Homework 2 Solutions

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Problem 1: Show that the iteration equation for the Secant method can be written in the following form:

$$p_n = \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}.$$

Solution: The Secant iteration is defined as

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

We have

$$\begin{aligned} p_n &= p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \\ &= \frac{(f(p_{n-1}) - f(p_{n-2}))p_{n-1} - f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \\ &= \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}. \quad \checkmark \end{aligned}$$

Problem 2: Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$, find p_3 using the Secant method.

Solution: The Secant iteration is defined as

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

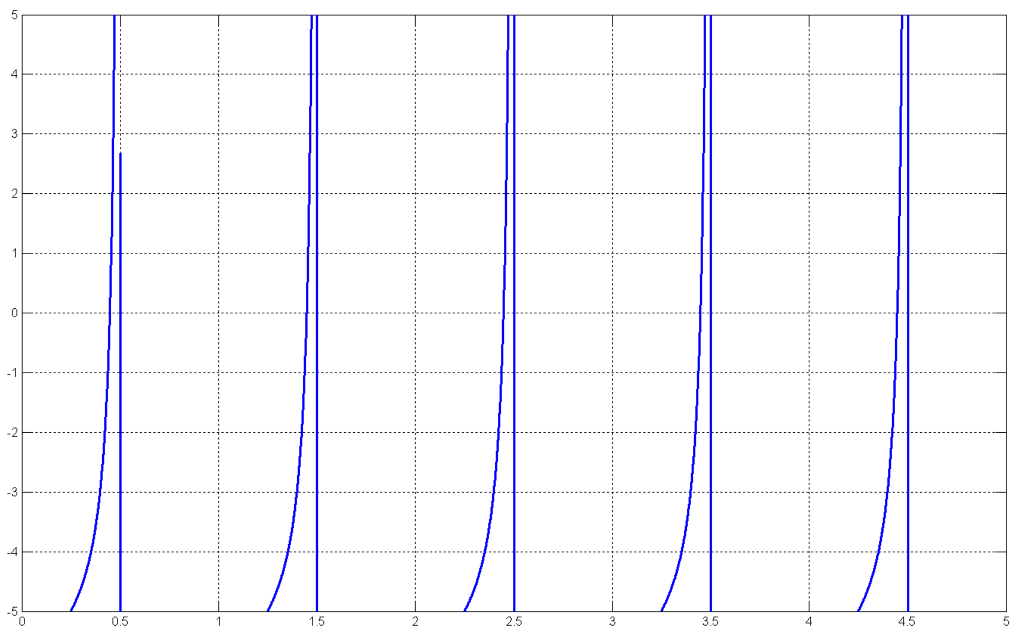
We have

$$\begin{aligned} p_2 &= p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 0 - \frac{f(0)(0+1)}{f(0) - f(-1)} = -\frac{f(0)}{f(0) - f(-1)} \\ &= -\frac{-\cos 0}{-\cos 0 - (-(-1)^3 - \cos(-1))} = -\frac{-1}{-1 - (1 - \cos(-1))} \\ &= \frac{1}{-2 + \cos(-1)} = \frac{1}{-2 + 0.5403} = -0.6851. \\ p_3 &= p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} = -0.6851 - \frac{f(-0.6851)(-0.6851 - 0)}{f(-0.6851) - f(0)} \\ &= -0.6851 - \frac{-0.6851(-(-0.6851)^3 - \cos(-0.6851))}{-(-0.6851)^3 - \cos(-0.6851) + \cos(0)} \\ &= -1.252. \quad \checkmark \end{aligned}$$

Note that $f(-1.252) = 1.649$, which is far from 0. We need to make three more iterations to get reasonably close to the answer.

Computational Problem:
The function $f(x) = \tan(\pi x) - 6$ has a zero at $\frac{\arctan(6)}{\pi} \approx 0.447431543$.
Use ten iterations of each of the following methods to approximate this root.
Which method is most successful?
a) Bisection method with initial interval $[0, 1]$.
b) Secant method with $p_0 = 0.4$ and $p_1 = 0.48$.
c) Newton's method with $p_0 = 0.4$.

Solution:
The plot of the function $f(x) = \tan(\pi x) - 6$ is shown below. It is important to note that f has several roots on the interval $[0, 5]$. This interval was considered in the earlier version of this homework. Even though you had to use the new version of the homework assignment, I will at least mention what happens if the old initial guesses were used.



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a) Note that since f has several roots in $[0, 5]$, the bisection method converges to a different root in this interval. Therefore, it would be a better idea to choose the interval to be $[0, 1]$. For such case, we have the following results:

