Homework 3 Solutions

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Problem 1: Compute the absolute error and relative error in approximations of p by p^*. (Use calculator!)
a) $p = \pi,\ \ p^* = 22/7$;
b) $p = \pi,\ \ p^* = 3.1416$.

Solution: For this exercise, you can use either calculator or Matlab.
a) Absolute error: $|p - p^*| \ = \ |\pi - 22/7| \ = \ 0.0012645.$
Relative error: $\frac{|p-p^*|}{|p|} \ = \ \frac{|\pi - 22/7|}{\pi} \ = \ 4.0250 \times 10^{-4}.$

b) Absolute error: $|p - p^*| \ = \ |\pi - 3.1416| \ = \ 7.3464 \times 10^{-6}.$
Relative error: $\frac{|p-p^*|}{|p|} \ = \ \frac{|\pi - 3.1416|}{\pi} \ = \ 2.3384 \times 10^{-6}.$

Problem 2: Find the largest interval in which p^* must lie to approximate $\sqrt{2}$ with relative error at most 10^{-5} for each value for p.

Solution: The relative error is defined as $\frac{|p-p^*|}{|p|}$, where in our case, $p = \sqrt{2}$. We have

$$\frac{|\sqrt{2} - p^*|}{\sqrt{2}} \ \le \ 10^{-5}.$$

Therefore,

$$|\sqrt{2} - p^*| \ \le \ \sqrt{2} \cdot 10^{-5},$$

or

$$\begin{aligned} -\sqrt{2}\cdot 10^{-5} \le & \quad \sqrt{2} - p^* && \le \ \sqrt{2}\cdot 10^{-5}, \\ -\sqrt{2} - \sqrt{2}\cdot 10^{-5} \le & \quad\ \ -p^* && \le \ -\sqrt{2} + \sqrt{2}\cdot 10^{-5}, \\ \sqrt{2} + \sqrt{2}\cdot 10^{-5} \ge & \quad\ \ \ p^* && \ge \ \sqrt{2} - \sqrt{2}\cdot 10^{-5}. \end{aligned}$$

Hence,

$$\sqrt{2} - \sqrt{2}\cdot 10^{-5} \le \ \ p^* \ \ \le \ \sqrt{2} + \sqrt{2}\cdot 10^{-5}. \qquad \checkmark$$

This interval can be written in decimal notation as $[1.41419942\ldots, 1.41422770\ldots]$.

Problem 3: Use the 64-bit long real format to find the decimal equivalent of the following floating-point machine numbers.
a) 0 10000001010 10010011000000$\cdots$0
b) 1 10000001010 01010011000000$\cdots$0

Solution:
a) Given a *binary number* (also known as a *machine number*)

$$\underbrace{0}_{s}\ \underbrace{10000001010}_{c}\ \underbrace{10010011000000\cdots0}_{f},$$

a *decimal number* (also known as a *floating-point decimal number*) is of the form:

$$(-1)^s 2^{c-1023}(1+f).$$

Therefore, in order to find a decimal representation of a binary number, we need to find s, c, and f.
The leftmost bit is zero, i.e. $s = 0$, which indicates that the number is positive.
The next 11 bits, 10000001010, giving the characteristic, are equivalent to the decimal number:

$$\begin{aligned} c &= 1\cdot 2^{10} + 0\cdot 2^9 + \cdots + 0\cdot 2^4 + 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0 \\ &= 1024 + 8 + 2 \ = \ 1034. \end{aligned}$$

The exponent part of the number is therefore $2^{1034-1023} = 2^{11}$.
The final 52 bits specify that the mantissa is

$$\begin{aligned} f &= 1\cdot\left(\frac{1}{2}\right)^1 + 1\cdot\left(\frac{1}{2}\right)^4 + 1\cdot\left(\frac{1}{2}\right)^7 + 1\cdot\left(\frac{1}{2}\right)^8 \\ &= 0.57421875. \end{aligned}$$

