18.317 Combinatorics, Probability, and Computations on Groups October 29, 2001

Lecture 0

Lecturer: Igor Pak *Scribe: M. Alekhnovich*

Mixing time & long paths in graphs

Let Γ be a Cayley graph of group G: $\Gamma = Caley(G, S)$, $|S| = D$, $|\Gamma| = n$. Recall the following notation from the previous lectures:

$\{x_t^v\}$ - random walk on Γ starting at $v \in \Gamma$.

$Q_v^t(g) = Pr(x_t^v = g)$

$\tau_4 = \min\{t : ||Q_v^t - u|| < \frac{1}{4}\ \forall v \in \Gamma\}$

In this lecture we will prove the following result:

Theorem 1 *Let Γ be a D-regular graph with $\tau_4 \leq k$ s.t. $D > 8k^2$. Then Γ contains a (self-avoiding) path of length greater than $\frac{|\Gamma|}{16k}$.*

This theorem holds for an arbitrary D-regular graph as well but in this lecture we will confine ourselves only to Cayley graphs.

Definition 2 *Let* $\alpha_v(A) = Pr\left(x_t^v \notin A,\ \forall\ t \in [1..k]\right),\ \ \beta_v(A) = 1 - \alpha_v(A)$.

Clearly, $\beta_v(A) \leq \sum_{t=1}^k Pr(x_t^v \in A) = \sum_{t=1}^k Q_v^t(A)$.

Proposition 3 *FOr any $A \subseteq G$, we have:* $\sum_{v\in\Gamma} \beta_v(A) \leq k|A|$.

Proof:

$$\sum_{v\in\Gamma} \beta_v(A) \leq \sum_{v\in\Gamma}\sum_{t=1}^{k} Q_v^t(A) = \sum_{t=1}^{k}\sum_{z\in A}\sum_{v\in\Gamma} Q_v^t(z) = k|A|$$

■

Lemma 4 *For any $A \subseteq G$ and any $\beta > 0$ we have:*

$$\#\{v \in \Gamma : \beta_v(A) < \beta\} \geq n - \frac{k|A|}{\beta}$$

Proof: Let $m = \#\{v \in \Gamma : \beta_v(A) \geq \beta\}$. Since $m\beta \leq \sum_{v\in\Gamma} \beta_v(A) \leq k|A|$ we have $m \leq \frac{k|A|}{\beta}$ and

$$\#\{v \in \Gamma : \beta_v(A) < \beta\} = n - m \geq n - \frac{k|A|}{\beta}.$$

■

Lemma 5 *Let*

$$\rho := \min_{\substack{|B| = k \\ v \notin B}} Pr(x_1^v, ..., x_k^v \notin B \wedge [\textit{all } x_i^v \textit{ are distinct}])$$

$$\delta := 1 - \rho$$

Then $\rho > 1 - \frac{2k^2}{D}$; $\delta < \frac{2k^2}{D}$.

Proof:

$$\rho \geq \left(1 - \frac{|B|+1}{D}\right) \cdot ... \cdot \left(1 - \frac{|B|+k}{D}\right) > \left(1 - \frac{2k}{D}\right)^k > 1 - \frac{2k^2}{D}$$

■

Lemma 6 *Let*

$$\xi_v(A) = \min_{\substack{B \subset G, |B| = k \\ v \notin A \cup B}} \alpha_v(A \cup B)$$

$$Z(A, \beta) = \{v \in \Gamma : \xi_v(A) > 1 - \beta - \frac{2k^2}{D}\}$$

Then $\forall A \subset \Gamma \ |Z(A,\beta)| > n - \frac{k|A|}{\beta}$.

Proof:

$$\xi_v(A) \geq 1 - \beta_v(A) - \delta >^{(Lemma\ 5)} 1 - \beta - \frac{2k^2}{D}$$

if $\beta_v(A) < \beta$. Hence, by Lemma 4

$$\xi_v(A) > 1 - \beta - \frac{2k^2}{D}$$

for more than $n - \frac{k|A|}{\beta}$ points. ■

Definition 7 (failure probability) *Assume that* v *and* A *are given. The random walk starting in* v *is* successful *iff all* x_i $i \in [1..k]$ *are distinct and don't belong to* A. *Let the* failure probability *be*

$$fp(v, A) = 1 - Pr(x_1^v, ..., x_k^v \notin A \wedge [\textit{all } x_i^v \textit{ are distinct}]).$$

Clearly $||A_v^k - U|| \geq \frac{|Z|}{|\Gamma|} - Q_v^k(Z)$. By the statement of the theorem $||A_v^k - U|| < \frac{1}{4}$. Thus $Q_v^k(Z) > \frac{|Z|}{n} - \frac{1}{4}$ for any $Z \subseteq \Gamma$.

Proof (of the Theorem):

Fix $\gamma := \frac{1}{2}$; $\beta := \gamma - \frac{2k^2}{D} \leq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. Let A be some set s.t. $\exists v \notin A$ for which failure probability is less or equal than γ. Let us denote

$$p = Pr(x_1^v, ..., x_k^v \notin A \wedge [\text{all } x_i \text{ are distinct}] \wedge fp(x_k^v, A \cup \{x_1^v, ..., x_{k-1}^v\}) < \gamma).$$

Then

$$p \geq Pr(x_k^v \in Z(A,\beta)) - \gamma \geq \left(\frac{|Z(A,\beta)|}{|\Gamma|} - \frac{1}{4}\right) - \gamma > 1 - \frac{k|A|}{\beta n} - \frac{1}{4} - \frac{1}{2} = \frac{1}{4} - \frac{k|A|}{\frac{1}{4}n},$$

which is greater than 0 as long as $|A| < \frac{n}{16k}$.

Let us set $A = \emptyset$ at the start. By Lemma 5, $\exists v \; fp(v, A) < \gamma$. Since $p > 0$ we can choose at least one self-avoiding path with "good" end-point. We can continue the process of constructing the path as long as $p > 0$ which is equivalent to $|A| < \frac{n}{16k}$. Thus there exists a path of length greater or equal $\frac{n}{16k}$. ■

At the end, I would like to mention about the following conjecture of Lovász:

Every Cayley graph has a Hamiltonian path.

The main Theorem shows that it is not very easy to produce a counterexample to this conjecture in class of groups with small mixing time.

Also, we proved Theorem 1 in a non-constructive way, but in fact using these ideas one can find a long path in polynomial time. For this, it is sufficient to test $x_k^v \in Z$ efficiently.