Homework 8 Solutions

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Problem 1: Consider the data in the following table

$$\begin{array}{ccccc} x_i & 1 & 2 & 4 & 5 \\ y_i & 2 & 3 & 5 & 6 \end{array} \tag{1}$$

We want to find the least squares polynomial of degree 2

$$P(x) = a_0 + a_1 x + a_2 x^2 \tag{2}$$

for the data in the following ways.

(a) Write the normal equations and solve them analytically.

(b) Write a linear least squares problem $\min_{u \in \mathbb{R}^3} E = ||Au - b||_2$ for the data, where $u = (a_0, a_1, a_2)^T$. Solve this linear least squares problem analytically with QR decomposition. Compute the error E.

(c) Write a program for part (b) to verify your solutions.

Solution:
Note that in the formulation above,

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix}, \quad u = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}.$$

Plugging in the values of x_i and y_i into $Au = b$, we obtain

$$Au = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 6 \end{bmatrix} = b.$$

It is better to approach part (b) first. From part (b), we see that the polynomial in (2) is a line.

a) Note: The solution to the normal equations described in this part is a simplification. Consult pages 484-486 of textbook for a proper procedure of writing out normal equations.
We have a linear least squares problem

$$\min_{u \in \mathbb{R}^3} E = ||Au - b||_2.$$

The normal equations associated to this problem are

$$A^T A u = A^T b.$$

For the problem above, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \\ 1 & 4 & 16 & 25 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix} u = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \\ 1 & 4 & 16 & 25 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \\ 6 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 12 & 46 \\ 12 & 46 & 198 \\ 46 & 198 & 898 \end{bmatrix} u = \begin{bmatrix} 16 \\ 58 \\ 244 \end{bmatrix}.$$

Thus,

$$u = (A^T A)^{-1} A^T b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \quad \checkmark$$

Thus the polynomial in (2) is written as

$$P(x) = 1 + x.$$

b) We now use Gram-Schmidt process to compute the QR decomposition of A. Below, a_1, a_2, a_3 are the columns of matrix A, not the elements of vector u. First compute the columns of Q:

$$\tilde{q}_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$q_1 = \frac{\tilde{q}_1}{||\tilde{q}_1||_2} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix},$$

$$\tilde{q}_2 = a_2 - (q_1^T a_2) q_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix} - 6 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix},$$

$$q_2 = \frac{\tilde{q}_2}{||\tilde{q}_2||_2} = \frac{1}{\sqrt{10}} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{bmatrix},$$

$$\tilde{q}_3 = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2 = \begin{bmatrix} 1 \\ 4 \\ 16 \\ 25 \end{bmatrix} - 23 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} - \frac{60}{\sqrt{10}} \begin{bmatrix} -\frac{2}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix},$$

$$q_3 = \frac{\tilde{q}_3}{||\tilde{q}_3||_2} = \frac{1}{3} \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

We now compute the elements of R. The elements of R on the diagonal are:

$$\begin{aligned} r_{11} &= ||\tilde{q_1}||_2 = 2, \\ r_{22} &= ||\tilde{q_2}||_2 = \sqrt{10}, \\ r_{33} &= ||\tilde{q_3}||_2 = 3. \end{aligned}$$

The elements of R in the upper triangular part are:

$$\begin{aligned} r_{12} = q_1^T a_2 &= 6, \\ r_{13} = q_1^T a_3 &= 23, \\ r_{23} = q_2^T a_3 &= \frac{60}{\sqrt{10}}. \end{aligned}$$

Thus,

$$\begin{aligned} Q &= \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{2}{\sqrt{10}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{10}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{10}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{2}{\sqrt{10}} & \frac{1}{2} \end{bmatrix}, \\ R &= \begin{bmatrix} ||\tilde{q_1}||_2 & q_1^T a_2 & q_1^T a_3 \\ 0 & ||\tilde{q_2}||_2 & q_2^T a_3 \\ 0 & 0 & ||\tilde{q_3}||_2 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 23 \\ 0 & \sqrt{10} & \frac{60}{\sqrt{10}} \\ 0 & 0 & 3 \end{bmatrix}. \end{aligned}$$