Therefore, this binary number represents the decimal number

$$\begin{aligned} (-1)^s 2^{c-1023}(1+f) &= (-1)^0 \cdot 2^{1034-1023} \cdot (1 + 0.57421875) \\ &= 2^{11} \cdot 1.57421875 \\ &= 3224. \quad \checkmark \end{aligned}$$

b) Given a *binary number*

$$\underbrace{1}_{s}\ \underbrace{10000001010}_{c}\ \underbrace{01010011000000\cdots 0}_{f},$$

a *decimal number* is of the form:

$$(-1)^s 2^{c-1023}(1+f).$$

Therefore, in order to find a decimal representation of a binary number, we need to find s, c, and f.
The leftmost bit is zero, i.e. $s = 1$, which indicates that the number is negative.
The next 11 bits, 10000001010, giving the characteristic, are equivalent to the decimal number:

$$\begin{aligned} c &= 1\cdot 2^{10} + 0\cdot 2^9 + \cdots + 0\cdot 2^4 + 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0 \\ &= 1024 + 8 + 2 \ = \ 1034. \end{aligned}$$

The exponent part of the number is therefore $2^{1034-1023} = 2^{11}$.
The final 52 bits specify that the mantissa is

$$\begin{aligned} f &= 1\cdot\left(\frac{1}{2}\right)^2 + 1\cdot\left(\frac{1}{2}\right)^4 + 1\cdot\left(\frac{1}{2}\right)^7 + 1\cdot\left(\frac{1}{2}\right)^8 \\ &= 0.32421875. \end{aligned}$$

Therefore, this binary number represents the decimal number

$$\begin{aligned} (-1)^s 2^{c-1023}(1+f) &= (-1)^1\cdot 2^{1034-1023}\cdot(1+0.32421875) \\ &= -2^{11}\cdot 1.32421875 \\ &= -2712. \quad \checkmark \end{aligned}$$

Problem 4: Find the next largest and smallest machine numbers in decimal form for the numbers given in the above problem.

Solution:
a) Consider a *binary number* (also known as a *machine number*)

$$0\ 10000001010\ 10010011000000\cdots00\ ,$$

- The next largest machine number is

$$0\ 10000001010\ 10010011000000\cdots01\ . \tag{1}$$

From problem 3(a), we know that $s=0$ and $c=1034$. We need to find f:

$$\begin{aligned} f &= 1\cdot\left(\frac{1}{2}\right)^{1} + 1\cdot\left(\frac{1}{2}\right)^{4} + 1\cdot\left(\frac{1}{2}\right)^{7} + 1\cdot\left(\frac{1}{2}\right)^{8} + 1\cdot\left(\frac{1}{2}\right)^{52} \\ &= 0.57421875\ +\ 2.220446049250313\ldots\cdot 10^{-16} \\ &= 0.57421875\ +\ 0.0000000000000002220446\ldots \\ &= 0.5742187500000002220446\ldots \end{aligned}$$

Therefore, this binary number (in (1)) represents the decimal number

$$\begin{aligned} (-1)^s 2^{c-1023}(1+f) &= (-1)^0\cdot 2^{1034-1023}\cdot(1+0.57421875\ +\ 0.0000000000000002220446\ldots) \\ &= 2^{11}\cdot(1.57421875\ +\ 2.220446049250313\ldots\times 10^{-16}) \\ &= 3224\ +\ 4.547473508864641\times 10^{-13} \\ &= 3224.00000000000045474735\ldots\quad\checkmark \end{aligned}$$

- The next smallest machine number is

$$0\ 10000001010\ 10010010111111\cdots11\ . \tag{2}$$

From problem 3(a), we know that $s=0$ and $c=1034$. We need to find f: [1]

$$\begin{aligned} f &= 1\cdot\left(\frac{1}{2}\right)^{1} + 1\cdot\left(\frac{1}{2}\right)^{4} + 1\cdot\left(\frac{1}{2}\right)^{7} + \sum_{n=9}^{52} 1\cdot\left(\frac{1}{2}\right)^{n} \\ &= \left(\frac{1}{2}\right)^{1} + \left(\frac{1}{2}\right)^{4} + \left(\frac{1}{2}\right)^{7} + \left(\frac{1}{2}\right)^{8} - \left(\frac{1}{2}\right)^{52} \\ &= 0.57421875\ -\ 2.220446049250313\ldots\cdot 10^{-16} \\ &= 0.57421875\ -\ 0.0000000000000002220446\ldots \\ &= 0.5742187499999999977795539\ldots \end{aligned}$$

