1}$ FOR ROUGH DATA

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ABSTRACT. We consider wave maps between Minkowski space $\mathbf{R}^{1+1}$ and an analytic manifold. Results include global existence for large data in Sobolev spaces H^s for $s > 3/4$, and in the scale-invariant norm $L^{1,1}$. We prove local well-posedness in H^s for $s > 1/2$, and a negative well-posedness result for wave maps on $\mathbf{R}^{n+1}$ with data in $H^{n/2}(\mathbf{R}^n)$, $n \geq 1$. Also included are positive and negative results for scattering.

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1. Introduction

Write $(\mathbf{R}^{n+1}, g)$ for $n+1$ dimensional Minkowski space with flat metric $g = \mathrm{diag}(1, 1, \ldots -1)$. In what follows $(\mathcal{M}, h)$ will denote a Riemannian manifold with metric h; for simplicity we will restrict our attention to those manifolds $(\mathcal{M}, h)$ which are uniformly analytic; that is, the manifold can be covered by a family of charts such that the metric and Christoffel symbol components are analytic in each chart, with uniform exponential bounds on the Taylor series coefficients. Examples include S^m, $\mathbf{R}^m$, the hyperbolic plane, or any compact analytic manifold.

We are interested in maps

$$\phi(x,t): \ (\mathbf{R}^{n+1}, g) \longrightarrow (\mathcal{M}, h). \tag{1}$$

which are stationary with respect to compact variations of the Lagrangian

$$L = \int_{\mathbf{R}^{n+1}} -\frac{1}{2} \mathrm{Tr}_g \phi^*(h) dv_g \tag{2}$$

$$= \int_{\mathbf{R}^{n+1}} -\frac{1}{2} g^{\mu\nu} h_{\alpha\beta}(\phi) \frac{\partial \phi^\alpha}{\partial x^\mu} \frac{\partial \phi^\beta}{\partial x^\nu} dv_g \tag{3}$$

In (3) we have written (2) with respect to the coordinates $x^1, x^2, \ldots x^n, x^{n+1} = t$ on $\mathbf{R}^{n+1}$ and local coordinates on $\mathcal{M}$. Stationary points of this Lagrangian are called wave maps, and can be parametrized by the Cauchy problem for the wave map equation - which is the Euler-Lagrange equation of (2),

$$\begin{aligned} \Box \phi^k + \Gamma^k_{\alpha\beta}(\phi) \partial_\mu \phi^\alpha \partial^\mu \phi^\beta &= 0. \\ \phi[0] &= (f, g) \end{aligned} \tag{4}$$

where $\Gamma^k_{\alpha\beta}$ are the Christoffel symbols corresponding to the Riemannian metric h, and $\phi[T] = (\phi(T), \phi_t(T))$ denotes the Cauchy data of ϕ at time T.

A model to keep in mind is the case with image $\mathcal{M} = S^{m-1} \subset \mathbf{R}^m$, where the equations (4) take the form (see e.g. [32])

$$\begin{aligned} \Box \phi + \phi \partial_\mu \phi^t \partial^\mu \phi &= 0, \\ \phi[0] &= (f, g) \\ f(x) &\in S^{m-1} \\ f(x)^t g(x) &= 0 \end{aligned} \tag{5}$$

where we think of $\mathbf{R}^m$ as an $m \times 1$ column vector, and write ϕ^t for its transpose. It is well known that smooth solutions to (4) will stay on the sphere; the same result will hold for rough solutions by a limiting argument assuming that the problem is well-posed in the rough space.

The equation (4) is invariant under the scaling

$$\begin{aligned} \phi^\lambda(x,t) &= \phi(\frac{x}{\lambda},\frac{t}{\lambda}) \\ f^\lambda(x) &= f(\frac{x}{\lambda}) \\ g^\lambda(x) &= \lambda^{-1} g(\frac{x}{\lambda}). \end{aligned} \tag{6}$$

for any $\lambda > 0$. For data in $H^s \times H^{s-1}$, $s > n/2$, the initial value problem (4) is called subcritical since $\|(f,g)\|_{\dot{H}^s \times \dot{H}^{s-1}}$ can be made small by choosing λ large in (6). A norm left invariant by the scaling (6) is called critical.

In this paper we will usually restrict ourselves to the one-dimensional case $n = 1$, where the analysis of the Cauchy problem (4) is simplified by introducing the null coordinates

$$u = x + t \quad v = x - t. \tag{7}$$

wherein the wave map equation is

$$\begin{aligned} \partial_u \partial_v \phi(u,v) &= -\Gamma_{\alpha\beta}(\phi) \cdot (\partial_u \phi^\alpha \partial_v \phi^\beta) \\ \phi[0] &= (f,g) \end{aligned} \tag{8}$$

and in the case of the sphere we have

$$\begin{aligned} \phi_{uv} &= -\phi \phi_u^t \phi_v \\ \phi[0] &= (f,g) \\ f(x) &\in S^{m-1} \\ f(x)^t g(x) &= 0 \end{aligned} \tag{9}$$

We aim to show that for various sub-critical and critical initial data spaces $\mathcal{D} \subset C(\mathbf{R})$, the Cauchy problem (8) is locally and globally well posed in $\mathcal{D}$, in the sense that the solution operator exists and maps data from $\mathcal{D}$ continuously into a unique solution in $C([0,T],\mathcal{D}) \cap X$ for all $T > 0$, where X is some auxilliary space to be specified. We also wish to show persistence of regularity, so that a solution in a rough space $\tilde{\mathcal{D}}$ whose initial data is in a smooth space $\mathcal{D}$ will stay in the smooth space $\mathcal{D}$. All the positive results are for the one-dimensional equation (8). Finally, we complement these results with negative results.

For our local results in Theorem 1.2 below, we may assume that we are working in a single coordinate chart since the norms used embed in the space of Hölder continuous functions. (See section 3 below.)

Definition 1.1. [24] Define the spaces $H^{s,\delta}$ for $s, \delta \in \mathbf{R}$ by the norm

$$\|\phi\|_{H^{s,\delta}} = \|\langle|\tau| + |\xi|\rangle^s \langle|\tau| - |\xi|\rangle^\delta \hat{\phi}\|_{L^2_{\xi,\tau}}$$

where ξ, τ are the dual variables to x, t, and $\langle x \rangle = (1 + |x|^2)^{1/2}$.

The $H^{s,\delta}$ spaces first appeared in [24], where Rauch-Reed study the propagation of singularities of hyperbolic equations. Our local well-posedness results are based on bilinear estimates in the $H^{s,\delta}$ spaces, versions of which first appeared in the higher dimensional work of [1, 18].

Theorem 1.2. *(Local theory and persistence) If $s > 1/2$, then the Cauchy problem* (8) *is locally well-posed in $H^s \times H^{s-1}$ on some nontrivial time interval $[0,T]$. For any $\tilde{s} > 1/2$ one may choose T to depend only on the $H^{\tilde{s}} \times H^{\tilde{s}-1}$ norm of the data. Furthermore the solution to this problem is locally in $H^{s,s}$ with norm depending only on the $H^s \times H^{s-1}$ norm of the data.*

In particular, if the H^s solution cannot be continued past some maximal time T^*, then the solution must blow up in $H^{\tilde{s}} \times H^{\tilde{s}-1}$ as $t \longrightarrow T^*$ for all $\tilde{s} > 1/2$. The last statement in Theorem 1.2 will be needed for Theorem 1.3 below.

Theorem 1.3. *(Global theory for Sobolev spaces) The Cauchy problem* (8) *is globally well-posed for large data in H^s for $s \geq 1$. In the case of the sphere, the Cauchy problem* (9) *is globally well-posed for large data in H^s for $1 > s > 3/4$.*

In practice the distinction between functions which are globally in H^s and those which are locally H^s is unimportant, due to finite speed of propagation.

Theorem 1.4. *(Global theory for $L^{1,1}$) The Cauchy problem* (8) *is globally well-posed with scattering for large data in the critical space $L^{1,1}$ defined by*

$$||(f,g)||_{L^{1,1}} = ||f'||_{L^1} + ||f||_{L^\infty} + ||g||_{L^1}. \tag{10}$$

Our negative results are in Section 8. Although we do not have a specific example of blow-up for the one-dimensional wave-map problem, we collect some previous observations on the ill-posedness of superficially similar equations, and also show that the wave map problem (4) in certain coordinates is analytically ill-posed in the critical space $\dot{H}^{n/2}$ for $n \geq 1$. We also show there is no scattering when $n = 1$ without a suitable decay condition on the data.

We now briefly discuss each of the positive results and their relationship with previous literature. Careful surveys of regularity results and open questions in spatial dimensions $n > 1$ can be found in [32],[16],[9], and [28].

The local theory of Theorem 1.2 is the $n = 1$ version of higher-dimensional results initiated in [18] and further studied in [9, 36, 19, 20, 33, 34]: for $n \geq 2$, the wave map equation is locally well-posed in the subcritical spaces $H^s(\mathbf{R}^n)$, $s > \frac{n}{2}$. Our methods are the same, but we have some additional simplifications due to the null-coordinates u, v which are only available in one dimension. We prove the local results in Section 3, after some abstract considerations in Section 2.

From Theorem 1.2 and energy conservation, one immediately obtains Theorem 1.3 for $s \geq 1$. For $s < 1$ the energy conservation law is not directly applicable, and to obtain our low regularity global existence results we adapt some ideas of Bourgain [3] and a pointwise version of energy conservation observed by Pohlmeyer[23]. This is the most involved part of the paper, and occupies Sections 5, 6, and 7. Theorem 1.3 can almost certainly be extended to more general compact manifolds. When $s \geq 2$, Theorem 1.3 was previously shown in [11], also [7, 27]. We have recently learned that for $s \geq 1$, the result appears in [35].

We note that the main interest of these global results is that there is no smallness condition on the data. If the data is small in some H^s norm, $s > \frac{1}{2}$, and hence small in the Besov space $B^{\frac{1}{2},1}$, then global existence follows from the critical regularity results of Tataru [33, 34] after specializing the argument there to the one-dimensional setting.

Theorem 1.4 follows the usual pattern of global well-posedness results for large-data in a critical space: one first proves global existence for small data, then shows that the solution does not concentrate. Due to the simple structure and symmetries of our equation and the data space, both of these steps are extremely elementary, especially when compared with other large-data critical results e.g. [8, 2, 13, 29, 6, 10, 30]. Scattering is obtained by conformal compactification. We prove this theorem in Section 9.

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2. Abstract local existence and persistence theory

Questions of local existence and persistence of regularity for nonlinear wave equations are very often handled by the method of Picard iteration, using estimates to control the nonlinearity. For wave maps the algebraic (or analytic) nature of the nonlinearity allows one to formalize these estimates quite explicitly; this was done for instance in Tataru [33, 34]. In this section we describe the well-known abstract machinery which allows one to obtain existence and regularity from these estimates. The statements of this section will be valid in every dimension $n \geq 1$.

We begin with the standard reduction of local existence and persistence questions to estimates, which we set abstractly. We consider the Cauchy problem

$$\begin{aligned} L(\phi) &= N(\phi) \qquad t \in (-T_1, T_2) \\ \phi[0] &= f \end{aligned} \tag{11}$$

where L is a linear evolution operator of order d, N is a nonlinear operator such that $N(0) = 0$, $0 \leq T_1, T_2 \leq \infty$ are times, and the Cauchy data $f = (f_0, \dots, f_{d-1})$ lies in some Banach space $\mathcal{D}$. We assume that a suitable coordinate chart has been chosen so that ϕ takes values in Euclidean space $\mathbf{R}^m$; this can be done (locally at least) if $\mathcal{D}$ embeds in the space of continuous functions.

We may rewrite this problem in integral form as

$$\phi = \eta(S(f) + L^{-1}N(\phi)),$$

where $S(f)$ is the solution to the homogeneous linear problem $L(\phi) = 0$ with Cauchy data f, $L^{-1}F$ is the solution to the inhomogeneous problem $L\phi = F$ with Cauchy data 0, and η is any function which equals 1 on $[-T_1, T_2]$. For a rough initial problem it will be advantageous to choose a smooth cutoff η (see [1]).

We will always assume that the free problem is well-posed in $\mathcal{D}$. For higher-dimensional wave equations this effectively restricts $\mathcal{D}$ to the L^2-based family of spaces, but in one dimension many more spaces are available.

From the contraction mapping theorem we have the following local existence metatheorem. As this result is well-known, we omit some details and rigor.

Lemma 2.1 (Local existence for small data). *Let the notation be as above. Suppose that there exists a reasonable*[1] *Banach space X of functions in spacetime which obeys the estimates*

$$\|\eta S(f)\|_X \lesssim \|f\|_{\mathcal{D}} \tag{12}$$

$$\|\phi[T]\|_{\mathcal{D}} \lesssim \|\phi\|_X \tag{13}$$

$$\left\|\eta L^{-1}(N(\phi) - N(\psi))\right\|_X \lesssim \|\phi - \psi\|_X \left(\|\phi\|_X + \|\psi\|_X\right) \tag{14}$$

for all data f, $T \in (-T_1, T_2)$, and all spacetime functions ϕ, ψ with sufficiently small X norm.

Then for sufficiently small ϵ depending only on the constants in the above estimates, the Cauchy problem (11) is well posed in $(-T_1, T_2)$ for data $f \in \mathcal{D}$, with a unique solution in $X \cap C((-T_1, T_2), \mathcal{D})$, providing that $\|f\|_{\mathcal{D}} \leq \epsilon$.

