Homework 3 Solutions

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Section 5.11, Problem 10: Show that the fourth-order Runge-Kutta method,

$$\begin{aligned}
k_1 &= hf\big(t_i, w_i\big),\\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right),\\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right),\\
k_4 &= hf\big(t_i + h, w_i + k_3\big),\\
w_{i+1} &= w_i + \frac{1}{6}\big(k_1 + 2k_2 + 2k_3 + k_4\big),
\end{aligned} \tag{1}$$

when applied to the differential equation $y' = \lambda y$, can be written in the form

$$w_{i+1} = \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right)w_i.$$

Solution: We have

$$\begin{aligned}
w_{i+1} &= w_i + \frac{1}{6}\big(k_1 + 2k_2 + 2k_3 + k_4\big)\\
&= w_i + \frac{1}{6}hf\big(t_i, w_i\big) + \frac{1}{3}hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}hf\big(t_i, w_i\big)\right)\\
&\quad + \frac{1}{3}hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}hf\big(t_i, w_i\big)\right)\right)\\
&\quad + \frac{1}{6}hf\left(t_i + h, w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}hf\big(t_i, w_i\big)\right)\right)\right).
\end{aligned}$$

Since $y' = \lambda y = f(t, y)$, we have

$$\begin{aligned}
w_{i+1} &= w_i + \frac{1}{6}h\lambda w_i + \frac{1}{3}h\lambda\left(w_i + \frac{1}{2}h\lambda w_i\right)\\
&\quad + \frac{1}{3}h\lambda\left(w_i + \frac{1}{2}h\lambda\left(w_i + \frac{1}{2}h\lambda w_i\right)\right)\\
&\quad + \frac{1}{6}h\lambda\left(w_i + h\lambda\left(w_i + \frac{1}{2}h\lambda\left(w_i + \frac{1}{2}h\lambda w_i\right)\right)\right)\\
&= w_i\left(1 + \frac{1}{6}h\lambda + \frac{1}{3}h\lambda + \frac{1}{6}(h\lambda)^2\right.\\
&\qquad + \frac{1}{3}h\lambda + \frac{1}{6}(h\lambda)^2 + \frac{1}{12}(h\lambda)^3\\
&\qquad \left. + \frac{1}{6}h\lambda + \frac{1}{6}(h\lambda)^2 + \frac{1}{12}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right)\\
&= \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right)w_i. \quad \checkmark
\end{aligned}$$

Section 5.11, Problem 15(a): Show that the Implicit Trapezoidal method

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + \frac{h}{2}\Big[f(t_{i+1}, w_{i+1}) + f(t_i, w_i)\Big], \end{aligned}$$

is A-stable.

Solution: The region R of absolute stability is $R = \{h\lambda \in \mathbb{C} \mid |Q(h\lambda)| < 1\}$, where $w_{i+1} = Q(h\lambda)w_i$. A numerical method is said to be A-stable if its region of stability R contains the entire left half-plane.

In other words, in order to show that the method is A-stable, we need to show that when it is applied to the scalar test equation $y' = \lambda y = f$, whose solutions tend to zero for $\lambda < 0$, all the solutions of the method also tend to zero for a fixed $h > 0$ as $i \to \infty$.

For the Implicit Trapezoidal method, we have

$$\begin{aligned} w_{i+1} &= w_i + \frac{h}{2}(\lambda w_{i+1} + \lambda w_i), \\ w_{i+1} - \frac{h\lambda}{2}w_{i+1} &= w_i + \frac{h\lambda}{2}w_i, \\ w_{i+1} &= \frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}w_i, \\ w_{i+1} &= \left(\frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}\right)^{n+1} w_0. \end{aligned}$$

Thus,

$$Q(h\lambda) = \frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}} = \frac{2+h\lambda}{2-h\lambda}.$$

Note that for $Re(h\lambda) < 0$, $|Q(h\lambda)| < 1$, and for $Re(h\lambda) > 0$, $|Q(h\lambda)| > 1$. Therefore, the region of absolute stability R for the Implicit Trapezoidal methods is the entire left half-plane, and hence, the method is A-stable.

Section 5.11, Problem 7(b): Solve the following stiff initial-value problem using the Trapezoidal Algorithm with $TOL = 10^{-5}$

$$\begin{aligned} y' &= -20(y-t)^2 + 2t, \qquad 0 \le t \le 1, \\ y(0) &= \frac{1}{3}, \end{aligned} \tag{2}$$

with $h = 0.1$. Compare the results with the actual solution $y(t) = t^2 + \frac{1}{3}e^{-20t}$.

Solution: Slightly modifying the code I posted on my homepage for the problem above and running it gives the following results:
$N = 10, h = 0.1, t = 1.0, w = 1.0488, y = 1.0000, error = 4.87754e-002.$

Section 5.6, Problem 6(a): THERE IS A TYPO IN THE BOOK. THE SOLUTION TO THE INITIAL VALUE PROBLEM DOES NOT MATCH THE ACTUAL SOLUTION. WE WILL BE USING A DIFFERENT ODE.