[1]Note that

$$\sum_{n=0}^{N} 2^n = 2^{N+1} - 1.$$

The formula above is a specific case of the following more general equation:

$$\sum_{n=M}^{N} 2^n = 2^{N+1} - 2^M.$$

Similarly, we also have a formula:

$$\sum_{n=M}^{N} \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{M-1} - \left(\frac{1}{2}\right)^{N}.$$

To get some intuition about these formulas, consider an example with $M=2$ and $N=5$, for instance.

Therefore, this binary number (in (2)) represents the decimal number

$$
\begin{aligned}
(-1)^s 2^{c-1023}(1+f) &= (-1)^0 \cdot 2^{1034-1023} \cdot (1 + 0.57421875 \ - \ 0.0000000000000002220446\ldots) \\
&= 2^{11} \cdot (1.57421875 \ - \ 2.220446\ldots \times 10^{-16}) \\
&= 3224 \ - \ 4.547473508 \times 10^{-13} \\
&= 3224 \ - \ 0.0000000000004547473508 \\
&= 3223.9999999999995452527\ldots \quad \checkmark
\end{aligned}
$$

b) Consider a *binary number*

1 10000001010 01010011000000 $\cdots$ 0

- The next largest (in magnitude) machine number is

1 10000001010 01010011000000 $\cdots$ 1 (3)

From problem 3(b), we know that $s = 1$ and $c = 1034$. We need to find f:

$$
\begin{aligned}
f &= 1 \cdot \left(\frac{1}{2}\right)^2 + 1 \cdot \left(\frac{1}{2}\right)^4 + 1 \cdot \left(\frac{1}{2}\right)^7 + 1 \cdot \left(\frac{1}{2}\right)^8 + 1 \cdot \left(\frac{1}{2}\right)^{52} \\
&= 0.32421875 \ + \ 2.220446049250313\ldots \cdot 10^{-16} \\
&= 0.32421875 \ + \ 0.0000000000000002220446\ldots \\
&= 0.3242187500000002220446\ldots
\end{aligned}
$$

Therefore, this binary number (in (3)) represents the decimal number

$$
\begin{aligned}
(-1)^s 2^{c-1023}(1+f) &= (-1)^1 \cdot 2^{1034-1023} \cdot (1 + 0.32421875 \ + \ 0.0000000000000002220446\ldots) \\
&= -2^{11} \cdot (1.32421875 \ + \ 2.220446049250313\ldots \times 10^{-16}) \\
&= -2712 \ - \ 4.547473508864641 \times 10^{-13} \\
&= -2712 \ - \ 0.0000000000004547473508 \\
&= -2712.00000000000045474735\ldots \quad \checkmark
\end{aligned}
$$

- The next smallest (in magnitude) machine number is

1 10000001010 01010010111111 $\cdots$ 1 (4)

From problem 3(b), we know that $s = 1$ and $c = 1034$. We need to find f:

$$
\begin{aligned}
f &= 1 \cdot \left(\frac{1}{2}\right)^2 + 1 \cdot \left(\frac{1}{2}\right)^4 + 1 \cdot \left(\frac{1}{2}\right)^7 + \sum_{n=9}^{52} 1 \cdot \left(\frac{1}{2}\right)^n \\
&= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^7 + \ \left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^{52} \\
&= 0.32421875 \ - \ 2.220446049250313\ldots \cdot 10^{-16} \\
&= 0.32421875 \ - \ 0.0000000000000002220446\ldots \\
&= 0.3242187499999999777795539\ldots
\end{aligned}
$$

Therefore, this binary number (in (4)) represents the decimal number

$$
\begin{aligned}
(-1)^s 2^{c-1023}(1+f) &= (-1)^1 \cdot 2^{1034-1023} \cdot (1 + 0.32421875 \ - \ 0.0000000000000002220446\ldots) \\
&= -2^{11} \cdot (1.32421875 \ - \ 2.220446049250313\ldots \times 10^{-16}) \\
&= -2712 \ + \ 4.547473508864641 \times 10^{-13} \\
&= -2712 \ + \ 0.0000000000004547473508 \\
&= -2711.9999999999995452527\ldots \quad \checkmark
\end{aligned}
$$

Problem 5: Use four-digit rounding arithmetic and the formulas to find the most accurate approximations to the roots of the following quadratic equations. Compute the relative error.
a) $\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} \;=\; 0$;
b) $1.002x^2 + 11.01x + 0.01265 \;=\; 0$.

