Homework 2 Solutions

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Section 5.3, Problem 1(b): Use Taylor's method of order two to approximate the solution for the following initial-value problem:

$$\begin{aligned} &y' = 1 + (t-y)^2, \qquad 2 \le t \le 3, \\ &y(2) = 1, \end{aligned} \tag{1}$$

with $h = 0.5$.

Solution: Let us first derive the Taylor's method or order two for general initial value problem

$$\begin{aligned} &y' = f(t,y), \qquad a \le t \le b, \\ &y(a) = \alpha. \end{aligned} \tag{2}$$

Expand the solution $y(t)$ in terms of its second Taylor polynomial about t_i and evaluate at t_{i+1}, we obtain

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \frac{h^3}{6} y'''(\xi_i), \tag{3}$$

for some $\xi_i \in (t_i, t_{i+1})$. Since we have $y'(t) = f(t, y(t))$, differentiating $y'(t)$ twice gives

$$\begin{aligned} y''(t) &= f'(t, y(t)), \\ y'''(t) &= f''(t, y(t)). \end{aligned}$$

Plugging these into (3) gives

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \frac{h^3}{6} f''(\xi_i, y(\xi_i)). \tag{4}$$

When h is small, the last term in (4) is small, and thus, we can ignore it. The Taylor's method of order two for general initial value problem (2) is therefore

$$\begin{aligned} &w_0 = \alpha, \\ &w_{i+1} = w_i + hf(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i). \end{aligned} \tag{5}$$

For the initial value problem (1), we have

$$\begin{aligned} f(t, y(t)) &= 1 + (t-y)^2, \\ f'(t, y(t)) &= \frac{d}{dt}(1 + (t-y)^2) = 2(t-y) \cdot (1 - y') = 2(t-y) \cdot (1 - (1 + (t-y)^2)) \\ &= -2(t-y)^3. \end{aligned}$$

Equation (5), with $t_i = 2 + 0.5i$, becomes

$$\begin{aligned} w_{i+1} &= w_i + h(1 + (t_i - w_i)^2) - \frac{h^2}{2}(2(t_i - w_i)^3) \\ &= w_i + 0.5(1 + (2 + 0.5i - w_i)^2) - 0.25(2 + 0.5i - w_i)^3. \end{aligned}$$

Hence, with $w_0 = 1$, we have

$$\begin{aligned}
w_1 &= w_0 + 0.5(1 + (2 - w_0)^2) - 0.25(2 - w_0)^3 = 1 + 0.5(1 + 1) - 0.25 = 1.75, \quad \checkmark \\
w_2 &= w_1 + 0.5(1 + (2 + 0.5 - w_1)^2) - 0.25(2 + 0.5 - w_1)^3 \\
&= 1.75 + 0.5(1 + (2 + 0.5 - 1.75)^2) - 0.25(2 + 0.5 - 1.75)^3 = 2.425781. \quad \checkmark
\end{aligned}$$

Section 5.3, Problem 1(c): Use Taylor's method of order two to approximate the solution for the following initial-value problem:

$$\begin{aligned}
&y' = 1 + \frac{y}{t}, \qquad 1 \le t \le 2, \\
&y(1) = 2,
\end{aligned} \tag{6}$$

with $h = 0.25$.

Solution: The Taylor's method of order two for general initial value problem (2) is given by equation (5). For the initial value problem (6), we have

$$\begin{aligned}
f(t, y(t)) &= 1 + \frac{y}{t}, \\
f'(t, y(t)) &= \frac{d}{dt}\Big(1 + \frac{y}{t}\Big) = \frac{y't - y}{t^2} = \frac{(1 + \frac{y}{t})t - y}{t^2} = \frac{1}{t}.
\end{aligned}$$

Equation (5), with $t_i = 1 + 0.25i$, becomes

$$\begin{aligned}
w_{i+1} &= w_i + h\Big(1 + \frac{w_i}{t_i}\Big) + \frac{h^2}{2}\Big(\frac{1}{t_i}\Big) \\
&= w_i + 0.25\Big(1 + \frac{w_i}{1 + 0.25i}\Big) + 0.03125\Big(\frac{1}{1 + 0.25i}\Big).
\end{aligned}$$

Hence, with $w_0 = 2$, we have

$$\begin{aligned}
w_1 &= w_0 + 0.25(1 + w_0) + 0.03125 = 2.78125, \quad \checkmark \\
w_2 &= w_1 + 0.25\Big(1 + \frac{w_1}{1.25}\Big) + 0.03125\Big(\frac{1}{1.25}\Big) = 3.6125, \quad \checkmark
\end{aligned}$$

and, similarly, calculate w_3 and w_4.