Proof If $\|f\|_D$ is sufficiently small, then the assumptions imply the Picard iteration map

$$\phi \mapsto \eta(S(f) + L^{-1}N(\phi)) \tag{15}$$

[1] In fact, it's enough to assume that X can be densely approximated by test functions.

will be a contraction on a small neighbourhood of the origin in X. The contraction mapping theorem thus gives a unique solution on this ball which depends continouously in X on $S(f)$. By (12) and (13) we get well-posedness. ∎

One can relax the $(\|\phi\|_X + \|\psi\|_X)$ factor in the condition (14), but we shall not need to do so in this paper.

A small modification of this argument allows one to get persistence of regularity as long as the solution stays in a rougher space $\tilde{D}$ or $\tilde{X}$, providing of course that one has the appropriate estimates.

Lemma 2.2 (Persistence of regularity). *Assume $X \subset \tilde{X}, \mathcal{D} \subset \widetilde{\mathcal{D}}$ are spaces such that X, $\mathcal{D}$ satisfy (12) and (13), and $\tilde{X}$, $\widetilde{\mathcal{D}}$ satisfy (12), and (14). Assume also that we have the estimate*

$$\begin{aligned}\left\|\eta L^{-1}(N(\phi) - N(\psi))\right\|_X \lesssim \|\phi - \psi\|_{\tilde{X}} (\|\phi\|_X + \|\psi\|_X) \\ + \|\phi - \psi\|_X (\|\phi\|_{\tilde{X}} + \|\psi\|_{\tilde{X}})\end{aligned} \tag{16}$$

for all spacetime functions ϕ, ψ with sufficiently small $\tilde{X}$ norm.

Then there exists $\epsilon > 0$ such that the problem (11) is well-posed in $(-T_1, T_2)$ for data $f \in \mathcal{D}$, with unique solution in $X \cap C((-T_1, T_2), \mathcal{D})$, providing that $\|f\|_{\widetilde{\mathcal{D}}} \leq \epsilon$.

Proof Define,

$$\|\phi\|_Z = c(\varepsilon \|\phi\|_X + C \|\phi\|_{\tilde{X}})$$

where $\varepsilon = 1/\|f\|_{\mathcal{D}}$ and c, C are large constants. Then the assumptions imply that the map (15) is a contraction on the unit ball in Z, providing that c, C are sufficiently large and ϵ is sufficiently small, hence the result. ∎

One can show that the X-solution persists as long as the $\tilde{X}$ norm stays finite, but we shall not need that here.

We now specialize to the case of the wave map equation, in which $L = \Box$ and

$$N(\phi) = \Gamma^k_{\alpha\beta}(\phi)\partial_\mu \phi^\alpha \partial^\mu \phi^\beta.$$

We will assume that the data is small in $\mathcal{D}$, and that $\mathcal{D}$ is embedded in the space $C(\mathbf{R})$ of continuous data; this allows us to use a single chart of coordinates. In this case the required estimates can be simplified by the identity

$$\partial_\mu \phi \partial^\mu \psi = \frac{1}{2}[\Box(\phi\psi) - \phi \Box \psi - \psi \Box \phi]. \tag{17}$$

If Γ depends polynomially on ϕ, then $N(\phi) - N(\psi)$ can be decomposed by (17) into quantities of the form $F \Box G$, where F and G are polynomials in ϕ, ψ, and at least one of F, G contains a factor of $\phi - \psi$. Combining this with the previous lemmas, one obtains

Lemma 2.3 (Wave map local existence). [33, 34] *Suppose that the Christoffel symbols $\Gamma(\phi)$ depend polynomially on ϕ. If $\mathcal{D} \subset C(\boldsymbol{R})$ and X obey* (12) *and* (13)*, as well as the estimates*

$$\|\phi\psi\|_X \lesssim \|\phi\|_X \cdot \|\psi\|_X \tag{18}$$

$$\left\|\eta\,\Box^{-1}\left(\phi\,\Box\,\psi\right)\right\|_X \lesssim \|\phi\|_X\,\|\psi\|_X \tag{19}$$

then the Cauchy problem (4) *is well-posed in $[-T_1, T_2]$ for data in $\mathcal{D}$ provided that $\|f\|_{\mathcal{D}}$ is sufficiently small.*

Lemma 2.4 (Wave map persistence of regularity). [33, 34] *Suppose that the Christoffel symbols $\Gamma(\phi)$ depend polynomially on ϕ. Assume $X \subset \tilde{X}, \mathcal{D} \subset \widetilde{\mathcal{D}} \subset C(\boldsymbol{R})$ are spaces such that X, $\mathcal{D}$ satisfy* (12) *and* (13)*, and $\tilde{X}$, $\widetilde{\mathcal{D}}$ satisfy* (12)*,* (18)*, and* (19)*. Assume also that we have the estimates*

$$\|\phi\psi\|_X \lesssim \|\phi\|_X\|\psi\|_{\tilde{X}} + \|\psi\|_X\|\phi\|_{\tilde{X}} \tag{20}$$

$$\left\|\eta\,\Box^{-1}\left(\phi\,\Box\,\psi\right)\right\|_X \lesssim \|\phi\|_X\|\psi\|_{\tilde{X}} + \|\psi\|_X\|\phi\|_{\tilde{X}}. \tag{21}$$

Then the Cauchy problem (4) *is well-posed in $[-T_1, T_2]$ for data in $\mathcal{D}$ provided that $\|f\|_{\widetilde{\mathcal{D}}}$ is sufficiently small.*

The same results hold if Γ is uniformly analytic on the target manifold $\mathcal{M}$, since one can obtain the desired estimates by expanding Γ as a power series. Note that the geometry of $\mathcal{M}$ does not play any role in these results.

3. Proof of Theorem 1.2

By finite speed of propagation and the fact (from Sobolev embedding) that H^s functions have some degree of Hölder continuity for $s > 1/2$, we may assume that the data is compactly supported[2] and stays within a single coordinate chart.

Choose δ such that $1/2 < \delta < \tilde{s}, s, 1$. Since the $H^{\tilde{s}} \times H^{\tilde{s}-1}$ norm of the data is bounded, the $H^\delta \times H^{\delta-1}$ norm is also bounded; we now show that by rescaling the data and shifting coordinates we may make the $H^\delta \times H^{\delta-1}$ norm arbitrarily small. (Our Sobolev norms are inhomogeneous and do not obey an exact scaling identity, so one must take a little care with this argument).

The equation (8) is invariant under the scaling (6). Eventually we will choose λ depending only on δ, $\tilde{s}$, and the $H^{\tilde{s}} \times H^{\tilde{s}-1}$ norm of the data. Thus to obtain well-posedness for the original data up to time $1/\lambda$ it suffices to get well-posedness up to time 1 for the data f^λ, g^λ. By finite speed of propagation we may restrict

[2]For a proof that H^s functions can be localized, see Corollary 3.4.

f^λ, g^λ to an interval of length 4 centered at some x_0; by translation invariance we may make $x_0 = 0$. By shifting the origin of the coordinate system we may replace f^λ by $f^\lambda - c$ for some constant c. In particular, it suffices to get well-posedness up to time 1 for the data

$$\tilde f(x) = \chi(x)(f(\frac{x}{\lambda}) - \overline{f})$$
$$\tilde g(x) = \lambda^{-1}\chi(x) g(\frac{x}{\lambda})$$

where χ is a standard compactly supported cutoff function, and $\overline{f} = \int f(\frac{x}{\lambda})\psi(x)\,dx$, where ψ is a standard bump function with unit mass supported near χ.

We claim that we may make the $H^\delta \times H^{\delta-1}$ norm of $(\tilde f, \tilde g)$ arbitrarily small, by choosing λ sufficiently large (but depending only on the $H^{\tilde s} \times H^{\tilde s - 1}$ norm of the original data). More precisely,

Lemma 3.1. *If $\delta < \tilde s$ and $\delta \leq 1$, we have*

$$\|\tilde f\|_{H^\delta} + \|\tilde g\|_{H^{\delta-1}} \lesssim \lambda^{-\varepsilon}(\|f\|_{H^{\tilde s}} + \|g\|_{H^{\tilde s - 1}}) \tag{22}$$

for all $\lambda \gg 1$, where $\varepsilon > 0$ is a small number depending only on δ, $\tilde s$.

Proof The contribution of g is easily handled by the rescaling properties of $H^{\delta-1}$:

$$\|\tilde g\|_{H^{\delta-1}} \lesssim \lambda^{-1}\|g(\frac{x}{\lambda})\|_{H^{\delta-1}} \lesssim \lambda^{-1}\lambda^{\frac{1}{2}}\|g\|_{H^{\delta-1}} \lesssim \lambda^{-1/2}\|g\|_{H^{\tilde s-1}},$$

so we may assume that $g = 0$. It suffices to check the cases when the Fourier transform of f are supported on $|\xi| \lesssim \lambda$ and $|\xi| \gtrsim \lambda$.

We first consider the case when $|\xi| \lesssim \lambda$. Since $\delta \leq 1$ and $\tilde f$ is compactly supported, the H^δ norm is controlled by the C^1 norm, and so it suffices to control the quantity

$$\|\chi(f(\frac{x}{\lambda}) - \overline{f})\|_{C^1}.$$

But a computation shows that this is majorized by $\lambda^{-1}\|f\|_{C^1}$, which by Sobolev embedding is majorized by $\lambda^{-1}\|f\|_{H^{3/2+\varepsilon}}$, which by the frequency support assumption on f is controlled by $\lambda^{1/2-\tilde s+\varepsilon}\|f\|_{H^{\tilde s}}$, which gives the desired estimate if ε is chosen sufficiently small.

We now consider the case when $|\xi| \gtrsim \lambda$. In this case we use the triangle inequality to estimate

$$\|\tilde f\|_{H^\delta} \lesssim \|\chi f(\frac{x}{\lambda})\|_{H^\delta} + \|\chi \overline{f}\|_{H^\delta} \lesssim \|f(\frac{x}{\lambda})\|_{H^\delta} + |\overline{f}|.$$

By the frequency support assumption on f and Plancherel's theorem, we have

$$\|f(\frac{x}{\lambda})\|_{H^\delta} \sim \|f(\frac{x}{\lambda})\|_{\dot H^\delta} \sim \lambda^{1/2-\delta}\|f\|_{\dot H^\delta} \lesssim \lambda^{1/2-\delta}\|f\|_{H^{\tilde s}}$$

as desired. To control $\overline{f}$, we use Plancherel's theorem to write

$$\overline{f} = \int \lambda \hat{f}(\lambda\xi)\hat{\psi}(\xi)\ d\xi = \int \hat{f}(\xi)\hat{\psi}(\frac{\xi}{\lambda})\ d\xi.$$

From the support hypothesis on $\hat{f}$ and Cauchy-Schwarz, this is estimated by

$$(\int_{|\xi|\gtrsim\lambda} |\hat{f}(\xi)|^2\langle\xi\rangle^{2\tilde{s}}\ d\xi)^{1/2}(\int_{|\xi|\gtrsim\lambda} \hat{\psi}(\frac{\xi}{\lambda})\langle\xi\rangle^{-2\tilde{s}}\ d\xi)^{1/2};$$

since ψ is rapidly decreasing, this is majorized by $\lambda^{1/2-\tilde{s}}\|f\|_{H^{\tilde{s}}}$, which is acceptable. ■

It is likely that a version of the above lemma can also be proven by Rellich's lemma and a compactness argument using the nonconcentration of H^δ norm for smooth functions, but we shall not do so here.

To finish the proof of Theorem 1.2, we have to show that the equation (8) is locally well-posed in H^s up to time 1 with a solution in $H^{s,s}$, whenever the $H^\delta \times H^{\delta-1}$ norm of the data is sufficiently small.

We apply Lemma 2.4, with $\mathcal{D} = H^s \times H^{s-1}$, $\tilde{\mathcal{D}} = H^\delta \times H^{\delta-1}$, $X = H^{s,\delta}$, $\tilde{X} = H^{\delta,\delta}$, $T_1 = T_2 = 1$, and a smooth cutoff η. Assuming we can verify all the estimates in the lemma, this gives well-posedness in H^s up to time 1 with a solution in $H^{s,\delta}$. At the end of this section we shall improve this to $H^{s,s}$.

Of course, it still remains to verify the hypotheses in Lemma 2.4. More precisely, we need to show that X, D satisfy (12), (13), that $\tilde{X}$, $\tilde{\mathcal{D}}$ satisfy (12), (18), and (19), and that (20), (21) hold; the inclusion $\mathcal{D} \subset C(\mathbf{R})$ follows from Sobolev embedding.

We first take advantage of the null coordinates to rewrite the $H^{s,\delta}$ norms in terms of product Sobolev spaces $H^{s_1}_u H^{s_2}_v = H^{s_2}_v H^{s_1}_u$ defined by

$$\|\phi\|_{H^{s_1}_u H^{s_2}_v} = \|D^{s_1}_u D^{s_2}_v \phi\|_{L^2_{u,v}},$$

where D_u and D_v are the Fourier multipliers corresponding to $\langle\mu\rangle$, $\langle\nu\rangle$ respectively, and μ, ν are the frequency variables dual to u, v. We define the one-dimensional Sobolev spaces H^s_u, H^s_v in the usual manner.

By Plancherel's theorem one can easily verify that

$$H^{s,\delta} = H^s_u H^\delta_v \cap H^s_v H^\delta_u \tag{23}$$

when $\delta \leq s$. Thus to prove estimates concerning the $H^{s,\delta}$ spaces in $\mathbf{R}^{1+1}$, it suffices to prove estimates on product Sobolev spaces. We collect the estimates we will need below, and then use them to prove the requirements of Lemma 2.4.

We first begin with a standard result regarding multiplication of one-dimensional Sobolev spaces; we will use variants of this argument in other places in this paper.