Q is 4×3 matrix with orthonormal columns. We can assume $[Q, q_4]$ is a square orthogonal matrix for a vector q_4. This vector can be obtained by choosing any nonzero vector a_4 such that a_1, a_2, a_3, a_4 are linearly independent, and then continuing the above Gram-Schmidt process.)
Thus, we have

$$\begin{aligned} ||Au - b||_2^2 &= ||QRu - b||_2^2 \\ &= \left|\left|[Q, q_4]^T (QRu - b)\right|\right|_2^2 \\ &= \left|\left| \begin{pmatrix} Ru - Q^T b \\ -q_4^T b \end{pmatrix} \right|\right|_2^2 \\ &= ||Ru - Q^T b||_2^2 + ||q_4^T b||_2^2, \end{aligned}$$

which is minimized with

$$u^* = R^{-1} Q^T b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \quad \checkmark$$

Thus the polynomial in (2) is written as

$$P(x) = 1 + x,$$

which agrees with part (a).
The error is

$$||Au^* - b||_2 = \left|\left| \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 5 \\ 6 \end{bmatrix} \right|\right|_2 = \left|\left| \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right|\right|_2 = 0.$$

That is, the error to this minimization is 0. But this is not surprising, since this polynomial describes the relationship between x_i and y_i exactly.

c) Implementing the QR factorization and running it to decompose A, we obtain the following matrices:

$$
\begin{aligned}
Q &= \begin{bmatrix} 0.5 & -0.6325 & 0.5 \\ 0.5 & -0.3162 & -0.5 \\ 0.5 & 0.3162 & -0.5 \\ 0.5 & 0.6325 & 0.5 \end{bmatrix}, \\
R &= \begin{bmatrix} 2.0 & 6.0 & 23.0 \\ 0 & 3.1623 & 18.9737 \\ 0 & 0 & 3.0 \end{bmatrix}.
\end{aligned}
$$

This agrees with our computation in part (b). The rest of calculations above can also be verified.

Problem 2: For the data in problem 1, construct the least squares approximation of the form be^{ax}, and compute the error.

Solution: Instead directly minimizing the least squares error to

$$y = be^{ax},$$

we can convert this problem to

$$\log y = \log b + ax. \tag{3}$$

The resulting system is

$$Au = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} \log b \\ a \end{bmatrix} = \begin{bmatrix} \log y_1 \\ \log y_2 \\ \log y_3 \\ \log y_4 \end{bmatrix} = b.$$

For the data given in (1), we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \log b \\ a \end{bmatrix} = \begin{bmatrix} \log 2 \\ \log 3 \\ \log 5 \\ \log 6 \end{bmatrix}.$$

QR factorization of A gives

$$Q = \begin{bmatrix} q_1 \; q_2 \; q_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{2}{\sqrt{10}} \\ \frac{1}{2} & -\frac{1}{\sqrt{10}} \\ \frac{1}{2} & \frac{1}{\sqrt{10}} \\ \frac{1}{2} & \frac{2}{\sqrt{10}} \end{bmatrix},$$

$$R = \begin{bmatrix} ||\tilde{q}_1||_2 & q_1^T a_2 \\ 0 & ||\tilde{q}_2||_2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 0 & \sqrt{10} \end{bmatrix}.$$

Thus, (don't confuse two different entities denoted as b below)

$$\underbrace{\begin{bmatrix} \log b \\ a \end{bmatrix}}_{u^*} = R^{-1}Q^T b = \begin{bmatrix} 0.4858 \\ 0.2708 \end{bmatrix}.$$

Hence, $\log b = 0.4858, \quad a = 0.2708$, or
$b = e^{0.4858}$, $a = 0.2708$.