Solution: The quadratic formula states that the roots of $ax^2 + bx + c = 0$ are

$$x_{1,2} \;=\; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

a) The roots of $\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} \;=\; 0$ are approximately

$$x_1 = 92.24457962731231, \qquad x_2 = 0.00542037268770.$$

We use four-digit rounding arithmetic to find approximations to the roots. We find the first root:

$$\begin{aligned} x_1^* &= \frac{\frac{123}{4} + \sqrt{\left(-\frac{123}{4}\right)^2 - 4\cdot\frac{1}{3}\cdot\frac{1}{6}}}{2\cdot\frac{1}{3}} = \frac{30.75 + \sqrt{30.75^2 - 4\cdot 0.3333\cdot 0.1667}}{2\cdot 0.3333} \\ &= \frac{30.75 + \sqrt{945.6 - 1.333\cdot 0.1667}}{0.6666} = \frac{30.75 + \sqrt{945.6 - 0.2222}}{0.6666} \\ &= \frac{30.75 + \sqrt{945.4}}{0.6666} = \frac{30.75 + 30.75}{0.6666} = \frac{61.50}{0.6666} = 92.26, \quad \checkmark \end{aligned}$$

which has the following relative error:

$$\frac{|x_1 - x_1^*|}{|x_1|} \;=\; \frac{|92.24457962731231 - 92.26|}{92.24457962731231} \;=\; 1.672\cdot 10^{-4}. \quad \checkmark$$

$$\begin{aligned} x_2^* &= \frac{\frac{123}{4} - \sqrt{\left(-\frac{123}{4}\right)^2 - 4\cdot\frac{1}{3}\cdot\frac{1}{6}}}{2\cdot\frac{1}{3}} = \frac{30.75 - \sqrt{30.75^2 - 4\cdot 0.3333\cdot 0.1667}}{2\cdot 0.3333} \\ &= \frac{30.75 - \sqrt{945.6 - 1.333\cdot 0.1667}}{0.6666} = \frac{30.75 - \sqrt{945.6 - 0.2222}}{0.6666} \\ &= \frac{30.75 - \sqrt{945.4}}{0.6666} = \frac{30.75 - 30.75}{0.6666} = 0. \end{aligned}$$

has the following relative error:

$$\frac{|x_2 - x_2^*|}{|x_2|} \;=\; \frac{|0.00542037268770 - 0|}{0.00542037268770} \;=\; 1.0.$$

We obtained a very large relative error, since the calculation for x_2^* involved the subtraction of nearly equal numbers. In order to get a more accurate approximation to x_2^*, we need to use an alternate quadratic formula, namely

$$x_{1,2} \;=\; \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}.$$

Using four-digit rounding arithmetic, we obtain:

$$x_2^* \;=\; \frac{-2\cdot\frac{1}{6}}{-\frac{123}{4} - \sqrt{\left(-\frac{123}{4}\right)^2 - 4\cdot\frac{1}{3}\cdot\frac{1}{6}}} \;=\; fl(0.00541951) \;=\; 0.005420, \quad \checkmark$$

which has the following relative error:

$$\frac{|x_2 - x_2^*|}{|x_2|} \;=\; \frac{|0.00542037268770 - 0.005420|}{0.00542037268770} \;=\; 6.876\cdot 10^{-5}. \quad \checkmark$$

b) The roots of $1.002x^2 + 11.01x + 0.01265 \; = \; 0$ are approximately

$$x_1 = -0.00114907565991, \qquad\qquad x_2 = -10.98687487643590.$$

We use four-digit rounding arithmetic to find approximations to the roots.
If we use the generic quadratic formula for the calculation of x_1^*, we will encounter the subtraction of nearly equal numbers (you may check). Therefore, we use the alternate quadratic formula to find x_1^*:

$$\begin{aligned} x_1 \quad &= \quad \frac{-2c}{b+\sqrt{b^2-4ac}} \quad = \quad \frac{-2\cdot 0.01265}{11.01-\sqrt{11.01^2+4\cdot 1.002\cdot 0.01265}} \\ &= \quad \frac{-0.02530}{11.01+11.00} \quad = \quad \frac{-0.02530}{22.01} \quad = \quad -0.001149, \quad \checkmark \end{aligned}$$

which has the following relative error:

$$\frac{|x_1 - x_1^*|}{|x_1|} \quad = \quad \frac{|-0.00114907565991-(-0.001149)|}{|-0.00114907565991|} \quad = \quad 6.584\cdot 10^{-5}. \quad \checkmark$$

We find the second root using the generic quadratic formula:

$$\begin{aligned} x_2^* \quad &= \quad \frac{-11.01-\sqrt{(-11.01)^2-4\cdot 1.002\cdot 0.01265}}{2\cdot 1.002} \quad = \quad \frac{-11.01-\sqrt{121.2-4.008\cdot 0.01265}}{2.004} \\ &= \quad \frac{-11.01-\sqrt{121.2-0.05070}}{2.004} \quad = \quad \frac{-11.01-\sqrt{121.1}}{2.004} \quad = \quad \frac{-11.01-11.00}{2.004} \\ &= \quad \frac{-22.01}{2.004} \quad = \quad -10.98, \quad \checkmark \end{aligned}$$

which has the following relative error:

$$\frac{|x_2 - x_2^*|}{|x_2|} \quad = \quad \frac{|-10.98687487643590-(-10.98)|}{|-10.98687487643590|} \quad = \quad 6.257\cdot 10^{-4}. \quad \checkmark$$

Similar Problem
The roots of $1.002x^2 - 11.01x + 0.01265 \;=\; 0$ are approximately

$$x_1 = 10.98687487643590, \qquad x_2 = 0.00114907565991.$$

We use four-digit rounding arithmetic to find approximations to the roots. We find the first root:

$$\begin{aligned}
x_1^* &= \frac{11.01 + \sqrt{(-11.01)^2 - 4 \cdot 1.002 \cdot 0.01265}}{2 \cdot 1.002} = \frac{11.01 + \sqrt{121.2 - 4.008 \cdot 0.01265}}{2.004} \\
&= \frac{11.01 + \sqrt{121.2 - 0.05070}}{2.004} = \frac{11.01 + \sqrt{121.1}}{2.004} = \frac{11.01 + 11.00}{2.004} \\
&= \frac{22.01}{2.004} = 10.98, \quad \checkmark
\end{aligned}$$

which has the following relative error:

$$\frac{|x_1 - x_1^*|}{|x_1|} = \frac{|10.98687487643590 - 10.98|}{10.98687} = 6.257 \cdot 10^{-4}. \quad \checkmark$$

If we use the generic quadratic formula for the calculation of x_2^*, we will encounter the subtraction of nearly equal numbers. Therefore, we use the alternate quadratic formula to find x_2^*:

$$\begin{aligned}
x_2 &= \frac{-2c}{b - \sqrt{b^2 - 4ac}} = \frac{-2 \cdot 0.01265}{-11.01 - \sqrt{(-11.01)^2 - 4 \cdot 1.002 \cdot 0.01265}} \\
&= \frac{-0.02530}{-11.01 - 11.00} = \frac{-0.02530}{-22.01} = 0.001149, \quad \checkmark
\end{aligned}$$

which has the following relative error:

$$\frac{|x_2 - x_2^*|}{|x_2|} = \frac{|0.00114907565991 - 0.001149|}{0.00114907565991} = 6.584 \cdot 10^{-5}. \quad \checkmark$$

Problem 6: Suppose that $fl(y)$ is a k-digit rounding approximation to y. Show that

$$\left|\frac{y - fl(y)}{y}\right| \;\leq\; 0.5 \times 10^{-k+1}.$$

(Hint: If $d_{k+1} < 5$, then $fl(y) = 0.d_1 \ldots d_k \times 10^n$.
If $d_{k+1} \geq 5$, then $fl(y) = 0.d_1 \ldots d_k \times 10^n + 10^{n-k}$.)