Lemma 3.2. *If s, s' are real numbers such that $s > 1/2$ and $s \geq s' \geq -s$, then for all test functions ϕ, ψ*

$$\|\phi\psi\|_{H_u^{s'}} \lesssim \|\phi\|_{H_u^s}\|\psi\|_{H_u^{s'}}.$$

Proof We may assume that the norms on the right-hand side are equal to one. By Plancherel's theorem it suffices to show that

$$\langle\mu\rangle^{s'}(\hat{\phi} * \hat{\psi})(\mu) = \int_{\mu_1+\mu_2=\mu} \langle\mu_1+\mu_2\rangle^{s'}\hat{\phi}(\mu_1)\hat{\psi}(\mu_2)\, d\mu_1 \tag{24}$$

is in L^2_μ. Since the right-hand side norms depend only on the size of $\hat{\phi}$, $\hat{\psi}$, we may assume that these functions are non-negative.

We observe the elementary inequality

$$\langle\mu_1+\mu_2\rangle^{s'} \lesssim \langle\mu_2\rangle^{s'} + \langle\mu_1\rangle^{s}\langle\mu_2\rangle^{s'-s} + \langle\mu_1+\mu_2\rangle^{-s}\langle\mu_1\rangle^{s}\langle\mu_2\rangle^{s'},$$

which is easily shown by checking the cases $\langle\mu_1\rangle \ll \langle\mu_2\rangle$, $\langle\mu_1\rangle \gg \langle\mu_2\rangle$, $\langle\mu_1\rangle \sim \langle\mu_2\rangle$ seperately. By applying this estimate to (24) and using Plancherel's theorem again, we see that it suffices to show that

$$\phi(D_u^{s'}\psi), (D_u^s\phi)(D_u^{s'-s}\psi), D_u^{-s}[(D_u^s\phi)(D_u^{s'}\psi)]$$

are each in L^2_u.

The first function is a product of an H^s_u and an L^2_u function, and is thus in L^2 by the Sobolev embedding $H^s_u \subset L^\infty_u$. The second function is a product of an L^2_u and an H^s_u function and is treated similarly. To show that the last function is in L^2, it suffices by the Sobolev embedding $D_u^{-s}L^1_u \subset L^2_u$ to show that $(D^s_u\phi)(D^{s'}_u\psi)$ is in L^1_u. But this follows from Hölder's inequality since the two factors are in L^2_u. Thus $D_u^{s'}(\phi\psi)$ is in L^2_u as desired. ■

The same argument applies of course to the v variable. Working in both the u and v variables we obtain,

Lemma 3.3. *If $s_1, s_2 > 1/2$, and $s_1 \geq s_1' \geq -s_1$, $s_2 \geq s_2' \geq -s_2$, then for all test functions ϕ, ψ*

$$\|\phi\psi\|_{H_u^{s_1'}H_v^{s_2'}} \lesssim \|\phi\|_{H_u^{s_1}H_v^{s_2}}\|\psi\|_{H_u^{s_1'}H_v^{s_2'}}. \tag{25}$$

$$\|\phi\psi\|_{H_u^{s_1'}H_v^{s_2'}} \lesssim \|\phi\|_{H_u^{s_1'}H_v^{s_2}}\|\psi\|_{H_u^{s_1}H_v^{s_2'}}. \tag{26}$$

Corollary 3.4. *The one-dimensional and product Sobolev spaces are stable under multiplication by bump functions. In particular, if η is a bump function and $\tilde{\eta}$ is a Schwarz function which is non-zero on the support of η, then $\eta\phi \in H_u^{s_1}H_v^{s_2}$ whenever $\tilde{\eta}\phi \in H_u^{s_1}H_v^{s_2}$.*

Finally, we need the following lemma on the smoothing properties of $\Box^{-1}$. For previous instances of this lemma in higher dimensions and for differential operators other than $\Box$, see [1, 14, 19].

Lemma 3.5. *If η is a fixed bump function and $s_1, s_2 \geq 1/2$, then*

$$\|\eta\, \Box^{-1}\, \phi\|_{H^{s_1}_u H^{s_2}_v} \leq C_\eta \|\phi\|_{H^{s_1-1}_u H^{s_2-1}_v}$$

for all test functions ϕ.

Proof If ϕ has compact frequency support then we have $H^\infty_{u,v}$ control on ϕ and we may estimate $\eta\, \Box^{-1}\, \phi$ in whatever norm we please (e.g. $H^N_{u,v}$ for some large even integer N). Thus we may assume that the Fourier transform of ϕ vanishes near the origin. By a partition of unity we may assume that ϕ has frequency support in the region $\mu \gg 1$, $|\nu| \lesssim \mu$, since the other regions are completely analogous. We may assume that the $H^{s_1-1}_u H^{s_2-1}_v$ norm of ϕ is equal to 1.

By Corollary 3.4 we may replace η by $\hat\Phi(x)\hat\Phi(t)$, where Φ is a one-dimensional bump function which equals 1 near the origin. Write $\psi = \Box^{-1}\phi$, and $F = \hat\Phi(x)\hat\Phi(t)\psi$.

Since ψ vanishes to first order on the line $t = 0$ we have

$$F = \hat\Phi(x)\hat\Phi(t)\psi = \hat\Phi(x)(\hat\Phi(t) - \delta(t))\psi,$$

where δ is the Dirac distribution. A computation then shows

$$\hat F(\mu,\nu) = C \int\int \Phi(\xi)\Phi(\tau)\hat\psi(\mu - \frac{\xi+\tau}{2}, \nu - \frac{\xi-\tau}{2})\; d\xi d\tau \tag{27}$$

$$= C \int\int \Phi(\xi)(1-\Phi(\tau))\hat\psi(\mu - \frac{\xi+\tau}{2}, \nu - \frac{\xi-\tau}{2})\; d\xi d\tau \tag{28}$$

where $\hat F$, $\hat\psi$ represent the Fourier transform in the u,v variables. We estimate $\hat F$ by dividing into two cases: $|\nu| \gg 1$ and $|\nu| \lesssim 1$. When $|\nu| \gg 1$, it suffices by (27) and Minkowski's inequality to show that the functions

$$\hat\psi(\mu - \frac{\xi+\tau}{2}, \nu - \frac{\xi-\tau}{2})$$

are uniformly in $H^{s_1}_u H^{s_2}_v$ for all ξ, τ in the support of Φ. But this follows from the hypothesis $\phi \in H^{s_1-1}_u H^{s_2-1}_v$ and the identity

$$\hat\psi(\mu,\nu) = C\frac{\hat\phi(\mu,\nu)}{\mu\nu}, \qquad \mu,\nu \neq 0 \tag{29}$$

which follows from the fact that $\Box\psi = \phi$.

It remains to consider the case $|\nu| \lesssim 1$; It suffices to show that $\langle\mu\rangle^{s_1}\hat F(\mu,\nu)$ is uniformly in L^2_ν for $|\nu| \lesssim 1$. By (28) and Minkowski's inequality in the ξ variable, we thus have to show that

$$\int_{\mu \gg 1} \langle \mu \rangle^{2s_1} \left| \int_{|\tau| \gg 1} |\hat{\psi}(\mu - \frac{\xi+\tau}{2}, \nu - \frac{\xi - \tau}{2})|\ d\tau \right|^2\ d\mu \lesssim 1$$

uniformly in $|\xi| \lesssim 1$. By a small translation of τ,μ we may take $\nu = \xi = 0$. By Cauchy-Schwarz and the hypothesis $s_2 > 1/2$ we may estimate this expression by

$$\int_{\mu \gg 1} \int_{|\tau| \gg 1} \langle \mu \rangle^{2s_1} \langle \tau \rangle^{2s_2} |\hat{\psi}(\mu - \frac{\tau}{2}, \frac{\tau}{2})|^2\ d\tau d\mu,$$

which by (29) and the frequency support of ϕ is estimated by

$$\int_{\mu \gg 1} \int_{|\tau| \gg 1} \langle \mu \rangle^{2(s_1-1)} \langle \tau \rangle^{2(s_2-1)} |\hat{\phi}(\mu - \frac{\tau}{2}, \frac{\tau}{2})|^2\ d\tau d\mu.$$

But by a change of variables we see that this is bounded by $\|\phi\|^2_{H^{s_1-1}_u H^{s_2-1}_v} = 1$, as desired. ∎

We can now prove the estimates necessary to apply Lemma 2.4.

We first prove (12), which in this context is

$$\|\eta S(f,g)\|_{H^{s,\delta}} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}. \tag{30}$$

We observe that $S(f,g)$ can be written as $F(u)+G(v)$ for some compactly supported H^s functions F, G. By Corollary 3.4 it thus suffices to show that $F(u)\eta(v)$ and $G(v)\eta(u)$ are in $H^{s,\delta}$ for one-dimensional cutoff functions η. But this follows from (23). A similar argument shows that $\tilde{X}$, $\widetilde{\mathcal{D}}$ also obey (12).

We next prove (13), which in this context is

$$\|\phi(T)\|_{H^s} + \|\phi_t(T)\|_{H^{s-1}} \lesssim \|\phi\|_{H^{s,\delta}}. \tag{31}$$

It suffices to show that $(D^s\phi)(T)$ is in L^2 for all multipliers D^s which are symbols of order s. Since the symbol of D^s is majorized by that of $D^s_u + D^s_v$, we can thus decompose $D^s\phi = \psi_1 + \psi_2$, where ψ_1 and ψ_2 are in $L^2_u H^\delta_v$ and $L^2_v H^\delta_u$ respectively (by (23)). The claim then follows from Sobolev embedding and the fact that the $t = T$ trace of an $L^2_u L^\infty_v$ or $L^2_v L^\infty_u$ function is in L^2.

We now prove (20) and (21); the proof that $\tilde{X}$, $\widetilde{\mathcal{D}}$ satisfy (18) and (19) will follow by specializing the following arguments (which do not need the hypotheses $\delta < 1, s$) to the case $s = \delta$. In our context, the estimates to prove are

$$\|\phi\psi\|_{H^{s,\delta}} \lesssim \|\phi\|_{H^{s,\delta}} \|\psi\|_{H^{\delta,\delta}} + \|\phi\|_{H^{\delta,\delta}} \|\psi\|_{H^{s,\delta}} \tag{32}$$

$$\|\eta \,\Box^{-1} (\phi \,\Box\, \psi)\|_{H^{s,\delta}} \lesssim \|\phi\|_{H^{s,\delta}} \|\psi\|_{H^{\delta,\delta}} + \|\phi\|_{H^{\delta,\delta}} \|\psi\|_{H^{s,\delta}}. \tag{33}$$

The estimate (32) follows immediately from (23) and (25), so it only remains to show (33). It suffices by (23) and u-v symmetry to estimate the $H^s_u H^\delta_v$ norm of $\eta\Box^{-1}(\phi\Box\psi)$, which by Lemma 3.5 is controlled by the $H^{s-1}_u H^{\delta-1}_v$ norm of $\phi D_u D_v \psi$.

We now divide into two cases. If $s-1 \leq \delta$, then by (25) (with ψ replaced by $D_u D_v \psi$) we have

$$\|\phi D_u D_v \psi\|_{H^{s-1}_u H^{\delta-1}_v} \lesssim \|\phi\|_{H^\delta_u H^\delta_v} \|D_u D_v \psi\|_{H^{s-1}_u H^{\delta-1}_v},$$

which gives (33). When $s-1 > \delta$, the proof is similar but (25) is replaced by the following lemma (with ψ replaced by $D_u D_v \psi$):

Lemma 3.6. *If $s-1 > \delta > 1/2$, then*

$$\|\phi\psi\|_{H^{s-1}_u H^{\delta-1}_v} \lesssim \|\phi\|_{H^s_u H^\delta_v} \|\psi\|_{H^{\delta-1}_u H^{\delta-1}_v} + \|\phi\|_{H^\delta_u H^\delta_v} \|\psi\|_{H^{s-1}_u H^{\delta-1}_v}.$$

Proof We repeat the argument in Lemma 3.2. It suffices to estimate

$$\int_{\mu_1+\mu_2=\mu} \int_{\nu_1+\nu_2=\nu} \langle \mu_1+\mu_2\rangle^{s-1} \langle \nu_1+\nu_2\rangle^{\delta-1} \hat\phi(\mu_1,\nu_1) \hat\psi(\mu_2,\nu_2)\ d\mu_1 d\mu_2 \tag{34}$$

in $L^2_\mu L^2_\nu$, and we may assume as before that $\hat\phi$, $\hat\psi$ are non-negative.