The error is

$$||Au^* - b||_2 = \left\| \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 0.4858 \\ 0.2708 \end{bmatrix} - \begin{bmatrix} \log 2 \\ \log 3 \\ \log 5 \\ \log 6 \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} 0.063482 \\ -0.071178 \\ -0.040394 \\ 0.048090 \end{bmatrix} \right\|_2 = 0.114195.$$

Note that this error corresponds to the modified problem (3).

Problem 3: In the Gram-Schmidt QR process for $A = [a_1, a_2, \cdots, a_n] \in \mathbb{R}^{m \times n}$, assume we have obtained $j-1\,(j<n)$ orthogonal columns $q_1, q_2, \cdots, q_{j-1}$. During step j we have the following formulas

$$\begin{aligned} \tilde{q}_j &= a_i - (q_1^T a_j)q_1 - (q_2^T a_j)q_2 - \cdots - (q_{j-1}^T a_j)q_{j-1}, \qquad r_{ij} = q_i^T a_j, \\ q_j &= \frac{\tilde{q}_j}{||\tilde{q}_j||_2}. \end{aligned} \tag{4}$$

Show that $||\tilde{q}_j||_2 = q_j^T a_j$.

Solution: Hint: One way to approach this problem is to observe that

$$q_j = \frac{P_j a_j}{||P_j a_j||_2},$$

where P_j denotes an orthogonal projector.

Also, note note a simple fact:

$$\tilde{q}_1 = a_1, \qquad q_1 = \frac{\tilde{q}_1}{||\tilde{q}_1||_2},$$

we have $||\tilde{q}_1||_2 = q_1^T a_1$.

Section 8.5, Problem 4: Find the general continuous least squares trigonometric polynomial $S_n(x)$ for $f(x) = e^x$ on $[-\pi, \pi]$.

Solution: The continuous least squares approximation $S_n(x)$ is in the form

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx, \end{aligned}$$

with $k = 1, 2, \ldots$. Thus,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi} e^x \Big|_{x=-\pi}^{x=\pi} = \frac{1}{\pi}(e^{\pi} - e^{-\pi}), \quad \checkmark \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos kx\,dx, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin kx\,dx. \end{aligned}$$

To determine a_k and b_k we need to evaluate $\int e^x \cos kx\,dx$ and $\int e^x \sin kx\,dx$, respectively. Integrating the first integral by parts twice, we have

$$\int e^x \cos kx\,dx = e^x \frac{\sin kx}{k} - \int e^x \frac{\sin kx}{k}\,dx = e^x \frac{\sin kx}{k} + e^x \frac{\cos kx}{k^2} - \int e^x \frac{\cos kx}{k^2}\,dx.$$

Multiplying both sides by k^2, we obtain

$$k^2 \int e^x \cos kx\,dx = k e^x \sin kx + e^x \cos kx - \int e^x \cos kx\,dx,$$

or

$$\int e^x \cos kx\,dx = \frac{k e^x \sin kx}{k^2+1} + \frac{e^x \cos kx}{k^2+1}.$$

Thus,

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos kx\,dx \\ &= \frac{1}{\pi(k^2+1)}\Big[k e^x \sin kx + e^x \cos kx\Big]_{x=-\pi}^{x=\pi} \\ &= \frac{(-1)^k}{\pi(k^2+1)}\left(e^{\pi} - e^{-\pi}\right). \quad \checkmark \end{aligned}$$

Similarly,

$$\int e^x \sin kx\,dx = -\frac{k e^x \cos kx}{k^2+1} + \frac{e^x \sin kx}{k^2+1}.$$

Thus,

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin kx\,dx \\
&= \frac{1}{\pi(k^2+1)}\Big[-ke^x\cos kx + e^x \sin kx\Big]_{x=-\pi}^{x=\pi} \\
&= -\frac{k\cdot(-1)^k}{\pi(k^2+1)}\big(e^{\pi}-e^{-\pi}\big). \quad \checkmark
\end{aligned}$$