iteration # 1 , p = 2.50000000e-001, $|p - p_{true}|$ = 1.97431543e-001, f(p) = -5.00000000e+000
iteration # 2 , p = 3.75000000e-001, $|p - p_{true}|$ = 7.24315433e-002, f(p) = -3.58578644e+000
iteration # 3 , p = 4.37500000e-001, $|p - p_{true}|$ = 9.93154329e-003, f(p) = -9.72660508e-001
iteration # 4 , p = 4.68750000e-001, $|p - p_{true}|$ = 2.13184567e-002, f(p) = 4.15317039e+000
iteration # 5 , p = 4.53125000e-001, $|p - p_{true}|$ = 5.69345671e-003, f(p) = 7.41452405e-001
iteration # 6 , p = 4.45312500e-001, $|p - p_{true}|$ = 2.11904329e-003, f(p) = -2.36857995e-001
iteration # 7 , p = 4.49218750e-001, $|p - p_{true}|$ = 1.78720671e-003, f(p) = 2.14987771e-001
iteration # 8 , p = 4.47265625e-001, $|p - p_{true}|$ = 1.65918289e-004, f(p) = -1.92260366e-002
iteration # 9 , p = 4.48242188e-001, $|p - p_{true}|$ = 8.10644211e-004, f(p) = 9.56908037e-002
iteration #10, p = 4.47753906e-001, $|p - p_{true}|$ = 3.22362961e-004, f(p) = 3.77002196e-002
iteration #11, p = 4.47509766e-001, $|p - p_{true}|$ = 7.82223363e-005, f(p) = 9.10590697e-003

b) Note that the Secant method diverges for $p_0 = 0$ and $p_1 = 0.48$.

The Secant method converges for some other choices of initial guesses, for example, $p_0 = 0.4$ and $p_1 = 0.48$, and gives the following results:

iteration # 1 , p = 4.18240450e-001, $|p-p_{true}|$ = 2.91910934e-002, f(p) = -2.19275325e+000
iteration # 2 , p = 4.29444232e-001, $|p-p_{true}|$ = 1.79873112e-002, f(p) = -1.56266385e+000
iteration # 3 , p = 4.57230361e-001, $|p-p_{true}|$ = 9.79881777e-003, f(p) = 1.39758405e+000
iteration # 4 , p = 4.44112051e-001, $|p-p_{true}|$ = 3.31949257e-003, f(p) = -3.63145086e-001
iteration # 5 , p = 4.46817663e-001, $|p-p_{true}|$ = 6.13880538e-004, f(p) = -7.05406509e-002
iteration # 6 , p = 4.47469928e-001, $|p-p_{true}|$ = 3.83844683e-005, f(p) = 4.46500003e-003
iteration # 7 , p = 4.47431099e-001, $|p-p_{true}|$ = 4.44116519e-007, f(p) = -5.16231960e-005
iteration # 8 , p = 4.47431543e-001, $|p-p_{true}|$ = 3.21333904e-010, f(p) = -3.73515059e-008
iteration # 9 , p = 4.47431543e-001, $|p-p_{true}|$ = 2.66453526e-015, f(p) = 3.15303339e-013
iteration #10, p = 4.47431543e-001, $|p-p_{true}|$ = 5.55111512e-017, f(p) = -5.32907052e-15
iteration #11, p = 4.47431543e-001, $|p-p_{true}|$ = 0.00000000e+000, f(p) = 3.55271368e-15

c) We have $f(x) = \tan(\pi x) - 6$, and so, $f'(x) = \frac{\pi}{\cos^2(\pi x)}$.
Since the function f has several roots, some initial guesses may lead to convergence to a different root. Indeed, for $p_0 = 0$, Newton's method converges to a different root. For Newton's method, therefore, it is suggested that you use $p_0 = 0.4$ in order to converge to $\frac{\arctan(6)}{\pi} \approx 0.447431543$. We obtain the following results:

iteration # 1 , p = 4.88826408e-001, $|p-p_{true}|$ = 4.13948647e-002, f(p) = -2.92231646e+000
iteration # 2 , p = 4.80014377e-001, $|p-p_{true}|$ = 3.25828340e-002, f(p) = 2.24759935e+001
iteration # 3 , p = 4.67600335e-001, $|p-p_{true}|$ = 2.01687920e-002, f(p) = 9.90600920e+000
iteration # 4 , p = 4.55142852e-001, $|p-p_{true}|$ = 7.71130844e-003, f(p) = 3.79052857e+000
iteration # 5 , p = 4.48555216e-001, $|p-p_{true}|$ = 1.12367274e-003, f(p) = 1.04904280e+000
iteration # 6 , p = 4.47455353e-001, $|p-p_{true}|$ = 2.38094477e-005, f(p) = 1.33441457e-001
iteration # 7 , p = 4.47431554e-001, $|p-p_{true}|$ = 1.06857097e-008, f(p) = 2.76882733e-003
iteration # 8 , p = 4.47431543e-001, $|p-p_{true}|$ = 2.10942375e-015, f(p) = 1.24209570e-006
iteration # 9 , p = 4.47431543e-001, $|p-p_{true}|$ = 5.55111512e-017, f(p) = 2.49578136e-013
iteration #10, p = 4.47431543e-001, $|p-p_{true}|$ = 0.00000000e+000, f(p) = -5.32907052e-15
iteration #11, p = 4.47431543e-001, $|p-p_{true}|$ = 5.55111512e-017, f(p) = 3.55271368e-15

We see that for these particular examples and initial guesses, the Newton's method and the Secant method give very similar convergence behaviors. The Newton's method converges slightly faster though. Newton's method and Secant method give $|p - p_{true}| \approx 0$ in 8 or 9 iterations, respectively. The bisection method converges much slower than the two other methods, as expected. In order to obtain similar accuracy, 45 iterations need to be made for the bisection method.