By Plancherel's theorem and the easily verified inequalities

$$\langle \mu_1+\mu_2\rangle^{s-1} \lesssim \langle\mu_1\rangle^s \langle\mu_2\rangle^{-1} + \langle\mu_2\rangle^{s-1}$$
$$\langle \nu_1+\nu_2\rangle^{\delta-1} \lesssim \langle\nu_2\rangle^{\delta-1} + \langle\nu_1\rangle^\delta \langle\nu_2\rangle^{-1} + \langle\nu_1+\nu_2\rangle^{-\delta} \langle\nu_1\rangle^\delta \langle\nu_2\rangle^{\delta-1},$$

the $L^2_\mu L^2_\nu$ norm of (34) is majorized by the $L^2_u L^2_v$ norms of

$$(D^s_u\phi)(D^{-1}_u D^{\delta-1}_v\psi), \quad (D^s_u D^\delta_v\phi)(D^{-1}_u D^{-1}_v\psi), \quad D^{-\delta}_v[(D^s_u D^\delta_v\phi)(D^{-1}_u D^{\delta-1}_v\psi)],$$
$$(\phi)(D^{s-1}_u D^{\delta-1}_v\psi), \quad (D^\delta_v\phi)(D^{s-1}_u D^{-1}_v\psi), \quad D^{-\delta}_v[(D^\delta_v\phi)(D^{s-1}_u D^{\delta-1}_v\psi)].$$

The $L^2_u L^2_v$ norms of the first three expressions are controlled by the $H^s_u H^\delta_v$ norm of ϕ and the $H^{\delta-1}_u H^{\delta-1}_v$ norm of ψ, using the Hölder and Sobolev inequalities (in particular, the fact that the product of an $L^2_u H^\delta_v$ and an $L^2_v H^\delta_u$ function is in $L^2_u L^2_v$) as in the proof of Lemma 3.2. The last three expressions are similarly controlled by the $H^\delta_u H^\delta_v$ norm of ϕ and the $H^{s-1}_u H^{\delta-1}_v$ norm of ψ. ∎

Finally, we show that the solution ϕ is locally in $H^{s,s}$. From the above we have that ϕ is locally in $H^s_u H^\delta_v$. Since this space is an algebra by Lemma 3.2 and Γ is analytic, we see that $\Gamma(\phi) \in H^s_u H^\delta_v$. Also we have $\phi_u \in H^{s-1}_u H^\delta_v$, while a symmetrical argument gives $\phi_v \in H^\delta_u H^{s-1}_v$. We now divide into the cases $\delta \geq s-1$ and $\delta < s-1$. If $\delta \geq s-1$ then (8) and Lemma 3.2 now gives

$$\phi_{uv} = \Gamma(\phi)\phi_u\phi_v \in (H^s_u H^\delta_v)(H^{s-1}_u H^\delta_v)(H^\delta_u H^{s-1}_v) \subset H^{s-1}_u H^{s-1}_v \text{ (locally)}$$

Since $\eta\phi = \eta S(f,g) + \eta\,\Box^{-1}\,\phi_{uv}$, the claim then follows from (30) (with $\delta = s$) and Lemma 3.5.

If $\delta < s-1$, then the above argument will only yield that ϕ_{uv} is in $H^{s-1}_u H^\delta_v \cap H^\delta_u H^{s-1}_v$, so that ϕ is in $H^{s,\delta+1}$. One then iterates the above argument, with δ replaced by $\delta+1$, until one eventually obtains $H^{s,s}$ control on ϕ.

4. A pointwise conservation law, and consequences

In this section we introduce a pointwise conservation law for the one-dimensional wave map equation which is special to the one-dimensional case. This law was first observed by Polhmeyer[23]. (See also [27].) An identity key to our work here, (37), is motivated by [4].

Lemma 4.1. *If ϕ is a smooth solution to (4), then the quantity $|\phi_u|_h$ is constant with respect to v, and the quantity $|\phi_v|_h$ is constant with respect to u, where we use $|x|_h$ to denote the length of a tangent vector x in $\mathcal{M}$ with respect to the Riemannian metric h.*

Proof The energy-momentum tensor $T_{\alpha\beta}$ for wave maps is

$$T_{\alpha\beta} = \frac{1}{2}\big(\langle \partial_\alpha \phi, \partial_\beta \phi\rangle - \frac{1}{2} g_{\alpha\beta} \langle \partial_\mu \phi, \partial^\mu \phi\rangle \big) \tag{35}$$

where $\langle \phi, \psi\rangle = h_{\mu\nu}\phi^\mu \psi^\nu$ is the inner product on $\mathcal{M}$. Recall that in all dimensions, the tensor T is divergence free,

$$\partial^\alpha T_{\alpha\beta} = 0. \tag{36}$$

In $\mathbf{R}^{1+1}$, $g_\alpha^{\ \alpha} = 2$ and so T is also trace free,

$$T_\alpha^{\ \alpha} = \frac{1}{2}\big(\langle \partial_\alpha \phi, \partial^\alpha \phi\rangle - \frac{1}{2} g_\alpha^{\ \alpha} \langle \partial_\mu \phi, \partial^\mu \phi\rangle \big) = 0.$$

We write these two facts in null coordinates u, v. The trace-free property gives $T_{uv} + T_{vu} = 0$; since T is symmetric we thus have $T_{uv} = T_{vu} = 0$. The divergence-free property then gives

$$\partial_v T_{uu} = \partial_u T_{vv} = 0,$$

so that T_{uu} is constant with respect to v, and T_{vv} is constant with respect to u. The claim then follows since $T_{uu} = \frac{1}{2}|\phi_u|_h^2$, and $T_{vv} = \frac{1}{2}|\phi_v|_h^2$. ■

Although this lemma is phrased for smooth solutions, the result extends to rough solutions by applying a limiting argument and using the local well-posedness theory from Theroem 1.2. Note that Lemma 4.1 obviously holds as well for solutions of the free wave equation.

In the case when the target manifold is a sphere, a more direct proof is available. Since the solution ϕ is on the sphere, we have $\phi^t\phi = 1$. Differentiating with respect to u we obtain $\phi^t \phi_u = 0$. Combining this with (9) we obtain the useful identity

$$\phi_{uv} = R\phi_u, \tag{37}$$

where R is the anti-symmetric matrix

$$R = \phi_v \phi^t - \phi \phi_v^t. \tag{38}$$

The anti-symmetry of R implies that $|\phi_u|^2$ is constant in the v direction:

$$\partial_v |\phi_u|^2 = 2\phi_u^t \phi_{uv} = 2\phi_u^t R \phi_u = 0, \tag{39}$$

and the other conservation law in Lemma 4.1 is proven similarly.

Lemma 4.1 can be viewed as a pointwise form of energy conservation, and has many consequences. For H^1 solutions it implies the estimates

$$\|\phi_u\|_{L^2_u L^\infty_v} \lesssim \|f\|_{\dot H^1} + \|g\|_2 \tag{40}$$

$$\|\phi_v\|_{L^2_v L^\infty_u} \lesssim \|f\|_{\dot H^1} + \|g\|_2, \tag{41}$$

which in turn show that the $H^1 \times L^2$ norm[3] of the solution $\phi[t]$ at time t is bounded uniformly in t. Combining this with Theorem 1.2 one obtains Theorem 1.3 for $s \geq 1$.

From (4), (40), (41), and the assumption that $\Gamma(\phi)$ is uniformly bounded, we get the spacetime estimate

$$\|\phi_{uv}\|_{L^2_u L^2_v} \lesssim (\|f\|_{\dot H^1} + \|g\|_2)^2. \tag{42}$$

The following Corollary to Lemma 4.1 states that when the initial data is essentially compactly supported, the solution to (9) resolves to an exact free solution in finite time.

Corollary 4.2. *Suppose that H^1 Cauchy data (F, G) are given such that F', G are supported on the interval $[-T, T]$. Then the global solution $\Phi(u, v)$ to (9) with this data is constant on the quadrants $[T, \infty) \times [T, \infty)$, $[T, \infty) \times (-\infty, -T]$, $(-\infty, -T] \times [T, \infty)$, and $(-\infty, -T] \times (-\infty, -T]$, is constant in the v direction on the strips $[-T, T] \times [T, \infty)$, $[-T, T] \times (-\infty, T]$, and is constant in the u direction in the strips $[T, \infty) \times [-T, T]$, $(-\infty, T] \times [T, T]$.*

In particular, we see that Φ scatters exactly to a free solution Φ_+ when $t > T$ and to another free solution Φ_- when $t < -T$. (See Figure 1).

[3] Of course, one can show energy conservation much more directly, but the above approach is more robust, and can be extended to regularities below the H^1 norm.

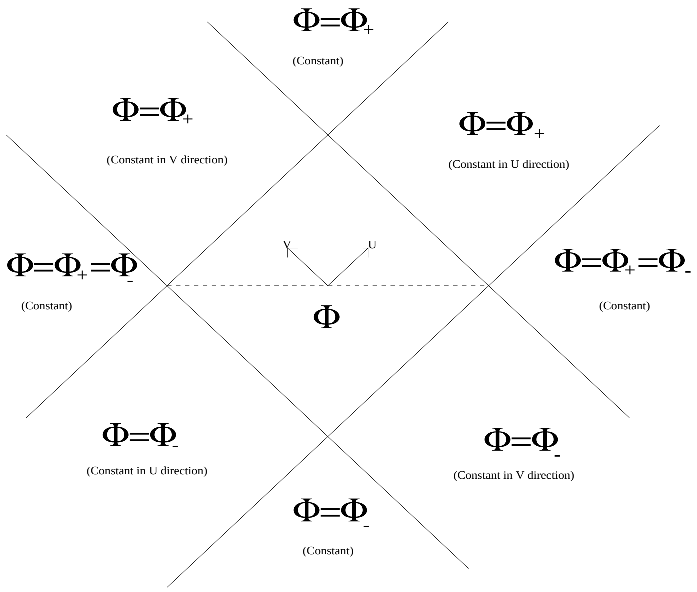

FIGURE 1. Scattering of solutions Φ with essentially compactly supported data.

5. Global existence in H^s, $3/4 < s < 1$: preliminaries

We now turn to the second claim in Theorem 1.3. Fix $3/4 < s < 1$. We have to show that the Cauchy problem (9) for the sphere is globally well-posed for data which is locally in H^s. It suffices to show local well-posedness on some time interval $(-T, T)$, where we fix T to be an arbitrary large time. By finite speed of propagation we may assume that the data becomes constant outside of the interval $[-4T, 4T]$. In particular, we have

$$f', g \text{ is supported on } [-4T, 4T]. \tag{43}$$

We also make the a priori assumption that the data is in H^1; this assumption will be removed by a density argument. More precisely, we will assume that

$$||f||_{H^s} + ||g||_{H^{s-1}} \leq C_0 \tag{44}$$

$$||f||_{H^1} + ||g||_{L^2} \leq M \tag{45}$$

where C_0, M are arbitrary constants. Henceforth all constants will be allowed to depend on C_0, *but not on* M. We will use the quantities C, N, ε to denote positive constants that vary from line to line.

Since the data is in H^1, there is a unique global H^1 solution ϕ. We aim to show the H^s norm of the solution is bounded by a quantity which depends polynomially on T but is independent of M:

$$||\phi(t)||_{H^s} + ||\phi_t(t)||_{H^{s-1}} \lesssim T^N \text{ for all } |t| \leq T. \tag{46}$$

Then by Theorem 1.2 and a limiting argument, the same estimate holds without the condition (45), and one obtains well-posedness in H^s on the interval $(-T, T)$.

It remains to prove (46). When $s = 1$ such an estimate can be obtained from (40) and (41), so it is natural to look for variants of (40), (41) (and perhaps (42)) which apply for data which are rougher than H^1.

For the free equation $\Box\phi = 0$, conservation of the H^s norm for $s < 1$ is shown by applying fractional integration operators to the equation and then applying the energy conservation law. Thus a first guess might be to apply the operators D_u^{s-1}, D_v^{s-1} to the above equations. The fact that these operators (for fractional s) are not local is inconvenient for technical reasons, and we will instead apply the following modified fractional integration operators.

Definition 5.1. For any $s \in \mathbf{R}$, let $m_s(\xi)$ be the convolution of $\langle\xi\rangle^s$ with $\hat{\eta}$, where $\eta(x)$ is a bump function with non-negative Fourier transform. We let $\tilde{D}_u^s$, $\tilde{D}_v^s$ be the Fourier multipliers corresponding to $m_s(\mu)$, $m_s(\nu)$ respectively.

These operators behave like the usual fractional differentiation and integration operators, but have a compactly supported kernel. Note that $m_s(\xi)$ is comparable to $\langle\xi\rangle^s$, so one can replace D_u^s, D_v^s by $\tilde{D}_u^s$, $\tilde{D}_v^s$ respectively in the definitions of the Sobolev spaces defined earlier. That the $\tilde{D}_u^s$ operators are not perfectly multiplicative in s is irrelevant for our purposes.

If one informally pretends that $\tilde{D}_u^{s-1}$, $\tilde{D}_v^{s-1}$ commute with the wave map equation, then (40), (41), (42) informally yield

$$\|\phi_u\|_{H_u^{s-1} L_v^\infty} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}} \tag{47}$$

$$\|\phi_v\|_{H_v^{s-1} L_u^\infty} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}} \tag{48}$$

$$\|\phi_{uv}\|_{H_u^{s-1} H_v^{s-1}} \lesssim (\|f\|_{H^s} + \|g\|_{H^{s-1}})^2, \tag{49}$$

where the spaces $H_u^{s-1} L_v^\infty$, $H_v^{s-1} L_u^\infty$ are defined[4]

$$\|\phi\|_{H_u^{s-1} L_v^\infty} = \|\tilde{D}_u^{s-1}\phi\|_{L_u^2 L_v^\infty}, \quad \|\phi\|_{H_v^{s-1} L_u^\infty} = \|\tilde{D}_v^{s-1}\phi\|_{L_v^2 L_u^\infty}. \tag{50}$$

The estimate (49) implies that ϕ is in $H^{s,s}$, and (46) would follow from (31). Conversely, when T is small then Theorem 1.2 implies that the solution ϕ is in $H^{s,s}$, and the above claims follow from Sobolev embedding.

Of course, this derivation of (47)- (49) is not rigorous since the wave map equation (37), which gave (40), does not commute with fractional integration operators as R is not constant coefficient. However one may hope to obtain some regularity control on R and thus obtain an approximate conservation law, using paraproduct type estimates to control the error. It turns out that when T is large one needs to first rescale the solution as in (6) in order to make this approach viable. We give rescaling precedence over differentiation, hence $\phi_u^\lambda = (\phi^\lambda)_u$.

[4]For technical reasons caused by the L^∞ norm, one has to take some care in defining these spaces; for instance, one cannot simply replace $\tilde{D}_u^{s-1}$ by D_u^{s-1}.

The rescaled versions of (47), (48), (49) that we will rigorously prove are as follows.

