Midterm Review

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1 Root-Finding Methods

Rootfinding methods are designed to find a zero of a function f, that is, to find a value of x such that

$$f(x) = 0.$$

1.1 Bisection Method

To apply Bisection Method, we first choose an interval $[a, b]$ where $f(a)$ and $f(b)$ are of different signs. We define a midpoint

$$p = \frac{a+b}{2}.$$

If $f(p) = 0$, then p is a root and we stop.
Else if $f(a)f(p) < 0$, then a root lies in $[a, p]$, and we assign $b = p$. Otherwise, $a = p$.
We then consider this new interval $[a, b]$, and repeat the procedure.

The formula for midpoint above generates values p_n. We can bound an error of each iterate p_n for the bisection method:

$$|p_n - p| = \frac{b-a}{2^n}.$$

Note, as $n \to \infty$, $p_n \to p$.

Practice Problem: The function $f(x) = \sin x$ satisfies $f(-\pi/2) = -1$ and $f(\pi/2) = 1$. Using bisection method, how many iterations are needed to find an interval of length at most 10^{-4} which contains a root of a function?

Solution: We need to find n such that:

$$\frac{b-a}{2^n} \le 10^{-4}.$$

We have

$$\frac{\pi}{2^n} \le 10^{-4}.$$

$$\begin{aligned}
2^n &\ge \frac{\pi}{10^{-4}}, \\
2^n &\ge \pi 10^4, \\
n \log 2 &\ge \log(\pi 10^4), \\
n &\ge \frac{\log\left(\pi 10^4\right)}{\log 2} = 14.94.
\end{aligned}$$

That is, $n = 15$ iterations are needed to find an interval of length at most 10^{-4} which contains the root.
See Problems 1 and 3 in Homework 1 for other examples of bisection method.

1.2 Newton's Method

The Newton's iteration is defined as:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}.$$

The iteration converges for smooth functions if $f'(p_0) \neq 0$ and $|p - p_0|$ is small enough.

Practice Problem: Consider the equation $x = x^2 + 5$. Write down an algorithm based on Newton's method to solve this equation.

Solution: We define $f(x) = x^2 - x + 5$, and we want to find x such that $f(x) = 0$. We have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - x_n + 5}{2x_n - 1}.$$

This equation can be and should be simplified further.

Practice Problem: What is the order of convergence of Newton's method?

Solution: Newton's method is quadratically convergent (second order of convergence), which means that

$$|x^* - x_{n+1}| \leq C|x^* - x_n|^2.$$

Practice Problem: Suppose $g(x)$, a smooth function, has a fixed point x^*; that is $g(x^*) = x^*$. Write a Taylor expansion of $g(x_n)$ around x^*.
Solution:

$$g(x_n) = g(x^*) + (x_n - x^*)g'(x^*) + \frac{(x_n - x^*)^2}{2} g''(\xi_n).$$

We can use the information that is given to us by the problem. A fixed point iteration is defined as $x_{n+1} = g(x_n)$. Also, $g(x^*) = x^*$. Using this, we can rewrite the equation as

$$x_{n+1} = x^* + (x_n - x^*)g'(x^*) + \frac{(x_n - x^*)^2}{2} g''(\xi_n),$$

or

$$x_{n+1} - x^* = (x_n - x^*)g'(x^*) + \frac{(x_n - x^*)^2}{2} g''(\xi_n).$$

Quite a few observations can be made using this equation if additional information is given by a problem.

1.3 Secant Method

The Secant method is derived from Newton's method by replacing $f'(p_{n-1})$ with the following approximation:

$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}}.$$

Then, the Newton's iteration can be rewritten as follows. This iteration is called the Secant method.

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-2} - p_{n-1})}{f(p_{n-2}) - f(p_{n-1})}.$$

2 Computer Arithmetic

Given a *binary number* (also known as a *machine number*), for example

$$\underbrace{0}_{s}\ \underbrace{10000001010}_{c}\ \underbrace{10010011000000\cdots0}_{f}$$

a *decimal number* (also known as a *floating-point decimal number*) is of the form:

$$(-1)^s 2^{c-1023}(1+f). \tag{1}$$

Therefore, in order to find a decimal representation of a binary number, we need to find s, c, and f and plug these into (1).
Problems 3 and 4 in Homework 3 are good applications of this idea. If you understand how to do such problems, consider a similar problem below.