Problem 2: Assume that $y(t) = 4\cos t - 7(t-4)^3$ is the solution to some initial value problem on $-2 \le t \le 3$.

a) Calculate the local truncation error when Euler's method is applied to this problem. Derive an upper bound for the local truncation error at $t = 3$ if the stepsize is $h = 0.001$.

b) For the same function $y(t)$, and also for $h = 0.001$, find an upper bound for the global error for Euler's method at $t = 3$, if we assume that the Lipschitz constant is $L = 2$. You would likely get a bigger number than the local truncation error.

Solution:
a) Since $y(t) = 4\cos t - 7(t-4)^3$, we have $y'(t) = -4\sin t - 21(t-4)^2$ and $y''(t) = -4\cos t - 42(t-4)$.
Thus, $f(t, y) = y'(t) = -4\sin t - 21(t-4)^2$.
Euler's method

$$\begin{aligned} &w_0 = \alpha, \\ &w_{i+1} = w_i + hf(t_i, w_i) \end{aligned} \tag{7}$$

has local truncation error

$$\begin{aligned} \tau_{i+1}(h) &= \frac{1}{h}(y_{i+1} - (y_i + hf(t_i, y_i))) = \\ &= \frac{y_{i+1} - y_i}{h} - f(t_i, y_i). \end{aligned}$$

For this problem, we have

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} + 4\sin t_i + 21(t_i - 4)^2. \quad \checkmark$$

Local truncation error measures the accuracy of the method at a specific step, assuming that the method was exact at the previous step. It is important to note that this error depends on the differential equation, the step size, and the particular step in the approximation. We know the differential equation, i.e., we know $f(t, y)$. Hence, the local truncation error is a function of the step size h and the particular step i. $\checkmark$

The local truncation error for Euler's method is:

$$\tau_{i+1}(h) = \frac{h}{2} y''(\xi_i),$$

for some ξ_i in $(t_i, ti + 1)$. The upper bound for the local truncation error is

$$|\tau_{i+1}(h)| \le \frac{h}{2} M.$$

At $t = 3$, we have

$$|y''(3)| = |-4\cos 3 - 42(3-4)| = 45.96.$$

Hence, for $h = 0.001$, at $t = 3$ we have

$$|\tau_{i+1}(h = 0.001)| \approx \frac{0.001}{2} \cdot 45.96 = 0.023. \quad \checkmark$$

b) The formula for calculating the global error is located on page 327 of the textbook:

$$|y(t_i) - w_i| \leq \frac{\tau(h)}{L} e^{L(t_i - a)},$$

where $\tau_i(h)$ satisfies $\tau_i(h) \leq \tau(h)$.
Since $y''(t) = -4\cos t - 42(t-4)$, we have

$$|y''(t)| \leq |-4\cos t - 42(t-4)| \leq 250.4 = M,$$

which gives

$$|\tau(h)| \quad \leq \quad \frac{h}{2} M = \frac{0.001}{2} \cdot 250.4 = 0.1252.$$

Thus, the global error is

$$|y(t_i) - w_i| \leq \frac{0.1252}{2} \cdot e^{2(3+2)} = 1379. \quad \checkmark$$

Problem 3: Use the midpoint method to approximate the solution to the initial value problem

$$\begin{aligned} &y' = 1 + (t-y)^2, \qquad 2 \le t \le 3, \\ &y(2) = 1, \end{aligned} \tag{8}$$

with stepsize $h = 0.5$. The exact solution is $y(t) = t + \dfrac{1}{1-t}$. Compare the approximate and exact solutions.

Solution: The Runge-Kutta Midpoint method for the solution of the initial value problem

$$\begin{aligned} &y' = f(t,y), \qquad a \le t \le b, \\ &y(a) = \alpha, \end{aligned} \tag{9}$$

can be written as

$$\begin{aligned} &w_0 = \alpha, \\ &w_{i+1} = w_i + hf\Big(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\Big), \qquad i = 0, 1, \ldots. \end{aligned} \tag{10}$$