Theorem 5.2. *Let $3/4 < s < 1$, $C, M > 0$, $T \gtrsim 1$ be fixed, and suppose that the initial data to the Cauchy problem (9) satisfies (43), (44), (45). Then the H^1 solution ϕ to (9) satisfies the global estimates*

$$\|\phi_u^\lambda\|_{H_u^{s-1} L_v^\infty} \leq C_1 \lambda^{\frac{1}{2}-s} \tag{51}$$

$$\|\phi_v^\lambda\|_{H_v^{s-1} L_u^\infty} \leq C_1 \lambda^{\frac{1}{2}-s} \tag{52}$$

$$\|\phi_{uv}^\lambda\|_{H_u^{s-1} H_v^{s-1}} \leq C_1 \lambda^{1-2s} \tag{53}$$

for $\lambda = C_2 T^{N_2}$, where $C_1, C_2, N_2 > 0$ are constants which do not depend on T or M.

Note that we have the scaling relationship

$$\lambda^{1/2} \|f\|_{H^{s-1}} \lesssim \|f^\lambda\|_{H^{s-1}} \lesssim \lambda^{\frac{3}{2}-s} \|f\|_{H^{s-1}} \tag{54}$$

when $\lambda \gg 1$, and similarly for ϕ. Thus the estimates (51), (52), (53) are implied by, but are weaker than, their $\lambda = 1$ counterparts (47), (48), (49), especially for the low frequency modes of ϕ. This will be enough to recover polynomial growth of the H^s norm, since for frequencies which are $\gg \lambda$ the two estimates are essentially equivalent.

The general approach implicit in Theorem 5.2, is motivated by that in [3], where Bourgain shows global well-posedness results (for the NLS and NLW equations) in spaces rougher than the energy space. Note that since H^1 solutions remain in H^1, the agent of blowup in H^s for $s < 1$ must be the migration of energy from high frequencies to low frequencies. The bounds (51)-(53) provide control on the movement from frequencies $|\xi| >> \lambda$ to lower frequencies.

Note however that the techniques in [3] do not apply directly to our situation since there is no smoothing for the one-dimensional wave-map equation; more precisely, the estimate

$$\|\phi(T) - S(f,g)(T)\|_{H^1} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}$$

which is central to the approach in [3] does not hold for any $s < 1$, even for short times T. Our approach relies on the very strong conservation laws in Lemma 4.1 to overcome this lack of smoothing.

Assuming Theorem 5.2 for the moment, let us conclude the proof of (46) and hence Theorem 1.3. By (31) it suffices to show that

$$\|\eta\phi\|_{H_u^s H_v^s} \lesssim T^N, \tag{55}$$

where η is a cutoff function adapted to the diamond $\{(u,v) : |u|, |v| \lesssim T\}$. On the other hand, applying (53) and (54), we obtain (after expanding λ in terms of T)

$$\|\phi_{uv}\|_{H_u^{s-1} H_v^{s-1}} \lesssim T^N.$$

Since $\phi = S(f,g) + \Box^{-1}\phi_{uv}$, the claim (55) follows from Lemma 3.5 and (30).

6. Localized H^s, and one-dimensional paraproduct estimates

In this section $1 > s > 1/2$ is fixed. In the local well-posedness theory developed in previous sections, estimates such as $H^s H^s \subset H^s$, $H^s H^{s-1} \subset H^{s-1}$ (together with product space analogues) were crucial. In order to show global well-posedness we will need to strengthen these inclusions in a number of ways.

Our first observation is that we may replace the space H^s by a localized variant, which we denote by L. We cover the real line by finitely overlapping intervals $\{I\} = \{J\}$ of length approximately 1, and for each I let η_I be a standard bump function adapted to I so that $\sum_I \eta_I \sim 1$.

Definition 6.1. If $f(u)$ is a test function, define the $L = L_u$ norm by

$$||f||_{L_u} = \sup_I ||\tilde{D}_u^s(\eta_I f)||_{L^2_u}.$$

where $\tilde{D}_u$ is defined in Definition 5.1. Similarly we define $L = L_v$ for functions of v.

Note that Corollary 3.4 implies that this definition is independent of the exact choice of η_I.

The Sobolev spaces $H^{\tilde{s}}$ can be described locally as follows.

Lemma 6.2. *Let $\tilde{s}$ be any real number. If $f(u)$ is a test function, we have*

$$||f||_{H^{\tilde{s}}_u} \sim (\sum_I ||\eta_I f||^2_{H^{\tilde{s}}_u})^{1/2}. \tag{56}$$

If $\phi(u,v)$ is a test function,

$$||\phi||_{H^{\tilde{s}}_u H^{\tilde{s}}_v} \sim (\sum_I \sum_J ||\eta_I(u)\eta_J(v)\phi||^2_{H^{\tilde{s}}_{u,v}})^{1/2}.$$

Proof We prove (56); the second estimate is proven similarly.

Suppose first that $f \in H^{\tilde{s}}$. By Plancherel's theorem, one can write $f = \tilde{D}_u^{-\tilde{s}} F$ for some $F \in L^2$, with $||F||_2 \sim ||f||_{H^{\tilde{s}}}$. We may write $F = \sum_I F_I$ where each F_I is supported in I and $||F||_2 \sim (\sum_I ||F_I||_2^2)^{1/2}$. Thus for any I

$$\eta_I f = \sum_J \eta_I (D_u^{-\tilde{s}} F_J).$$

Since $\tilde{D}_u^{-\tilde{s}}$ has compactly supported kernel, the summands will vanish unless $\mathrm{dist}(I,J) \lesssim 1$. If we now invoke the triangle inequality and discard the η_I cutoff by Corollary

3.4, we have

$$\|\eta_I f\|_{H^{\tilde{s}}} \lesssim \sum_{J:\mathrm{dist}(I,J)\lesssim 1} \|\tilde{D}_u^{-\tilde{s}} F_J\|_{H^{\tilde{s}}} \sim \sum_{J:\mathrm{dist}(I,J)\lesssim 1} \|F_J\|_2.$$

and (56) follows since there are only a finite number of J for each I.

Conversely, suppose that $(\sum_I \|\eta_I f\|^2_{H^{\tilde{s}}})^{1/2}$ is finite. We may write

$$f = \sum_I \tilde{\eta}_I \eta_I f$$

for some cutoffs $\tilde{\eta}_I$ which are adapted to slight dilates of I. We have to estimate

$$\|f\|_{H^{\tilde{s}}} \sim \|\tilde{D}_u^{\tilde{s}} f\|_2 = \|\sum_I \tilde{D}_u^{\tilde{s}} \tilde{\eta}_I \eta_I f\|_2.$$

Since $\tilde{D}_u^{\tilde{s}}$ has compactly supported kernel, the summands are supported on slight dilates of I, and are therefore finitely overlapping. Thus we have

$$\|f\|_{H^{\tilde{s}}} \lesssim (\sum_I \|\tilde{D}_u^{\tilde{s}} \tilde{\eta}_I \eta_I f\|_2^2)^{1/2} \sim (\sum_I \|\tilde{\eta}_I \eta_I f\|_2^2)^{1/2},$$

and the claim then follows from Corollary 3.4. ∎

There is a slight subtlety involved in product norms involving the space L. Note that in the definition below the sup is inside the sum.

Definition 6.3. If $\phi(u,v)$ is a test function and $\tilde{s} \in \mathbf{R}$, we define the norms $H_u^{\tilde{s}} L_v$, $H_v^{\tilde{s}} L_u$, $L_u L_v = L_v L_u$ by

$$\|\phi\|_{H_u^{\tilde{s}} L_v} = (\sum_I \sup_J \|\eta_I(u)\eta_J(v)\phi\|^2_{H_u^{\tilde{s}} H_v^s})^{1/2}$$
$$\|\phi\|_{H_v^{\tilde{s}} L_u} = (\sum_J \sup_I \|\eta_I(u)\eta_J(v)\phi\|^2_{H_u^s H_v^{\tilde{s}}})^{1/2}$$
$$\|\phi\|_{L_u L_v} = \sup_I \sup_J \|\tilde{D}_u^s \tilde{D}_v^s (\eta_I(u)\eta_J(v)\phi)\|_{L_u^2 L_v^2}.$$

We now prove some algebraic relationships between L and the Sobolev space H^{s-1}. The following Lemma contains localized variants of the embedding $H^s H^{s-1} \subset H^{s-1}$ given in Lemma 3.2 above.

Lemma 6.4. *If $f(u)$ and $g(u)$ are test functions, then*

$$\|fg\|_{H^{s-1}} \lesssim \|f\|_L \|g\|_{H^{s-1}}. \tag{57}$$

Furthermore, if $\phi(u,v)$ and $\psi(u,v)$ are test functions, then

$$\|\phi\psi\|_{H_u^{s-1} H_v^{s-1}} \lesssim \|\phi\|_{L_u L_v} \|\psi\|_{H_u^{s-1} H_v^{s-1}} \tag{58}$$
$$\|\phi\psi\|_{H_u^{s-1} H_v^{s-1}} \lesssim \|\phi\|_{H_u^{s-1} L_v} \|\psi\|_{H_v^{s-1} L_u} \tag{59}$$

Proof We prove only (57); the other two estimates follow by arguing similarly in both u and v.

From (56) (with η_I replaced by η_I^2) we have

$$\|fg\|_{H^{s-1}} \lesssim (\sum_I \|\eta_I f \eta_I g\|_{H^{s-1}}^2)^{1/2}.$$

But from Lemma 3.2 and the definition of L we have

$$\|\eta_I f \eta_I g\|_{H^{s-1}} \lesssim \|\eta_I f\|_{H^s} \|\eta_I g\|_{H^{s-1}} \lesssim \|f\|_L \|\eta_I g\|_{H^{s-1}}.$$

Combining this with the above estimate and using (56) again one obtains (57). ■

In the sequel we will attempt to commute integration operators such as $\tilde{D}_u^{s-1}$ with identities such as (37). In doing so it will be natural to try to control paraproduct expressions such as

$$\tilde{D}_u^{s-1}(\phi\psi) - \phi \tilde{D}_u^{s-1}(\psi),$$

in terms of $\phi(u,v)$, $\psi(u,v)$. This quantity is of comparable strength to $\tilde{D}_u^{s-1}(\phi\psi)$, but exhibits cancellation when ϕ is constant or slowly varying.

In the next section we will need to estimate the above quantity in $L_u^2 H_v^{s-1}$. If one ignored the cancellation and used the triangle inequality, one obtains

$$\|\tilde{D}_u^{s-1}(\phi\psi) - \phi \tilde{D}_u^{s-1}\psi\|_{L_u^2 H_v^{s-1}} \lesssim \|\phi\psi\|_{H_u^{s-1} H_v^{s-1}} + \|\phi \tilde{D}_u^{s-1}\psi\|_{L_u^2 H_v^{s-1}}.$$

Combining this with Lemma 3.3 one obtains the bound

$$\|\tilde{D}_u^{s-1}(\phi\psi) - \phi \tilde{D}_u^{s-1}\psi\|_{L_u^2 H_v^{s-1}} \lesssim \|\phi\|_{H_u^s H_v^{s-1}} \|\psi\|_{H_u^{s-1} H_v^s}. \tag{60}$$

By the previous discussion, we may improve this estimate by localizing the H_v^s norm to an L_v norm. We could also improve the H_u^s norm in this manner, but we will instead pursue a different improvement which tries to take advantage of the cancellation if ϕ has low frequency. In fact, we have

Lemma 6.5. *If $\phi(u,v)$ and $\psi(u,v)$ are test functions, then*

$$\|\tilde{D}_u^{s-1}(\phi\psi) - \phi \tilde{D}_u^{s-1}(\psi)\|_{L_u^2 H_v^{s-1}} \lesssim \|\phi_u\|_{H_u^{s-1} H_v^{s-1}} \|\psi\|_{H_u^{s-1} L_v}. \tag{61}$$

Note that $\|f_u\|_{H^{s-1}}$ is essentially the same as $\|f\|_{H^s}$ when f consists of high frequencies, but is somewhat smaller for low frequencies, in accordance with the previous heuristics concerning the cancellation.

Proof The first step is to replace the L_v norm with the stronger H_v^s norm. Let $\tilde{\eta}_J$ be a cut-off function which is one on the support of η_J, as in the proof of Lemma 6.2. It suffices to prove the estimate

(62)

$$\|\eta_J(v)[\tilde{D}_u^{s-1}(\phi\psi) - \phi \tilde{D}_u^{s-1}(\psi)]\|_{L_u^2 H_v^{s-1}} \lesssim \|\tilde{\eta}_J(v)\phi_u\|_{H_u^{s-1} H_v^{s-1}} \|\eta_J(v)\psi\|_{H_u^{s-1} H_v^s}.$$

uniformly in J, since (61) can be recovered by square-summing (62) in J, using the compact support of the kernel of $\tilde{D}_u^{s-1}$, and applying Lemma 6.2. By the support properties of $\eta, \tilde{\eta}$, we may rewrite (62) as

$$\|[\tilde{D}_u^{s-1}(\tilde{\eta}_J(v)\phi\eta_J(v)\psi) - \tilde{\eta}_J(v)\phi\tilde{D}_u^{s-1}(\eta_J(v)\psi)]\|_{L^2_u H^{s-1}_v} \lesssim \|\tilde{\eta}_J(v)\phi_u\|_{H^{s-1}_u H^{s-1}_v}\|\eta_J(v)\psi\|_{H^{s-1}_u H^s_v}.$$

Replacing $\tilde{\eta}_J\phi$ with ϕ and $\eta_J\psi$ with ψ, it suffices to show

$$\|\tilde{D}_u^{s-1}(\phi\psi) - \phi\tilde{D}_u^{s-1}(\psi)\|_{L^2_u H^{s-1}_v} \lesssim \|\phi_u\|_{H^{s-1}_u H^{s-1}_v}\|\psi\|_{H^{s-1}_u H^s_v}. \tag{63}$$

for arbitrary test functions ϕ, ψ. (This estimate should be compared with (61)).