Thus, the continuous least squares trigonometric polynomial is

$$\begin{aligned}
S_n(x) &= \frac{1}{2\pi}(e^{\pi}-e^{-\pi}) + \sum_{k=1}^{n}\left[\frac{(-1)^k}{\pi(k^2+1)}\big(e^{\pi}-e^{-\pi}\big)\cos kx - \frac{k\cdot(-1)^k}{\pi(k^2+1)}\big(e^{\pi}-e^{-\pi}\big)\sin kx\right] \\
&= \frac{e^{\pi}-e^{-\pi}}{\pi}\left[\frac{1}{2} + \sum_{k=1}^{n}\frac{(-1)^k}{(k^2+1)}(\cos kx - k\sin kx)\right]. \quad \checkmark
\end{aligned}$$

Section 8.5, Problem 6: Find the general continuous least squares trigonometric polynomial $S_n(x)$ for

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x < 0, \\ 1, & \text{if } 0 \le x \le \pi. \end{cases}$$

Solution: The continuous least squares approximation $S_n(x)$ is in the form

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx),$$

where, for $k = 1, 2, \ldots,$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = -1 + 1 = 0, \quad \checkmark \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -1\cdot\cos kx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\cos kx\,dx, \\
&= -\frac{1}{\pi}\frac{\sin kx}{k}\Big|_{x=-\pi}^{x=0} + \frac{1}{\pi}\frac{\sin kx}{k}\Big|_{x=0}^{x=\pi} = 0 + 0 = 0, \quad \checkmark \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -1\cdot\sin kx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\sin kx\,dx, \\
&= \frac{1}{\pi}\frac{\cos kx}{k}\Big|_{x=-\pi}^{x=0} - \frac{1}{\pi}\frac{\cos kx}{k}\Big|_{x=0}^{x=\pi} \\
&= \frac{1}{\pi}\left(\frac{1-(-1)^k}{k}\right) - \frac{1}{\pi}\left(\frac{(-1)^k - 1}{k}\right) \\
&= \frac{1}{\pi}\left(\frac{1-(-1)^k}{k}\right) + \frac{1}{\pi}\left(\frac{1-(-1)^k}{k}\right) \\
&= \frac{2}{\pi}\left(\frac{1-(-1)^k}{k}\right). \quad \checkmark
\end{aligned}$$

Thus,

$$S_n(x) = \frac{2}{\pi}\sum_{k=1}^{n}\left(\frac{1-(-1)^k}{k}\right)\sin kx. \quad \checkmark$$

Section 8.5, Problem 7(a): Determine the discrete least squares trigonometric polynomial $S_n(x)$ for $f(x) = \cos 2x$, using $m = 4$, $n = 2$, on the interval $[-\pi, \pi]$.

Solution: We use the notation of Theorem 8.13, which states that the constants in the summation

$$S_n(x) = \frac{a_0}{2} + a_n \cos nx + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx),$$

are

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos kx_j, \qquad \text{for each } k = 0, 1, \ldots, n,$$

and

$$b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin kx_j, \qquad \text{for each } k = 1, 2, \ldots, n-1.$$

For our problem, we will find the discrete least squares polynomial of degree $n = 2$, $S_2(x)$. For $m = 4$, the nodes are

$$x_j = -\pi + \frac{j}{m}\pi \quad \text{and} \quad y_j = f(x_j) = \cos 2x_j, \quad \text{for } \; j = 0, 1, \ldots, 7. \;\; (2m-1 = 7)$$

The trigonometric polynomial is

$$S_2(x) = \frac{a_0}{2} + a_2 \cos 2x + (a_1 \cos x + b_1 \sin x), \tag{5}$$

where

$$\begin{aligned} a_k &= \frac{1}{4} \sum_{j=0}^{7} y_j \cos kx_j = \frac{1}{4} \sum_{j=0}^{7} \cos 2x_j \cos kx_j, \qquad \text{for each } k = 0, 1, 2. \;\; (n = 2), \\ b_k &= \frac{1}{4} \sum_{j=0}^{7} y_j \sin kx_j = \frac{1}{4} \sum_{j=0}^{7} \cos 2x_j \sin kx_j, \qquad \text{for } k = 1, \end{aligned}$$

