Midterm Review Problems

Igor Yanovsky (Math 151B TA)

These sample review problems do not necessarily represent the content, length, or depth of the material you will be tested on.

Among other things, it is also a good idea to go over the homework sets.

Main Concepts

A one-step difference equation method with local truncation error $\tau_i(h)$ at the ith step is said to be **consistent** if

$$\lim_{h\to 0} \max_{1\le i\le N} |\tau_i(h)| = 0. \tag{1}$$

A one-step difference equation method is said to be **convergent** if

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| = 0, \tag{2}$$

where y_i is the exact solution and w_i is the approximation obtained from the difference method. Recall that for Euler's method, we have

$$\max_{1\le i\le N} |w_i - y(t_i)| \le \frac{Mh}{2L}|e^{L(b-a)} - 1|, \tag{3}$$

and, therefore, Euler's method is convergent with the linear (first order) rate of convergence of $O(h)$.

A method is **stable** if its results depend continuously on the initial data.

Problem: To approximate the initial value problem

$$y' = f(t, y) \tag{4}$$

for $t > 0$, consider a multistep method

$$w_{i+1} = 2w_{i-1} - w_i + h\Big[\frac{5}{2}f(t_i, w_i) + \frac{1}{2}f(t_{i-1}, w_{i-1})\Big].$$

Is this method stable?

Solution: For a multistep method to be stable, it has to satisfy the root condition. A multistep method is said to satisfy the root condition if all roots λ_i of the characteristic polynomial $P(\lambda)$ (for a general form of $P(\lambda)$ see equation (5.57) in the book) are such that $|\lambda_i| \leq 1$, and if $|\lambda_i| = 1$, then λ_i is simple.
The characteristic polynomial of the following multistep method

$$w_{i+1} + w_i - 2w_{i-1} = h\Big[\frac{5}{2}f(t_i, w_i) + \frac{1}{2}f(t_{i-1}, w_{i-1})\Big].$$

is

$$P(\lambda) = -2 + \lambda + \lambda^2,$$

which has roots

$$\lambda_1 = 1, \;\; \lambda_2 = -2.$$

Thus this multistep method does not satisfy the root condition, and therefore is unstable. ✓
Note that a quick way of writing a characteristic polynomial is to associate the coefficient a_0 to the leftmost grid point in the method's stencil. In the example above, the leftmost grid point has an index $i-1$, and therefore, $a_0 = -2$, $a_1 = 1$, $a_2 = 1$.

Problem: To approximate the initial value problem

$$y' = f(t, y) \tag{5}$$

for $t > 0$, consider a multistep method

$$w_{i+1} = 2w_{i-1} - w_i + h\Big[\frac{5}{2}f(t_i, w_i) + \frac{1}{2}f(t_{i-1}, w_{i-1})\Big].$$

Find the local truncation error.

Solution: Expanding terms in Taylor's series around t_i, we obtain the following local truncation error $\tau(h)$:

$$\begin{aligned}
\tau_{i+1}(h) &= \frac{1}{h}\bigg(y(t_{i+1}) + y(t_i) - 2y(t_{i-1}) - h\Big[\frac{5}{2}f(t_i, y(t_i)) + \frac{1}{2}f(t_{i-1}, y(t_{i-1}))\Big]\bigg) = \\
&= \frac{1}{h}\Big[y(t_{i+1}) + y(t_i) - 2y(t_{i-1})\Big] - \frac{5}{2}y'(t_i) - \frac{1}{2}y'(t_{i-1}) \\
&= \frac{1}{h}\Big[\ y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(\xi_{i1}) \\
&\qquad +y(t_i) \\
&\qquad -2\big(y(t_i) - hy'(t_i) + \frac{h^2}{2}y''(t_i) - \frac{h^3}{6}y'''(\xi_{i2})\big)\Big] \\
&\qquad\qquad -\frac{5}{2}y'(t_i) \\
&\qquad\qquad -\frac{1}{2}\Big[y'(t_i) - hy''(t_i) + \frac{h^2}{2}y'''(\xi_{i3})\Big] \\
&= \frac{1}{4}h^2 y'''(\xi_i),
\end{aligned}$$

where $t_{i-1} < \xi_i < t_{i+1}$.

Problem: Show that the Backward Euler (or Implicit Euler) method

$$w_{i+1} = w_i + hf(t_{i+1}, w_{i+1})$$

is A-stable.

Solution: The region R of absolute stability is $R = \{h\lambda \in \mathbb{C} \mid |Q(h\lambda)| < 1\}$, where $w_{i+1} = Q(h\lambda)w_i$. A numerical method is said to be A-stable if its region of stability R contains the entire left half-plane.

In other words, in order to show that the method is A-stable, we need to show that when it is applied to the scalar test equation $y' = \lambda y = f$, whose solutions tend to zero for $\lambda < 0$, all the solutions of the method also tend to zero for a fixed $h > 0$ as $i \to \infty$.

For the Backward Euler method, we have

$$\begin{aligned}
&w_{i+1} = w_i + h\lambda w_{i+1},\\
&w_{i+1} - h\lambda w_{i+1} = w_i,\\
&w_{i+1}(1 - h\lambda) = w_i,\\
&w_{i+1} = \frac{1}{1 - h\lambda} w_i,\\
&w_{i+1} = \left(\frac{1}{1 - h\lambda}\right)^{n+1} w_0.
\end{aligned}$$

Thus,

$$Q(h\lambda) = \frac{1}{1 - h\lambda}.$$

Note that for $Re(h\lambda) < 0$, $|Q(h\lambda)| < 1$. Therefore, the region of absolute stability R for the Backward Euler method contains the entire left half-plane, and hence, the method is A-stable.
Note that the region of absolute stability contains the interval $(2, +\infty)$.