By Plancherel's theorem, the left-hand side is equal to the $L^2_\mu L^2_\nu$ norm of

$$C\int_{\mu_1+\mu_2=\mu}\int_{\nu_1+\nu_2=\nu}[m_{s-1}(\mu_1+\mu_2) - m_{s-1}(\mu_2)]m_{s-1}(\nu_1+\nu_2)\hat{\phi}(\mu_1,\nu_1)\hat{\psi}(\mu_2,\nu_2)\ d\mu_1 d\nu_1. \tag{64}$$

Define Φ, Ψ by $\hat{\Phi}(\mu_1,\nu_1) = |\mu_1|\langle\mu_1\rangle^{-1}|\hat{\phi}(\mu_1,\nu_1)|$, $\hat{\Psi}(\mu_2,\nu_2) = |\hat{\psi}(\mu_2,\nu_2)|$; note that

$$\|\Phi\|_{H^s_u H^{s-1}_v} \sim \|\phi_u\|_{H^{s-1}_u H^{s-1}_v}, \quad \|\Psi\|_{H^{s-1}_u H^s_v} \sim \|\psi\|_{H^{s-1}_u H^s_v}. \tag{65}$$

The quantity (64) is majorized by

$$\int_{\mu_1+\mu_2=\mu}\int_{\nu_1+\nu_2=\nu}\frac{|m_{s-1}(\mu_1+\mu_2) - m_{s-1}(\mu_2)|}{|\mu_1|}\langle\mu_1\rangle\langle\nu_1+\nu_2\rangle^{s-1}\hat{\Phi}(\mu_1,\nu_1)\hat{\Psi}(\mu_2,\nu_2)\ d\mu_1 d\nu_1. \tag{66}$$

When $|\mu_1| \gtrsim 1$ we have

$$\frac{|m_{s-1}(\mu_1+\mu_2) - m_{s-1}(\mu_2)|}{|\mu_1|}\langle\mu_1\rangle \lesssim \langle\mu_2\rangle^{s-1} + \langle\mu_1+\mu_2\rangle^{s-1},$$

while when $|\mu_1| \lesssim 1$ the mean-value theorem gives

$$\frac{|m_{s-1}(\mu_1+\mu_2) - m_{s-1}(\mu_2)|}{|\mu_1|}\langle\mu_1\rangle \lesssim \langle\mu_2\rangle^{s-2}.$$

Thus in either case we have

$$\frac{|m_{s-1}(\mu_1+\mu_2) - m_{s-1}(\mu_2)|}{|\mu_1|}\langle\mu_1\rangle \lesssim \langle\mu_2\rangle^{s-1} + \langle\mu_1+\mu_2\rangle^{s-1}.$$

Inserting this into (66) and using Plancherel's theorem, we see that the $L^2_\mu L^2_\nu$ norm of (66) is majorized by

$$\|D_v^{s-1}(\Phi D_u^{s-1}\Psi)\|_{L^2_u L^2_v} + \|D_u^{s-1}D_v^{s-1}(\Phi\Psi)\|_{L^2_u L^2_v} = \|\Phi D_u^{s-1}\Psi\|_{H^{s-1}_u L^2_v} + \|\Phi\Psi\|_{H^{s-1}_u H^{s-1}_v}.$$

By Lemma 3.3 this is majorized by

$$\|\Phi\|_{H^s_u H^{s-1}_v}\|\Psi\|_{H^{s-1}_u H^s_v},$$

and the claim now follows from (65). ∎

To close this section we give some elementary estimates which connect the L space to H^{s-1} and the L^∞ norm; this will allow us to translate the estimates in Theorem 5.2 to ones involving L.

Lemma 6.6. *If $f(u)$ is a test function, then*

$$\|f\|_{L_u} \lesssim \|f\|_\infty + \|f_u\|_{H_u^{s-1}}. \tag{67}$$

If $\phi(u,v)$ is a test function, then

$$\|\phi\|_{H_v^{s-1}L_u} \lesssim \|\phi\|_{H_v^{s-1}L_u^\infty} + \|\phi_u\|_{H_v^{s-1}H_u^{s-1}}$$
$$\|\phi\|_{H_u^{s-1}L_v} \lesssim \|\phi\|_{H_u^{s-1}L_v^\infty} + \|\phi_v\|_{H_u^{s-1}H_v^{s-1}}$$
$$\|\phi\|_{L_uL_v} \lesssim \|\phi\|_{L_u^\infty L_v^\infty} + \|\phi_u\|_{H_u^{s-1}L_v^\infty} + \|\phi_v\|_{H_v^{s-1}L_u^\infty} + \|\phi_u\phi_v\|_{H_u^{s-1}H_v^{s-1}}$$

where the norms $H_u^{s-1}L_v^\infty$, $H_v^{s-1}L_u^\infty$ were defined in (50).

Proof We prove only (67); the other estimates follow by applying a similar argument applied to both variables at once.

It suffices to show that

$$\|\eta_I f\|_{H^s} \lesssim \|f\|_\infty + \|f_u\|_{H_u^{s-1}}.$$

We may partition frequency space and divide f into a piece with frequency support on $|\mu| \lesssim 1$, and a piece with frequency support on $|\mu| \gtrsim 1$. To handle the first piece we use the estimate

$$\|\eta_I f\|_{H^s} \lesssim \|\eta_I f\|_{H^N} \lesssim \|f\|_{C^N} \lesssim \|f\|_\infty$$

for some large integer N, where the last inequality follows from the frequency support hypothesis.

To handle the second piece we use Lemma 3.2 to obtain

$$\|\eta_I f\|_{H^s} \lesssim \|f\|_{H^s} \sim \|f_u\|_{H^{s-1}}$$

where the last inequality follows from the frequency support hypothesis. ∎

Lemma 6.7. *Suppose that $f(u)$ is a test function supported on an interval I of length $\gtrsim 1$. Then*

$$\|f\|_{L^\infty} \lesssim \|f\|_{L^\infty(I')} + |I|^{1/2}\|f_u\|_{H_u^{s-1}},$$

where I' is any nonempty subinterval of I.

Proof It suffices to show that

$$|f(u) - f(u_0)| \lesssim |I|^{1/2} \|f_u\|_{H_u^{s-1}}$$

whenever $u \in I$, $u_0 \in I'$. But by the fundamental theorem of calculus the left hand side is majorized by

$$|\langle f_u, \chi_{[u_0,u]}\rangle| \lesssim \|f_u\|_{H^{s-1}} \|\chi_{[u_0,u]}\|_{H^{1-s}},$$

and the result follows from the hypothesis $s > 1/2$ and the inequality $\|\chi_{[u_0,u]}\|_{H^{1-s}} \lesssim \langle u_0 - u\rangle^{1/2} \lesssim |I|^{1/2}$. When $u_0 = -1$, $u = 1$ this inequality follows from direct computation, and the general case follows by rescaling and translation invariance. ∎

7. Proof of Theorem 5.2

Fix C_0, T, M, $3/4 < s < 1$. We will let C_1 be a large constant to be chosen later, and C_2, N_2 to be large constants depending on C_1, also to be chosen later. In particular, λ is also fixed. The quantities N, C, ε and the implicit constants in the estimates will vary from line to line, but will not depend on C_2.

We shall use the continuity method. Let B denote the set

$$B = \{(f,g) : (43), (44), (45) \text{ hold.}\}.$$

We give B the induced topology from $H^1 \times L^2$. Consider the subset of B

$$E = \{(f,g) \in B : (51), (52), (53) \text{ hold.}\}.$$

We wish to show that $E = B$. To this end we introduce the weaker versions of (51), (52), (53)

$$\|\phi_u^\lambda\|_{H_u^{s-1} L_v^\infty} \lesssim \lambda^{\frac{1}{2} - s + \varepsilon} \tag{68}$$

$$\|\phi_v^\lambda\|_{H_v^{s-1} L_u^\infty} \lesssim \lambda^{\frac{1}{2} - s + \varepsilon} \tag{69}$$

$$\|\phi_{uv}^\lambda\|_{H_u^{s-1} H_v^{s-1}} \lesssim \lambda^{1-2s+2\varepsilon}. \tag{70}$$

and define the subset of B

$$\tilde{E} = \{(f,g) \in B : (68), (69), (70) \text{ hold}\}.$$

Clearly $E \subset \tilde{E}$ if C_2, N_2 are sufficiently large. Furthermore, we claim the following:

- If C_2, N_2 are sufficiently large, then there exists an $\epsilon_M > 0$ which can depend on T, M, λ such that the following holds: If $(f,g) \in E$ and $(\tilde{f}, \tilde{g})$ is within ϵ_M of (f,g) in $H^1 \times L^2$ norm, then $(\tilde{f}, \tilde{g})$ is in $\tilde{E}$.
- If (f,g) is in $\tilde{E}$, then (f,g) is in E.

Combining these two statements we see that E is both open and closed in $H^1 \times L^2$. Since E contains the origin and B is connected, we will be done.

To prove the first claim, we first observe that (51), (52), (53) are trivial to verify outside of the diamond $\{|u|, |v| \lesssim AT\}$, by Lemma 4.2. Thus we may restrict our attention to the diamond, which is a compact set.

From Theorem 1.2 we see that the $H^{1,1}$ norm of ϕ^λ on the diamond depends in a Lipschitz manner on the $H^1 \times L^2$ norm of the data (with a large Lipschitz constant depending on M, T, λ). Since the $H^{1,1}$ norm controls the norms present in the definition of E, $\tilde{E}$ by Sobolev embedding, the claim follows by elementary topology.

The remainder of this section is devoted to proving the second claim. Accordingly, we fix $(f,g) \in B$, assume that (68), (69), (70) hold, and try to prove (51), (52), and (53).

Since ϕ^λ stays on the sphere, we have

$$\|\phi^\lambda\|_{L^\infty_u L^\infty_v} \lesssim 1. \tag{71}$$

Since λ is large and $\frac{1}{2} - s + \varepsilon < 0$ for ε sufficiently small, we can use Lemma 6.6 and (71), (68), (69), (70) to obtain estimates involving the space L. More precisely, we have

$$\|\phi^\lambda\|_{L_u L_v} \lesssim 1 \tag{72}$$

$$\|\phi^\lambda_u\|_{H^{s-1}_u L_v} \lesssim \lambda^{\frac{1}{2}-s+\varepsilon} \tag{73}$$

$$\|\phi^\lambda_v\|_{H^{s-1}_v L_u} \lesssim \lambda^{\frac{1}{2}-s+\varepsilon} \tag{74}$$

$$\|\phi^\lambda_{uv}\|_{H^{s-1}_u H^{s-1}_v} \lesssim \lambda^{1-2s+2\varepsilon}. \tag{75}$$

We now show (51). This is the same (if C_1 is chosen sufficiently large) as

$$\|\tilde{D}^{s-1}_u \phi^\lambda_u\|_{L^2_u L^\infty_v} \lesssim \lambda^{\frac{1}{2}-s}. \tag{76}$$

When $s = 1$ this was proven in Section 4 by the computation (39). The argument here will be an adaptation of this computation.

We first prove (76) for short times $|t| \lesssim \varepsilon\lambda$, i.e. we show

$$\|\chi^\lambda \tilde{D}^{s-1}_u \phi^\lambda_u\|_{L^2_u L^\infty_v} \lesssim \lambda^{\frac{1}{2}-s},$$

where χ is a cutoff which equals one the slab $|t| \lesssim \varepsilon$, and vanishes on a dilate of this slab. Since $\tilde{D}^{s-1}_u$ has compactly supported kernel, we may write this as

$$\|\chi^\lambda \tilde{D}^{s-1}_u (\tilde{\chi}\phi)^\lambda_u\|_{L^2_u L^\infty_v} \tag{77}$$

where $\tilde{\chi}$ equals 1 on a dilate on the support of χ, and vanishes outside of an even larger dilate. Discarding the χ^λ term and using (54), we reduce ourselves to showing that

$$\|(\tilde{\chi}\phi)_u\|_{H^{s-1}_u L^\infty_v} \lesssim 1.$$

However, for short times $|t| \lesssim \varepsilon$ Theorem 1.2 applies, and we have

$$\|\tilde{\chi}\phi\|_{H^s_u H^s_v} \lesssim 1.$$

The claim then follows by Sobolev embedding.