Use Adams Fourth-Order Predictor-Corrector algorithm of section 5.6 to approximate the solutions to the initial-value problem

$$\begin{aligned} &y' = t^2 - 2e^{-2t}, \quad 0 \le t \le 1, \\ &y(0) = 1, \end{aligned} \tag{3}$$

with $h = 0.1$. Compare the results with the actual solution $y(t) = \dfrac{t^3}{3} + e^{-2t}$.

Solution: For this problem, we compute starting values w_i, $i = 1, 2, 3$ using the fourth order Runge-Kutta method:

$$\begin{aligned} k_1 &= hf\big(t_i, w_i\big), \\ k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right), \\ k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right), \\ k_4 &= hf\big(t_i + h, w_i + k_3\big), \\ w_{i+1} &= w_i + \frac{1}{6}\big(k_1 + 2k_2 + 2k_3 + k_4\big). \end{aligned} \tag{4}$$

For $i = 4, 5, \ldots$ we use Adams Fourth-Order Predictor-Corrector method, which consists of the predictor Adams-Bashforth, and corrector Adams-Moulton techniques.
The fourth-order Adams-Bashforth technique, an explicit four-step method, is defined as:

$$w_{i+1} = w_i + \frac{h}{24}\big[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})\big]. \tag{5}$$

The fourth-order Adams-Moulton technique, an implicit three-step method, is defined as:

$$w_{i+1} = w_i + \frac{h}{24}\big[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\big]. \tag{6}$$

Running the Adams Fourth-Order Predictor-Corrector algorithm gives the following results at the final step:
$N = 10, h = 1.0000000e - 001, t = 1.00,$
$w = 4.6864787414e - 001, y = 4.6866861657e - 001, error = 2.0742429498e - 005.$

You can verify that the solutions obtained with the method are indeed satisfying the fourth order accuracy. Check this, for example, running the code with $h = 0.01$ and $h = 0.005$ and calculate the order of convergence using the formula from homework 2.

Section 5.6, Problem 12: Derive the Adams-Bashforth three-step explicit method

$$w_{i+1} = w_i + \frac{h}{12}\big[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})\big] \tag{7}$$

by the following method. Set

$$y(t_{i+1}) = y(t_i) + ahf(t_i, y(t_i)) + bhf(t_{i-1}, y(t_{i-1})) + chf(t_{i-2}, y(t_{i-2})). \tag{8}$$

Expand $y(t_{i+1})$, $f(t_{i-2}, y(t_{i-2}))$, and $f(t_{i-1}, y(t_{i-1}))$ in Taylor series about $(t_i, y(t_i))$, and equate the coefficients of h, h^2, and h^3 to obtain a, b, and c.

Solution: Since $y'(t_i) = f(t_i, y(t_i))$, we can write equation (8) as

$$y(t_{i+1}) = y(t_i) + ahy'(t_i) + bhy'(t_{i-1}) + chy'(t_{i-2}). \tag{9}$$

Expanding both sides of (9) in Taylor series about t_i, we obtain

$$\begin{aligned}
&y(t_i) + hy'(t_i) + \tfrac{1}{2}h^2y''(t_i) + \tfrac{1}{6}h^3y'''(t_i) + O(h^4) \\
&= y(t_i) + ahy'(t_i) + bh\Big(y'(t_i) - hy''(t_i) + \tfrac{1}{2}h^2y'''(t_i) + O(h^3)\Big) \\
&\qquad\qquad\qquad + ch\Big(y'(t_i) - 2hy''(t_i) + \tfrac{4}{2}h^2y'''(t_i) + O(h^3)\Big), \\
&= y(t_i) + \big(a + b + c\big)hy'(t_i) + \big(-b - 2c\big)h^2y''(t_i) + \big(\tfrac{1}{2}b + 2c\big)h^3y'''(t_i) + O(h^4).
\end{aligned}$$

Thus, equating the coefficients, we obtain

$$\begin{aligned}
1 &= a + b + c, \\
\frac{1}{2} &= -b - 2c, \\
\frac{1}{6} &= \frac{1}{2}b + 2c,
\end{aligned}$$

which gives $a = \dfrac{23}{12},\ b = -\dfrac{16}{12},\ c = \dfrac{5}{12}$. Plugging these into (8), we obtain

$$y(t_{i+1}) = y(t_i) + \frac{h}{12}\big[23f(t_i, y(t_i)) - 16f(t_{i-1}, y(t_{i-1})) + 5f(t_{i-2}, y(t_{i-2}))\big] + O(h^4),$$

or

$$w_{i+1} = w_i + \frac{h}{12}\big[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})\big].$$

The order of the local truncation for the Adams-Bashforth three-step explicit method is, therefore, $\tau(h) = O(h^3)$.