Solution: We have to look at two cases separately.

Case ① : $d_{k+1} < 5$.

$$\begin{aligned}
\left|\frac{y - fl(y)}{y}\right| &= \left|\frac{0.d_1 \ldots d_k d_{k+1} \ldots \times 10^n \;-\; 0.d_1 \ldots d_k \times 10^n}{0.d_1 \ldots d_k d_{k+1} \ldots \times 10^n}\right| \\
&= \left|\frac{0.\overbrace{0 \ldots 0}^{k\ zeros}\, d_{k+1} \ldots \times 10^n}{0.d_1 \ldots d_k d_{k+1} \ldots \times 10^n}\right| \\
&= \left|\frac{0.d_{k+1} d_{k+2} \ldots \times 10^{n-k}}{0.d_1 d_2 \ldots \times 10^n}\right| \\
&= \left|\frac{0.d_{k+1} d_{k+2} \ldots}{0.d_1 d_2 \ldots}\right| \times 10^{-k} \\
&= \frac{|0.d_{k+1} d_{k+2} \ldots|}{|0.d_1 d_2 \ldots|} \times 10^{-k} \\
&\leq \frac{|0.d_{k+1} d_{k+2} \ldots|}{0.1} \times 10^{-k} \qquad \text{since } d_1 \geq 1, \text{ so } |0.d_1 d_2 \ldots| \geq 0.1 \\
&\leq \frac{0.5}{0.1} \times 10^{-k} \qquad \text{since } d_{k+1} \leq 5, \text{ by assumption} \\
&= 5 \times 10^{-k} \;=\; 0.5 \times 10^{-k+1}. \quad \checkmark
\end{aligned}$$

Case ② : $d_{k+1} \geq 5$.

$$\begin{aligned}
\left|\frac{y - fl(y)}{y}\right| &= \left|\frac{0.d_1 \ldots d_k d_{k+1} \ldots \times 10^n \;-\; (0.d_1 \ldots d_k \times 10^n + 10^{n-k})}{0.d_1 \ldots d_k d_{k+1} \ldots \times 10^n}\right| \\
&= \left|\frac{0.d_1 \ldots d_k d_{k+1} \ldots \times 10^n \;-\; (0.d_1 \ldots d_k + 10^{-k}) \times 10^n}{0.d_1 d_2 \ldots \times 10^n}\right| \\
&= \left|\frac{0.d_1 \ldots d_k d_{k+1} \ldots \times 10^n \;-\; 0.d_1 \ldots \delta_k \times 10^n}{0.d_1 d_2 \ldots \times 10^n}\right| \qquad \text{where } \delta_k = d_k + 1 \\
&= \frac{|0.d_1 \ldots d_k d_{k+1} \ldots \;-\; 0.d_1 \ldots \delta_k|}{|0.d_1 d_2 \ldots|} \\
&= \frac{|0.\overbrace{0 \ldots 0}^{k-1\ 0's}\, d_k d_{k+1} \ldots - 0.\overbrace{0 \ldots 0}^{k-1\ 0's}\, \delta_k|}{0.d_1 d_2 \ldots}
\end{aligned}$$

Note that $0.0 \ldots 0\, \delta_k > 0.0 \ldots 0\, d_k d_{k+1} \ldots$,
and $\delta_k = d_k + 1$, $d_{k+1} \geq 5$.
For example, $d_k = 1$, $d_{k+1} = 6$, $\delta_k = 2$.

$$\begin{aligned}
&\leq \frac{0.\overbrace{0 \ldots 0}^{k-1\ 0's}\,05}{0.d_1 d_2 \ldots} \;=\; \frac{0.5}{0.d_1 d_2 \ldots} \times 10^{-k} \qquad \text{since } d_1 \geq 1, \text{ so } 0.d_1 d_2 \ldots \geq 0.1 \\
&\leq \frac{0.5}{0.1} \times 10^{-k} \;=\; 5 \times 10^{-k} \;=\; 0.5 \times 10^{-k+1}. \quad \checkmark
\end{aligned}$$