We now prove the full estimate (76). By squaring, we obtain

$$\| |\tilde{D}_u^{s-1}\phi_u^\lambda|^2 \|_{L^1_u L^\infty_v} \lesssim \lambda^{1-2s}. \tag{78}$$

Since this estimate was just proven for short times, we may invoke Lemma 6.7, and reduce ourselves to showing that

$$\|\partial_v |\tilde{D}_u^{s-1}\phi_u^\lambda|^2 \|_{L^1_u H^{s-1}_v} \lesssim (\lambda T)^{-1/2}\lambda^{1-2s}.$$

By evaluating the v derivative and using (37), it thus suffices to show that

$$\|(\tilde{D}_u^{s-1}\phi_u^\lambda)^t(\tilde{D}_u^{s-1}R\phi_u^\lambda)\|_{L^1_u H^{s-1}_v} \lesssim (\lambda T)^{-1/2}\lambda^{1-2s} \tag{79}$$

Since R is anti-symmetric,

$$(\tilde{D}_u^{s-1}\phi_u^\lambda)^t R(\tilde{D}_u^{s-1}\phi_u^\lambda) = 0.$$

Thus it suffices to show that

$$\|(\tilde{D}_u^{s-1}\phi_u^\lambda)^t[(\tilde{D}_u^{s-1}R\phi_u^\lambda) - R(\tilde{D}_u^{s-1}\phi_u^\lambda)]\|_{L^1_u H^{s-1}_v} \lesssim (\lambda T)^{-1/2}\lambda^{1-2s}.$$

However, from (68) we have

$$\|\tilde{D}_u^{s-1}\phi_u^\lambda\|_{L^2_u L_v} \lesssim \lambda^{\frac{1}{2}-s+\varepsilon}.$$

Thus by (57) it suffices to show that

$$\|\tilde{D}_u^{s-1}(R\phi_u^\lambda) - R\tilde{D}_u^{s-1}\phi_u^\lambda\|_{L^2_u H^{s-1}_v} \lesssim (\lambda T)^{-1/2}\lambda^{\frac{1}{2}-s-\varepsilon}.$$

By Lemma 6.5 this reduces to

$$\|R_u\|_{H^{s-1}_v H^{s-1}_u}\|\phi_u^\lambda\|_{H^{s-1}_v L_u} \lesssim (\lambda T)^{-1/2}\lambda^{\frac{1}{2}-s-\varepsilon}.$$

By (68) again, it thus suffices to show that

$$\|R_u\|_{H^{s-1}_v H^{s-1}_u} \lesssim (\lambda T)^{-1/2}\lambda^{-2\varepsilon}.$$

By expanding out R_u, we need only show

$$\|(\phi^\lambda)^t\phi_{uv}^\lambda\|_{H^{s-1}_v H^{s-1}_u} + \|(\phi_u^\lambda)^t\phi_v^\lambda\|_{H^{s-1}_v H^{s-1}_u} \lesssim (\lambda T)^{-1/2}\lambda^{-2\varepsilon}.$$

But by (58), (59), the left-hand side of this is majorized by

$$\|\phi^\lambda\|_{L_v L_u}\|\phi_{uv}^\lambda\|_{H^{s-1}_v H^{s-1}_u} + \|\phi_u^\lambda\|_{H^{s-1}_u L_v}\|\phi_v^\lambda\|_{H^{s-1}_v L_u},$$

which is bounded by $\lambda^{1-2s+2\varepsilon}$ by (72), (74), (73), (75). Since $s > 3/4$, the claim is thus proven if ε is sufficiently small and C_2, N_2 are sufficiently large. This concludes the proof of (51). Note that if one used (60) instead of Lemma 6.5 then we'd need $s > 1$ instead of $s > \frac{3}{4}$.

The proof of (52) is similar, so we turn to (53). It suffices to show that (if C_1 is sufficiently large)

$$\|\phi_{uv}^\lambda\|_{H^{s-1}_u H^{s-1}_v} \lesssim \lambda^{1-2s}.$$

From (9) the left-hand side is

$$\|\phi^\lambda (\phi^\lambda_u)^t \phi^\lambda_v\|_{H^{s-1}_u H^{s-1}_v}.$$

But by (58), (59), this is majorized by

$$\|\phi^\lambda\|_{L_u L_v} \|\phi^\lambda_u\|_{H^{s-1}_u L_v} \|\phi^\lambda_v\|_{H^{s-1}_v L_u}.$$

Thus by (72) it suffices to show that

$$\|\phi^\lambda_u\|_{H^{s-1}_u L_v} \lesssim \lambda^{\frac{1}{2}-s}$$

and similarly for ϕ^λ_v. But this follows from the definition of $H^{s-1}_u L_v$, the estimate (51) just proved, and (70) (if ε is sufficiently small). ■

8. Negative results

In this section we give some rather simple negative results regarding ill-posedness of the wave map equation and similar equations.

The nonlinearity in the wave map equation (8) contains the null form $Q_0^{\alpha\beta}(\phi,\phi) \equiv \phi^\alpha_u \phi^\beta_v$. That the quadratic form Q_0 has this null structure is important for low regularity well-posedness, as the following simple example shows. (See [21], [19] for a similar situation in dimension $n = 3$.)

Proposition 8.1. *The scalar equation*

$$\Box \phi = |\phi_v|^2$$

is locally well-posed in H^s if and only if $s > 3/2$.

Proof By making the substitution $\psi = \phi_v$, it suffices to show that the equation

$$\psi_u = |\psi|^2$$

is locally well posed in H^s if and only if $s > 1/2$. But from the explicit solution

$$\psi(x,t) = \frac{\psi(x-t,0)}{1-\psi(x-t,0)t},$$

we see that ψ only stays regular for a non-zero time when the initial date $\psi(\cdot, 0)$ is bounded. This is only guaranteed when $s > 1/2$, hence the result. ■

With the null form structure, one can do much better, as the following example of Nirenberg shows.

Proposition 8.2. [18] *The scalar equation*

$$\Box\phi = \phi_u \phi_v$$
$$\phi(x,0) = f \quad \phi_t(x,0) = g$$

is locally well-posed in H^s if and only if $s > 1/2$.

Proof For completeness, we sketch the argument given in [18] here. Take data $f = 0$, $g \in H^{s-1}$. By making the substitution $\psi = 1 - e^{\phi}$, it suffices to show that the solution to

$$\Box\psi = 0$$

remains in H^s and satisfies $\|\psi(t,x)\|_{L^\infty(\mathbf{R})} < 1$ for a non-zero amount of time. This is true for $s > 1/2$ since ψ is then uniformly continuous by Sobolev embedding. For $s \leq 1/2$ it is easy to construct discontinuous ψ which becomes large instantaneously. ∎

By Theorem 1.2 and previously mentioned work in higher dimensions, one has local well-posedness in $H^s(\mathbf{R}^n)$ for wave maps from $\mathbf{R}^{n+1}$ when $s > n/2$. For $s < n/2$ the problem is supercritical and well-posedness seems very unlikely. (For $n \geq 3$, the supercritical wave map problem is ill-posed for certain manifolds: see Shatah, Shatah-Zadeh [27, 30].) The critical case $s = n/2$ seems very subtle, as the following example demonstrates.

Proposition 8.3. *If the target manifold of the wave map (1) is the sphere S^{m-1}, $m \geq 2$, then there exist coordinates for which the solution operator to the wave map equation (4) is not twice differentiable on the data space $H^{\frac{n}{2}}(\boldsymbol{R}^n)$.*

Remark: In particular, Proposition 8.3 shows that the solution operator in $\mathbf{R}^{2+1}$ does not depend smoothly on the data in the energy norm, and so there exists a coordinate system on the target manifold in which one cannot prove a critical H^1 result by the usual Picard iteration argument. The proposition also holds if the inhomogeneous norm $H^{n/2}$ is replaced by the homogeneous version $\dot{H}^{n/2}$.

Proof It is well known (e.g. [31]) that composition of a solution of the free wave equation with a geodesic yields a wave map. Let $\psi : \mathbf{R}^n \to \mathbf{R}$ satisfy the free wave equation with data

$$(\psi, \psi_t) = (0, \epsilon g). \tag{80}$$

where $g \in H^{\frac{n}{2}-1}$ and $\epsilon \in \mathbf{R}$. The mapping $\mathbf{R} \to S^1 \subset S^{m-1}$ given by $x \to e^{ix}$ is a geodesic, hence the function

$$\phi_\epsilon(x,t) = e^{i\psi(x,t)} = e^{i\epsilon\frac{\sin(\sqrt{-\Delta}t)}{\sqrt{-\Delta}}g} \tag{81}$$

is a solution of the wave-map system (5) with initial velocity $i\epsilon g$.

Assume for contradiction that the mapping taking initial velocity to the solution (81) at time 1

$$S : i\epsilon g \to \phi_\epsilon(x,1)$$

is twice differentiable as a mapping $S : H^{\frac{n}{2}-1}(\mathbf{R}^n) \to H^{\frac{n}{2}}(\mathbf{R}^n)$. This implies

$$\frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} \phi_\epsilon(x,1) \in H^{\frac{n}{2}}(\mathbf{R}^n). \tag{82}$$

Hence

$$\left(\frac{\sin(\sqrt{-\Delta})}{\sqrt{-\Delta}} g\right)^2 \in H^{\frac{n}{2}}(\mathbf{R}^n) \tag{83}$$

for all $g \in H^{\frac{n}{2}-1}(\mathbf{R}^n)$.

Consider the preliminary function $G(x)$ defined by

$$\hat{G}(\xi) = \frac{1}{\log^{\frac{1}{2}+}(\langle\xi\rangle) \cdot \langle\xi\rangle^{n-1}}.$$

One easily verifies $G \in H^{\frac{n}{2}-1}(\mathbf{R}^n)$. If we set $g(x) = \sin(\sqrt{-\Delta})G(x)$ then clearly we also have $g \in H^{\frac{n}{2}-1}(\mathbf{R}^n)$. A straightforward computation gives

$$\begin{aligned}
\| \left(\frac{\sin(\sqrt{-\Delta})}{\sqrt{-\Delta}} g\right)^2 \|_{H^{\frac{n}{2}}}^2 &= \int \langle\xi\rangle^n \Big(\int \frac{\sin^2(|\xi-\eta|)}{|\xi-\eta|} G(\xi-\eta) \frac{\sin^2(|\eta|)}{|\eta|} G(\eta) d\eta \Big)^2 d\xi \\
&\gtrsim \int \langle\xi\rangle^n \left(\int \frac{1}{|\xi-\eta|} G(\xi-\eta) \frac{1}{|\eta|} G(\eta) d\eta \right)^2 d\xi \\
&\gtrsim \int_{|\xi|\gtrsim 1} \langle\xi\rangle^n \Big(\int_{1\lesssim|\eta|\lesssim|\xi|} \frac{1}{|\xi-\eta|} G(\xi-\eta) \frac{1}{|\eta|} G(\eta) d\eta \Big)^2 d\xi \\
&\gtrsim \int_{|\xi|\gtrsim 1} \langle\xi\rangle^n \Big(\frac{1}{|\xi|} G(\xi) \int_1^{|\xi|} \frac{1}{\rho} G(\rho)\rho^{n-1} d\rho \Big)^2 d\xi \\
&\gtrsim \int_{|\xi|\gtrsim 1} |\xi|^n \left(\frac{1}{|\xi|} \frac{1}{\log^{\frac{1}{2}+}(|\xi|) \cdot |\xi|^{n-1}} \log^{\frac{1}{2}-}(|\xi|) \right)^2 d\xi \\
&= \int_1^\infty \frac{1}{\rho^n \log^{0+}(\rho)} \rho^{n-1} d\rho \\
&= \infty
\end{aligned}$$

which contradicts (83) as desired. ∎

One can be much more precise on the nature of the solution operator. For instance, when $n=1$, the operator is continuous but not uniformly continuous on $\dot{H}^{1/2}$, and is neither Lipschitz nor everywhere differentiable.

This example points out that the choice of coordinates or frames on the target manifold is important. For instance, if one uses intrinsic arclength coordinates on S^1 rather than extrinsic complex coordinates, then the wave map equation becomes the free wave equation, which is of course analytically well-posed in virtually any data space. These conclusions are consistent with [12, 10, 6, 30], which work in special coordinate systems.

We conclude this section with a negative scattering result. Scattering appears unlikely for the one-dimensional wave map, since there is no obvious decay in the equation, and furthermore the solution stays on a manifold $\mathcal{M}$ while free solutions almost never do. The following result reinforces these heuristics, at least for data which does not have a conditionally integrable velocity. In the converse direction, if the data is compactly supported, scattering was shown in Corollary 4.2, and we'll show in Section 9 that one also has scattering when the velocity and the derivative of the position are absolutely integrable.

Proposition 8.4. *If the target manifold is the unit circle S^1 in the complex plane, the initial position is $f \equiv 1$, and the initial velocity ig is smooth, then the solution asymptotically approaches a free solution in $\dot{H}^1(\boldsymbol{R})$ if and only if the limits $G_\pm = \lim_{x\to\pm\infty} G(x)$ exist, where G is a primitive of g. In particular, there is data in $\dot{H}^1 \times L^2$ which does not scatter.*

Proof We have the explicit solution

$$\phi(x,t) = e^{iG(x+t)/2} e^{-iG(x-t)/2}. \tag{84}$$

where $G(x) = \int_0^x g(\lambda) d\lambda$. If the limits $G_\pm$ exist, then it's easy to see ϕ approaches the free solution

$$\psi(x,t) = e^{iG(x+t)/2} e^{-iG_-/2} + e^{iG_+/2} e^{-iG(x-t)/2} - e^{iG_+/2} e^{-iG_-/2}$$

in $\dot{H}^1(\mathbf{R})$ as $t \to \infty$. For example,

$$\begin{aligned} \|\partial_u \left(\phi(u,v) - \psi(u,v)\right)\|_{L^2} &= \|g(u) e^{iG(u)/2} \left(e^{-iG(v)/2} - e^{-iG_-/2}\right)\|_{L^2} \\ &\to 0 \end{aligned}$$

by dominated convergence.

Suppose conversely that the solution (84) approaches a free solution in $\dot{H}^1 \times L^2$:

$$\|\nabla_{x,t}(\phi(x,t) - f_+(x+t) - f_-(x-t))\|_{L^2} \to 0 \quad \text{as } t \to \infty.$$

The convergence of the u derivative in L^2 gives,

$$\|\frac{ig(u)}{2} e^{iG(u)/2} e^{-iG(u-2t)/2} - f'_+(u)\|_{L^2} \to 0 \text{ as } t \to \infty \tag{85}$$

where we've changed variables $x = u - t$. Suppose we restrict the u integration in (85) to a compact set on which $g(u)$ is non-zero. Then the above convergence is only possible if $G(u-2t)$ converges as $t \to \infty$, which means that G_- must exist. A similar argument shows that G_+ must also exist. ∎

9. Global existence and scattering for large data in the critical space $L^{1,1}$

We give here an elementary proof of Theorem 1.4, which gives global existence and scattering for arbitrary target manifolds in the critical (i.e. scale-invariant) data space $\mathcal{D} = L^{1,1}$ defined by (10).

