Boundary Value Problem Example

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Section 11.3, Problem 3(b): Write the discretization of the following boundary value problem

$$
\begin{aligned}
&y'' = -\frac{4}{x}y' + \frac{2}{x^2}y - \frac{2}{x^2}\log x,\\
&1 \le x \le 2,\\
&y(1) = -\frac{1}{2}, \quad y(2) = \log 2,
\end{aligned}
$$

in matrix-vector notation $A\mathbf{w} = \mathbf{b}$.

Solution: At the interior points x_i, for $i = 1, 2, \ldots, N$, the differential equation to be approximated is

$$y''(x_i) = -\frac{4}{x_i}y'(x_i) + \frac{2}{x_i^2}y(x_i) - \frac{2}{x_i^2}\log x_i. \tag{1}$$

Since

$$
\begin{aligned}
y''(x_i) &= \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12}y^{(4)}(\xi_i),\\
y'(x_i) &= \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6}y'''(\eta_i),
\end{aligned}
$$

we can write the numerical approximation to (1) as

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + \frac{4}{x_i}\left(\frac{w_{i+1} - w_{i-1}}{2h}\right) - \frac{2}{x_i^2}w_i = -\frac{2}{x_i^2}\log x_i. \tag{2}$$

Multiplying both sides of (2) by $-h^2$ gives

$$-(w_{i+1} - 2w_i + w_{i-1}) - \frac{2h}{x_i}(w_{i+1} - w_{i-1}) + \frac{2h^2}{x_i^2}w_i = \frac{2h^2}{x_i^2}\log x_i.$$

Collecting w_{i-1}, w_i, and w_{i+1} terms, we obtain

$$-\left(1 - \frac{2h}{x_i}\right)w_{i-1} + \left(2 + \frac{2h^2}{x_i^2}\right)w_i - \left(1 + \frac{2h}{x_i}\right)w_{i+1} = \frac{2h^2}{x_i^2}\log x_i. \tag{3}$$

The resulting system of equations can be expressed in the tridiagonal $N \times N$ matrix form

$$A\mathbf{w} = \mathbf{b}, \quad \text{where}$$

$$
A = \begin{bmatrix}
2 + \frac{2h^2}{x_1^2} & -1 - \frac{2h}{x_1} & 0 & \cdots & 0\\
-1 + \frac{2h}{x_2} & 2 + \frac{2h^2}{x_2^2} & -1 - \frac{2h}{x_2} & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & 0\\
\vdots & \ddots & \ddots & \ddots & -1 - \frac{2h}{x_{N-1}}\\
0 & \cdots & 0 & -1 + \frac{2h}{x_N} & 2 + \frac{2h^2}{x_N^2}
\end{bmatrix},
$$

$$
\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} \dfrac{2h^2}{x_1^2}\log x_1 + \left(1 - \dfrac{2h}{x_1}\right) w_0 \\ \dfrac{2h^2}{x_2^2}\log x_2 \\ \vdots \\ \dfrac{2h^2}{x_{N-1}^2}\log x_{N-1} \\ \dfrac{2h^2}{x_N^2}\log x_N + \left(1 + \dfrac{2h}{x_N}\right) w_{N+1} \end{bmatrix}. \quad \checkmark
$$

In order to see that this system satisfies (3), look at a couple of rows of matrix A, for example, the second row:

$$
-\left(1 - \frac{2h}{x_2}\right) w_1 + \left(2 + \frac{2h^2}{x_2^2}\right) w_2 - \left(1 + \frac{2h}{x_2}\right) w_3 = \frac{2h^2}{x_2^2}\log x_2.
$$

Also, first and last elements of $\mathbf{b}$ might be a little daunting. However, if we look at the first row (for example), we see that

$$
\left(2 + \frac{2h^2}{x_1^2}\right) w_1 - \left(1 + \frac{2h}{x_1}\right) w_2 = \frac{2h^2}{x_1^2}\log x_1 + \left(1 - \frac{2h}{x_1}\right) w_0,
$$

or

$$
-\left(1 - \frac{2h}{x_1}\right) w_0 + \left(2 + \frac{2h^2}{x_1^2}\right) w_1 - \left(1 + \frac{2h}{x_1}\right) w_2 = \frac{2h^2}{x_1^2}\log x_1,
$$

which satisfies equation (3).

Also, note that the book considers the general second-order boundary value problem:

$$
y''(x) = p(x)y'(x) + q(x)y(x) + r(x).
$$

For our problem, $p(x) = -\dfrac{4}{x}$, $q(x) = \dfrac{2}{x^2}$, and $r(x) = -\dfrac{2}{x^2}\log x$. Plugging these values into the formulas in the book, we can verify whether our calculations are correct.