For the initial value problem in (8), $t_i = 2 + ih$, i.e. $t_0 = 2.0, t_1 = 2.5, t_2 = 3.0$, and we need to take only two steps, calculating w_1 at time t_1, and w_2 at time t_2. We have:

$$\begin{aligned}
w_0 &= 1, && (11)\\
w_1 &= w_0 + hf\Big(t_0 + \frac{h}{2}, w_0 + \frac{h}{2}f(t_0, w_0)\Big) && (12)\\
&= 1 + 0.5f\big(2 + 0.25, 1 + 0.25(1 + (2-1)^2)\big) && (13)\\
&= 1 + 0.5f(2.25, 1.5) && (14)\\
&= 1 + 0.5 \cdot (1 + (2.25 - 1.5)^2) = 1.78125, && (15)\\
w_2 &= w_1 + hf\Big(t_1 + \frac{h}{2}, w_1 + \frac{h}{2}f(t_1, w_1)\Big) && (16)\\
&= 1.78125 + 0.5f\big(2.5 + 0.25, 1.78125 + 0.25f(2.5, 1.78125)\big) && (17)\\
&= 1.78125 + 0.5f\big(2.75, 1.78125 + 0.25 \cdot (1 + (2.5 - 1.78125)^2)\big) && (18)\\
&= 1.78125 + 0.5f(2.75, 2.16040) && (19)\\
&= 1.78125 + 0.5(1 + (2.75 - 2.16040)^2) && (20)\\
&= 2.455064. && (21)
\end{aligned}$$

The exact solution gives the following values: $y(t_1 = 2.5) = 1.8333$ and $y(t_2 = 3) = 2.5$. Note that these results are more accurate than the results obtained with Euler's method in Homework 1.

Problem 4: Consider the initial value problem

$$\begin{aligned} y' &= 1 + \frac{y}{t}, \qquad 1 \le t \le 2, \\ y(1) &= 2. \end{aligned} \tag{22}$$

The exact solution to this problem is $y(t) = t \log t + 2t$.

a) If Euler's method is used to solve this problem and an accuracy of 10^{-4} is desired for the final value $y(2)$, what stepsize h should be used approximately?

b) Write a code for Euler's method and use it to solve this problem using the h in part (a). Plot both the numerical and exact solutions at all intermediate mesh points.

c) Repeat part (b) with midpoint method.

d) Repeat part (c) with step with stepsize $\frac{h}{2}$. Then use the numerical results from (c) and (d) to estimate the order p of the midpoint method.
Hint: $O(h^p)$ is the global error for the method when a stepsize h is used.

Solution:
a) We will use the following formula to obtain the stepsize h:

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right]. \tag{23}$$

We first find M and L. We have $y'(t) = \log t + 3$, and $y''(t) = \frac{1}{t}$. The bound M is obtained from: $|y''(t)| = \left|\frac{1}{t}\right| \le 1 = M,\ \ 1 \le t \le 2$. Also, since $f(t,y) = 1 + \frac{y}{t}$, we have

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{1}{t}\right| = 1 = L,\ \ 1 \le t \le 2. \tag{24}$$

Plugging these into (23), we get

$$\begin{aligned} 10^{-4} &= \frac{h}{2}\left[e^1 - 1\right], \\ h &= 1.164 \cdot 10^{-4}. \end{aligned}$$

b) Running the code gives:
$N = 8591, h = 1.1640088e - 004, t = 2.00, w = 5.3862361624e + 000, y = 5.3862943611e + 000, error = 5.8198752389e - 005$.

As expected, with stepsize $h = 1.164 \cdot 10^{-4}$ from part (a), we obtained the desired accuracy of 10^{-4}.
Check the plots of the exact solution and the solution obtained using Euler's method. The plots of the functions are overlaid due to sufficient accuracy.

c) The Runge-Kutta Midpoint method for the solution of the initial value problem

$$\begin{aligned} &y' = f(t, y), \qquad a \le t \le b, \\ &y(a) = \alpha, \end{aligned} \tag{25}$$

is

$$\begin{aligned} &w_0 = \alpha, \\ &w_{i+1} = w_i + hf\Big(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\Big), \qquad i = 0, 1, \dots . \end{aligned} \tag{26}$$

Running the Runge-Kutta Midpoint method program gives:

$N = 8591, h = 1.1640088e - 004, t = 2.00, w = 5.3862943603e + 000, y = 5.3862943611e + 000, error = 8.5053120102e - 010$.
Thus, RK Midpoint rule gives considerably more accurate results than Euler's method.

d) Running the Midpoint program gives the following results:

$N = 17182, h = 5.8200442e - 005, t = 2.00, w = 5.3862943609e + 000, y = 5.3862943611e + 000, error = 2.0656898414e - 010$.

Order of convergence:

$$p = \frac{\log_{10}\left(\frac{w_h - y_{exact}}{w_{h/2} - y_{exact}}\right)}{\log_{10}(2.0)} = \frac{\log_{10}\left(\frac{8.5053120102e-010}{2.0656898414e-010}\right)}{\log_{10}(2.0)} \approx 2.041. \quad \checkmark$$

We obtain the 2^{rd}, or quadratic, order of convergence, for the second order Runge-Kutta Midpoint method, which is expected.