As with other critical global existence results (e.g. [8]), the proof follows a familiar pattern:

- Prove global well-posedness for small $L^{1,1}$ data.
- Extend to global well-posedness for large $L^{1,1}$ data by a nonconcentration argument.

Scattering will be obtained by a conformal compactification argument in Section 9.4 below.

9.1. Global existence for small data.

9.1. Global existence for small data. Suppose the initial data f, g is small in $L^{1,1}$ when measured in a single coordinate chart.

We apply Lemma 2.3 with $T_1 = T_2 = \infty$ and the space X given by

$$\|\phi\|_X \equiv \|\partial_u \partial_v \phi\|_{L^1_{u,v}} + \|\phi(0)\|_{\mathcal{D}}. \tag{86}$$

We need to check that (12), (13), (18), and (19) hold.

The preliminary bounds

$$\begin{aligned} \|\partial_u \phi\|_{L^1_u L^\infty_v} &\lesssim \|\phi\|_X \\ \|\partial_v \phi\|_{L^1_v L^\infty_u} &\lesssim \|\phi\|_X \end{aligned} \tag{87}$$

follow from the fundamental theorem of calculus. For instance, we have

$$\begin{aligned} \|\partial_u \phi\|_{L^1_u L^\infty_v} &= \int_{-\infty}^{\infty} \sup_v |\partial_u \phi(u,v)| du \\ &\leq \int_{-\infty}^{\infty} \left(\int |\partial_u \partial_v \phi(u,v')| dv' + |\partial_u \phi(u,u)| \right) du \\ &\leq \|\phi\|_X . \end{aligned}$$

The property (12) is trivial from the definition of $S(f)$, so we turn to (13). Using (87),

$$\begin{aligned} \|\partial_x \phi\|_{L^1_x}(T) + \|\partial_t \phi\|_{L^1_x}(T) &\lesssim \|\partial_u \phi(u, u-2T)\|_{L^1_u} + \|\partial_v \phi(v+2T, v)\|_{L^1_v} \\ &\leq \|\partial_u \phi\|_{L^1_u L^\infty_v} + \|\partial_v \phi\|_{L^1_v L^\infty_u} \\ &\lesssim \|\phi\|_X . \end{aligned}$$

To finish the proof of (13) we need to bound the term $\|\phi(\cdot,T)\|_{L^\infty(\mathbf{R})}$ in the norm $\|\phi(\cdot,T)\|_{\mathcal{D}}$. We have

$$\int_v^u \int_{v'}^u \partial_u\partial_v\phi(u',v')du'dv' = \phi(u,u) - \phi(u,v) - \int_v^u \partial_v\phi(v',v')dv'. \tag{88}$$

Hence

$$\begin{aligned} \|\phi\|_{L^\infty_{u,v}} &\le \|\phi(x,0)\|_{L^\infty_x} + \|\partial_t\phi(x,0)\|_{L^1_x} + \|\partial_x\phi(x,0)\|_{L^1_x} + \|\partial_u\partial_v\phi\|_{L^1_{u,v}} \\ &\le \|\phi\|_X . \end{aligned} \tag{89}$$

To prove (18), it suffices to bound $\partial_u\partial_v(\phi\psi)$ in $L^1_{u,v}$, since $\mathcal{D}$ is easily seen to be an algebra. We compute using (87) and (89):

$$\begin{aligned} \|\partial_u\partial_v(\phi\psi)\|_{L^1_{u,v}} &\le \|\partial_u\partial_v\phi\cdot\psi\|_{L^1_{u,v}} + \|\partial_u\phi\cdot\partial_v\psi\|_{L^1_{u,v}} \\ &\quad + \|\partial_v\phi\cdot\partial_u\psi\|_{L^1_{u,v}} + \|\partial_u\partial_v\psi\cdot\phi\|_{L^1_{u,v}} \\ &\le \|\partial_u\partial_v\phi\|_{L^1_{u,v}}\|\psi\|_{L^\infty_{u,v}} + \|\partial_u\phi\|_{L^1_uL^\infty_v}\|\partial_v\psi\|_{L^1_vL^\infty_u} \\ &\quad + \|\partial_v\phi\|_{L^1_vL^\infty_u}\|\partial_u\psi\|_{L^1_uL^\infty_v} + \|\partial_u\partial_v\psi\|_{L^1_{u,v}}\|\phi\|_{L^\infty_{u,v}} \\ &\lesssim \|\phi\|_X\|\psi\|_X \end{aligned}$$

It remains to prove (19). We compute:

$$\begin{aligned} \left\|\Box^{-1}(\phi\,\Box\,\psi)\right\|_X &= \left\|\partial_u\partial_v\left(\Box^{-1}(\phi\,\Box\,\psi)\right)\right\|_{L^1_{u,v}} \\ &= \|\phi\partial_u\partial_v\psi\|_{L^1_{u,v}} \\ &\le \|\phi\|_{L^\infty_{u,v}}\|\partial_u\partial_v\psi\|_{L^1_{u,v}} \\ &\le \|\phi\|_X\|\psi\|_X . \end{aligned}$$

This concludes the proof of global well-posedness for data which is small in $\mathcal{D}$ and whose image lies in a single coordinate chart. The following elementary lemma allows us to localize large data. Let $\chi(x)$ be a smooth bump function supported on $[-2,2]$ with $\chi = 1$ on $[-1,1]$.

Lemma 9.2. *Given $\epsilon > 0$ and data $(f,g) \in L^{1,1}$ there exists $\delta = \delta(f,g,\epsilon)$ so that for all $x_0 \in \boldsymbol{R}$, $\chi(\frac{x-x_0}{\delta})f$ takes values in a single coordinate chart of $\mathcal{M}$ and*

$$\|\chi(\frac{x-x_0}{\delta})(f,g)\|_{\mathcal{D}} \le \epsilon. \tag{90}$$

Proof Fix $x_0 \in \mathbf{R}$, and pick δ so that

$$\int_{|x-x_0|\le\delta} |\partial_x f|\,dx + \int_{|x-x_0|\le\delta} |g|\,dx < \epsilon \tag{91}$$

for all $x_0 \in \mathbf{R}$. We may assume a coordinate chart on $\mathcal{M}$ around $f(x_0)$ is centered at 0. Then together with (91), the fundamental theorem of calculus gives

$$\|\chi(\frac{x-x_0}{\delta})f\|_{L^\infty_x} + \|\chi(\frac{x-x_0}{\delta})g\|_{L^1_x} \leq \epsilon.$$

Finally,

$$\begin{aligned}\|\frac{d}{dx}(\chi(\frac{x-x_0}{\delta})f)\|_{L^1_x} &\lesssim \|\frac{1}{\delta}f\|_{L^1(|x-x_0|\leq\delta)} + \|f'\|_{L^1(|x-x_0|\leq\delta)} \\ &\leq \|f\|_{L^\infty(|x-x_0|\leq\delta)} + \epsilon \\ &\lesssim \epsilon\end{aligned}$$

■

Together with the small data argument preceeding it, Lemma (9.2) and finite speed of propagation give local well-posedness for large $L^{1,1}$ data, but with a time of existence depending upon how concentrated the data is in $L^{1,1}$.

9.3. Nonconcentration of $L^{1,1}$ norm. We now turn to the question of global well-posedness for large data. Suppose for contradiction that there exists large $L^{1,1}$ data f, g for which a solution ϕ could only be continued in X up to a maximal time of existence $0 < T^* < \infty$. By finite speed of propagation we may assume that ϕ is compactly supported.

From the small-data well-posedness theory (which in particular implies uniqueness), finite speed of propagation, and a Lemma 9.2 this implies the existence of a point x_0 such that the solution concentrates on intervals near x_0 on every coordinate chart[5]:

$$\limsup_{\tau\to 0} \|\phi(T^* - \tau)\|_{\mathcal{D}(x_0-4\tau, x_0+4\tau)} \geq \delta > 0.$$

By translation invariance we may take $x_0 = 0$. We may pick our coordinate charts at each time $T^* - \tau$ so that $\phi(0, T^* - \tau) = 0$. By the fundamental theorem of calculus, the concentration thus becomes

$$\limsup_{\tau\to 0} \|\phi_x(T^* - \tau)\|_{L^1(-4\tau,4\tau)} + \|\phi_t(T^* - \tau)\|_{L^1(-4\tau,4\tau)} \gtrsim \delta > 0.$$

We can rewrite these derivatives in terms of u, v derivatives to obtain

$$\limsup_{\tau\to 0} \|\phi_u(T^* - \tau)\|_{L^1(-4\tau,4\tau)} + \|\phi_v(T^* - \tau)\|_{L^1(-4\tau,4\tau)} \gtrsim \delta > 0.$$

Thus in order to obtain a contradiction we need only show that the L^1 norms of $|\phi_u|_h$ and $|\phi_v|_h$ do not concentrate.

But this is an immediate consequence of Lemma 4.1. Indeed, as the data is in $L^{1,1}$, the quantities $|\phi_u|_h$ and $|\phi_v|_h$ are travelling waves of L^1 functions and therefore do not concentrate.

[5]Note that we are using the hypothesis that the Christoffel symbols are uniformly analytic to make these estimates independent of the choice of chart.

9.4. Conformal compactification and scattering. Let ϕ denote a global $L^{1,1}$ solution to the wave map equation (4), and let ϕ^+, ϕ^- denote global $L^{1,1}$ solutions to the free wave equation. Note that these solutions are continuous, since the solution space X used earlier embeds into the space of continuous functions. To show scattering and asymptotic completeness we have to prove the following two claims:

- For any ϕ, there exists a ϕ^+ such that $||\phi(T) - \phi^+(T)||_{L^{1,1}} \to 0$ as $T \to \infty$.
- For any ϕ^-, there exists a ϕ such that $||\phi(T) - \phi^-(T)||_{L^{1,1}} \to 0$ as $T \to -\infty$.

In $\mathbf{R}^{1+1}$, the conformal compactification transformation (see [5, 22]) takes the form

$$(P\phi)(U, V) = \phi(\tan U, \tan V)$$

which takes functions on $\mathbf{R}^{1+1}$ to functions on the Einstein diamond $\{|U|, |V| < \pi/2\}$. Since the wave map equation and the free wave equation are both conformally invariant in one dimension, the function $P\phi$ is also a solution to (4), and $P\phi^+$, $P\phi^-$ are solutions to the free wave equation.

A quick computation shows that the $L^{1,1}$ norm of $(P\phi)(0)$ is equal to the $L^{1,1}$ norm of $\phi(0)$ since the Jacobian factor and chain rule factor cancel. Thus by the global well-posedness theory just proved, $(P\phi)$ extends to a solution Φ to (4) on all of $\mathbf{R}^{1+1}$, where we may continuously extend the initial data so that the initial position $\Phi(0)$ is constant and the initial velocity $\Phi_t(0)$ is zero on the intervals $(-\infty, -\pi/2)$ and $(\pi/2, \infty)$.

By Lemma 4.2, we see that Φ is exactly equal to an $L^{1,1}$ solution Φ^+ (resp. Φ^-) to the free wave equation for $U \geq \pi/2$ or $V \geq \pi/2$ (resp. $U \leq -\pi/2$ or $V \leq -\pi/2$). See Figure 1. We may of course extend $\psi^\pm$ to be $L^{1,1}$ solutions to the free wave equation on all of $\mathbf{R}^{1+1}$. This gives a well-defined map from Φ to $\Phi^\pm$; the corresponding inverse map also exists by the same reasoning.

We now define the scattering maps $W_\pm : \phi \to \phi_\pm$ by defining $\phi^\pm = L^{-1}\Phi^\pm$ on the Einstein diamond; it is easy to see from the above discussion that this map is well-defined and invertible, and that $\phi^\pm$ are global $L^{1,1}$ solutions to the free wave equation. To complete the proof of scattering it suffices to show that

$$||\phi(T) - \phi^\pm(T)||_{L^{1,1}} \to 0$$

as $T \to \pm\infty$.

By time reversal symmetry it suffices to do this for ϕ^+. Since by construction $\phi^+(T)$ and $\phi(T)$ agree at the boundary of the Einstein diamond (i.e. when $x \to \pm\infty$), it suffices by the fundamental theorem of calculus to show that

$$||(\phi - \phi^+)_u(T)||_{L^1_x} + ||(\phi - \phi^+)_v(T)||_{L^1_x} \to 0$$

as $T \to \infty$. We show this only for the first term, as the second is analogous. We have

$$\|(\phi - \phi^+)_u(T)\|_{L^1_x} = \int |(\phi - \phi^+)_u(u, u - 2T)| \, du. \tag{92}$$

By changing to the U and V coordinates, (92) is

$$\int_{-\pi/2}^{\pi/2} |(\Phi - \Phi^+)_U(U, \tan^{-1}(\tan(U) - 2T)| \, dU.$$

By the fundamental theorem of calculus and the fact that $\Phi - \Phi^+$ vanishes at the upper boundary of the Einstein diamond, this is majorized by

$$\int\int_{|U|,|V| \le \pi/2, V > \tan^{-1}(\tan(U) - 2T)} |(\Phi - \Phi^+)_{UV}(U, V)| \, dU dV.$$

By the monotone convergence theorem, this will go to zero as $T \to 0$ provided that

$$\int\int_{|U|,|V| \le \pi/2} |(\Phi - \Phi^+)_{UV}(U, V)| \, dU dV < \infty.$$

But since Φ obeys (9), $|\Phi_{UV}| \lesssim |\Phi_U||\Phi_V|$. Since Φ^+ is a free solution, $\Phi^+_{UV} = 0$ and so our integral is majorized by

$$\int\int_{|U|,|V| \le \pi/2} |\Phi_U|_h |\Phi_V|_h \, dU dV,$$

But by Lemma 4.1 this is majorized by the square of the $L^{1,1}$ norm of the data of Φ, which is finite.

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