Boundary Value Problems

Given the second-order boundary-value problem

$$y''(x) = p(x)y'(x) + q(x)y(x) + r(x),$$
$$a \le x \le b,$$
$$y(a) = \alpha, \quad y(b) = \beta,$$

the differential equation to be approximated at the interior points x_i is

$$y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i). \tag{6}$$

It might be helpful to know how to derive the following approximations to the first and second derivatives of $y(x_i)$ (pages 656-657 in the book). Expanding $y(x_{i+1})$ and $y(x_{i-1})$ in Taylor polynomials about x_i, and doing some arithmetic manipulations (as in the book), we obtain the following formulas:

$$\begin{aligned} y''(x_i) &= \frac{1}{h^2}\big[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\big] - \frac{h^2}{12}y''''(\xi_i), \\ y'(x_i) &= \frac{1}{2h}\big[y(x_{i+1}) - y(x_{i-1})\big] - \frac{h^2}{6}y'''(\eta_i). \end{aligned}$$

The approximation to (6) is therefore:

$$\left(\frac{-w_{i+1} + 2w_i - w_{i-1}}{h^2}\right) + p(x_i)\left(\frac{w_{i+1} - w_{i-1}}{2h}\right) + q(x_i)w_i = -r(x_i). \tag{7}$$

Problem: Write the discretization of the following boundary value problem

$$y'' = -\frac{4}{x}y' + \frac{2}{x^2}y - \frac{2}{x^2}\log x,$$
$$1 \le x \le 2,$$
$$y(1) = -\frac{1}{2}, \quad y(2) = \log 2,$$

in matrix-vector notation $A\mathbf{w} = \mathbf{b}$.

Solution: At the interior points x_i, for $i = 1, 2, \dots, N$, the differential equation to be approximated is

$$y''(x_i) = -\frac{4}{x_i}y'(x_i) + \frac{2}{x_i^2}y(x_i) - \frac{2}{x_i^2}\log x_i. \tag{8}$$

Since

$$\begin{aligned} y''(x_i) &= \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12}y^{(4)}(\xi_i), \\ y'(x_i) &= \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6}y'''(\eta_i), \end{aligned}$$

we can write the numerical approximation to (8) as

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + \frac{4}{x_i}\left(\frac{w_{i+1} - w_{i-1}}{2h}\right) - \frac{2}{x_i^2}w_i = -\frac{2}{x_i^2}\log x_i. \tag{9}$$

Multiplying both sides of (9) by $-h^2$ gives

$$-(w_{i+1} - 2w_i + w_{i-1}) - \frac{2h}{x_i}(w_{i+1} - w_{i-1}) + \frac{2h^2}{x_i^2}w_i = \frac{2h^2}{x_i^2}\log x_i.$$

Collecting w_{i-1}, w_i, and w_{i+1} terms, we obtain

$$-\left(1-\frac{2h}{x_i}\right)w_{i-1}+\left(2+\frac{2h^2}{x_i^2}\right)w_i-\left(1+\frac{2h}{x_i}\right)w_{i+1}=\frac{2h^2}{x_i^2}\log x_i. \tag{10}$$

The resulting system of equations can be expressed in the tridiagonal $N \times N$ matrix form

$$A\mathbf{w}=\mathbf{b}, \qquad \text{where}$$

$$A=\begin{bmatrix} 2+\frac{2h^2}{x_1^2} & -1-\frac{2h}{x_1} & 0 & \cdots & 0 \\ -1+\frac{2h}{x_2} & 2+\frac{2h^2}{x_2^2} & -1-\frac{2h}{x_2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1-\frac{2h}{x_{N-1}} \\ 0 & \cdots & 0 & -1+\frac{2h}{x_N} & 2+\frac{2h^2}{x_N^2} \end{bmatrix},$$

$$\mathbf{w}=\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{bmatrix}, \quad \text{and} \quad \mathbf{b}=\begin{bmatrix} \frac{2h^2}{x_1^2}\log x_1+\left(1-\frac{2h}{x_1}\right)w_0 \\ \frac{2h^2}{x_2^2}\log x_2 \\ \vdots \\ \frac{2h^2}{x_{N-1}^2}\log x_{N-1} \\ \frac{2h^2}{x_N^2}\log x_N+\left(1+\frac{2h}{x_N}\right)w_{N+1} \end{bmatrix}. \quad \checkmark$$

In order to see that this system satisfies (10), look at a couple of rows of matrix A, for example, the second row:

$$-\left(1-\frac{2h}{x_2}\right)w_1+\left(2+\frac{2h^2}{x_2^2}\right)w_2-\left(1+\frac{2h}{x_2}\right)w_3=\frac{2h^2}{x_2^2}\log x_2.$$

Also, first and last elements of $\mathbf{b}$ might be a little daunting. However, if we look at the first row (for example), we see that

$$\left(2+\frac{2h^2}{x_1^2}\right)w_1-\left(1+\frac{2h}{x_1}\right)w_2=\frac{2h^2}{x_1^2}\log x_1+\left(1-\frac{2h}{x_1}\right)w_0,$$

or

$$-\left(1-\frac{2h}{x_1}\right)w_0+\left(2+\frac{2h^2}{x_1^2}\right)w_1-\left(1+\frac{2h}{x_1}\right)w_2=\frac{2h^2}{x_1^2}\log x_1,$$

which satisfies equation (10).

Also, note that the book considers the general second-order boundary value problem:

$$y''(x)=p(x)y'(x)+q(x)y(x)+r(x).$$

For our problem, $p(x)=-\frac{4}{x}$, $q(x)=\frac{2}{x^2}$, and $r(x)=-\frac{2}{x^2}\log x$. Plugging these values into the formulas in the book, we can verify whether our calculations are correct.