and the coefficients are

$$\begin{aligned} a_0 &= \frac{1}{4} \sum_{j=0}^{7} \cos \left[2\Big(-\pi + \frac{j}{4}\pi \Big) \right] \cos \left[0 \cdot \Big(-\pi + \frac{j}{4}\pi \Big) \right] \\ &= \frac{1}{4} \bigg(\cos(-2\pi) + \cos\Big(-\frac{3\pi}{2} \Big) + \cos(-\pi) + \cos\Big(-\frac{\pi}{2} \Big) + \cos 0 + \cos\frac{\pi}{2} + \cos\pi + \cos\frac{3\pi}{2} \bigg) \\ &= 0, \\ a_1 &= \frac{1}{4} \sum_{j=0}^{7} \cos \left[2\Big(-\pi + \frac{j}{4}\pi \Big) \right] \cos \left[1 \cdot \Big(-\pi + \frac{j}{4}\pi \Big) \right] \\ &= \frac{1}{4} \bigg(\cos(-2\pi)\cos(-\pi) + \cos\Big(-\frac{3\pi}{2} \Big) \cos\Big(-\frac{3\pi}{4} \Big) + \cos(-\pi) \cos\Big(-\frac{\pi}{2} \Big) \\ &\qquad + \cos\Big(-\frac{\pi}{2} \Big) \cos\Big(-\frac{\pi}{4} \Big) + \cos 0 \cos 0 + \cos\frac{\pi}{2} \cos\frac{\pi}{4} + \cos\pi \cos\frac{\pi}{2} + \cos\frac{3\pi}{2} \cos\frac{3\pi}{4} \bigg) \\ &= 0, \end{aligned}$$

$$
\begin{aligned}
a_2 &= \frac{1}{4}\sum_{j=0}^{7}\cos\left[2\Big(-\pi+\frac{j}{4}\pi\Big)\right]\cos\left[2\cdot\Big(-\pi+\frac{j}{4}\pi\Big)\right] \\
&= \frac{1}{4}\bigg(\cos(-2\pi)\cos(-2\pi)+\cos\Big(-\frac{3\pi}{2}\Big)\cos\Big(-\frac{3\pi}{2}\Big)+\cos(-\pi)\cos(-\pi) \\
&\qquad +\cos\Big(-\frac{\pi}{2}\Big)\cos\Big(-\frac{\pi}{2}\Big)+\cos 0\cos 0+\cos\frac{\pi}{2}\cos\frac{\pi}{2}+\cos\pi\cos\pi+\cos\frac{3\pi}{2}\cos\frac{3\pi}{2}\bigg) \\
&= 1, \\
b_1 &= \frac{1}{4}\sum_{j=0}^{7}\cos\left[2\Big(-\pi+\frac{j}{4}\pi\Big)\right]\sin\left[1\cdot\Big(-\pi+\frac{j}{4}\pi\Big)\right] \\
&= \frac{1}{4}\bigg(\cos(-2\pi)\sin(-\pi)+\cos\Big(-\frac{3\pi}{2}\Big)\sin\Big(-\frac{3\pi}{4}\Big)+\cos(-\pi)\sin\Big(-\frac{\pi}{2}\Big) \\
&\qquad +\cos\Big(-\frac{\pi}{2}\Big)\sin\Big(-\frac{\pi}{4}\Big)+\cos 0\sin 0+\cos\frac{\pi}{2}\sin\frac{\pi}{4}+\cos\pi\sin\frac{\pi}{2}+\cos\frac{3\pi}{2}\sin\frac{3\pi}{4}\bigg) \\
&= 0.
\end{aligned}
$$

Thus, the trigonometric least squares polynomial $S_2(x)$ defined in equation (5) is $S_2(x) = \cos 2x$. ✓

Section 8.5, Problem 12:
a) Determine the discrete least squares trigonometric polynomial $S_4(x)$, using $m = 16$, for $f(x) = x^2 \sin x$ on the interval $[0, 1]$.
b) Compute $\int_0^1 S_4(x)\,dx$.
c) Compare the integral in part (b) to $\int_0^1 x^2 \sin x\,dx$.