Practice Problem: Consider a *binary number* (also known as a *machine number*)

0 10000001010 10010011000000 $\cdots$ 00

Find the floating point decimal number it represents as well as the next largest floating point decimal number.

Solution: A *decimal number* (also known as a *floating-point decimal number*) is of the form:

$$(-1)^s 2^{c-1023}(1+f).$$

Therefore, in order to find a decimal representation of a binary number, we need to find s, c, and f.
The leftmost bit is zero, i.e. $s=0$, which indicates that the number is positive.
The next 11 bits, 10000001010, giving the characteristic, are equivalent to the decimal number:

$$\begin{aligned} c &= 1\cdot 2^{10} + 0\cdot 2^9 + \cdots + 0\cdot 2^4 + 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0 \\ &= 1024 + 8 + 2 \;=\; 1034. \end{aligned}$$

The exponent part of the number is therefore $2^{1034-1023} = 2^{11}$.
The final 52 bits specify that the mantissa is

$$f = 1\cdot\left(\frac{1}{2}\right)^1 + 1\cdot\left(\frac{1}{2}\right)^4 + 1\cdot\left(\frac{1}{2}\right)^7 + 1\cdot\left(\frac{1}{2}\right)^8.$$

Therefore, this binary number represents the decimal number

$$\begin{aligned} (-1)^s 2^{c-1023}(1+f) &= (-1)^0\cdot 2^{1034-1023}\cdot\left(1+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^7+\left(\frac{1}{2}\right)^8\right) \\ &= 2^{11}\cdot\left(1+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^7+\left(\frac{1}{2}\right)^8\right) \\ &= 2^{11}+2^{10}+2^7+2^4+2^3. \end{aligned}$$

It won't be necessary to further simplify this number on the test.

The next largest machine number is

0 10000001010 10010011000000 $\cdots$ 01 .

We already know that $s=0$ and $c=1034$ for this number. We find f:

$$f = 1\cdot\left(\frac{1}{2}\right)^1 + 1\cdot\left(\frac{1}{2}\right)^4 + 1\cdot\left(\frac{1}{2}\right)^7 + 1\cdot\left(\frac{1}{2}\right)^8 + 1\cdot\left(\frac{1}{2}\right)^{52}.$$

Therefore, this binary number represents the decimal number

$$\begin{aligned} (-1)^s 2^{c-1023}(1+f) &= 2^{11}\left(1+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^7+\left(\frac{1}{2}\right)^8+\left(\frac{1}{2}\right)^{52}\right) \\ &= 2^{11}+2^{10}+2^7+2^4+2^3+\left(\frac{1}{2}\right)^{41}. \end{aligned}$$

It won't be necessary to further simplify this number on the test.
Note how these two numbers differ.

3 Interpolation

3.1 Lagrange Polynomials

We can construct a polynomial of degree at most n that passes through $n+1$ points:

$$(x_0, f(x_0)),\ (x_1, f(x_1)),\ \ldots,\ (x_n, f(x_n)).$$

Such polynomial is unique.

Linear (first order) interpolation is achieved by constructing the Lagrange polynomial P_1 of order 1, connecting the two points. We have:

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1),$$

where [1]

$$\begin{aligned} L_0(x) &= \frac{x - x_1}{x_0 - x_1}, \\ L_1(x) &= \frac{x - x_0}{x_1 - x_0}. \end{aligned}$$

Quadratic (second order) interpolation is achieved by constructing the Lagrange polynomial P_2 of order 2, connecting the three points. We have:

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2),$$

where

$$\begin{aligned} L_0(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}, \\ L_1(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}, \\ L_2(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}. \end{aligned}$$

In general, to construct a polynomial of order n, connecting $n+1$ points, we have

$$P_n(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + \ldots + L_n(x)f(x_n),$$

where

$$L_k(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}.$$

L_k are called the $k-$Lagrange basis functions.

[1] Note that in some sources, $L_{n,k}$ notation is used for functions below, where n designates the order of polynomial. To avoid confusion, I omit n-index since it is usually obvious what order of the polynomial we are considering. I write those functions as L_k.

3.2 Newton's Divided Differences

The polynomial of degree n, interpolating $n+1$ points, can be written in terms of Newton's divided differences:

$$\begin{aligned} P_n(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ \ldots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}). \end{aligned}$$

The zeroth divided difference of f with respect to x_i is

$$f[x_i] = f(x_i).$$

The first divided difference of f with respect to x_i and x_{i+1} is

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

The second divided difference is

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$