Solution: Note that here you have to make a choice how to define m and the values j can take.
To find the discrete least squares approximation $S_4(x)$ for the data $\{(x_j, y_j)\}_{j=0}^{15}$, where [1]

$$x_j = 0 + \frac{j}{16} \cdot 1 = \frac{j}{16} \quad \text{and} \quad y_j = f(x_j) = x_j^2 \sin x_j, \qquad j = 0, 1, \ldots, 15,$$

first requires a transformation from $[0, 1]$ to $[-\pi, \pi]$. This linear transformation is

$$z_j = \pi(2x_j - 1).$$

Thus, given z_j we can transform to x_j with $x_j = \dfrac{z_j}{2\pi} + \dfrac{1}{2}$ which maps $[-\pi, \pi]$ back to $[0, 1]$.
The transformed data is of the form

$$\left\{\left(z_j, f\left(\frac{z_j}{2\pi} + \frac{1}{2}\right)\right)\right\}_{j=0}^{15}.$$

(This transformation distributes the data of f defined on $[0, 1]$ onto $[-\pi, \pi]$.)
The least squares trigonometric polynomial is, consequently,

$$S_4(z) = \frac{a_0}{2} + a_4 \cos 4z + \sum_{k=1}^{3} (a_k \cos kz + b_k \sin kz),$$

where

$$a_k = \frac{1}{16} \sum_{j=0}^{15} f\left(\frac{z_j}{2\pi} + \frac{1}{2}\right) \cos kz_j, \qquad \text{for each } k = 0, 1, 2, 3, 4, \quad (n = 4)$$

and

$$b_k = \frac{1}{16} \sum_{j=0}^{15} f\left(\frac{z_j}{2\pi} + \frac{1}{2}\right) \sin kz_j, \qquad \text{for each } k = 1, 2, 3.$$

Evaluating these sums produces the approximation $S_4(z)$, which can be converted back to the variable x.

[1] Note that $j = m - 1$, and not $2m - 1$, as in the book, since the problem is different.

Section 8.5, Problem 15: Show that the functions

$$\begin{array}{l} \phi_0(x) = \frac{1}{2}, \\ \phi_1(x) = \cos x, \ldots, \phi_n(x) = \cos nx, \\ \phi_{n+1}(x) = \sin x, \ldots, \phi_{2n-1}(x) = \sin(n-1)x \end{array} \tag{6}$$

are orthogonal on $[-\pi, \pi]$ with respect to $w(x) \equiv 1$.

Solution: $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is said to be an orthogonal set of functions for the interval $[a, b]$ with respect to the weight function w if

$$\int_a^b \phi_j(x)\phi_k(x)w(x)\,dx = \begin{cases} 0, & \text{when } j \neq k, \\ \alpha_k > 0, & \text{when } j = k. \end{cases}$$

Note that

$$\int_{-\pi}^{\pi} \cos jx \cos kx\,dx = \begin{cases} 0, & \text{when } j \neq k, \\ \pi, & \text{when } j = k. \end{cases}$$

$$\int_{-\pi}^{\pi} \sin jx \sin kx\,dx = \begin{cases} 0, & \text{when } j \neq k, \\ \pi, & \text{when } k = k. \end{cases}$$

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{1}{2} \cdot \frac{1}{2}\,dx &= \frac{\pi}{2}, \\ \int_{-\pi}^{\pi} \cos jx \sin kx\,dx &= 0, \quad \text{for all } j,\ k \\ \int_{-\pi}^{\pi} \frac{1}{2} \cos kx\,dx &= 0, \quad \text{for all } k \\ \int_{-\pi}^{\pi} \frac{1}{2} \sin kx\,dx &= 0, \quad \text{for all } k \end{aligned}$$

Thus, the functions in (6) are orthogonal on $[-\pi, \pi]$ with respect to $w(x) \equiv 1$.